Partial Hölder continuity for solutions of subquadratic elliptic systems in low dimensions

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Abstract: We consider weak solutions of second order nonlinear elliptic systems in divergence form under standard subquadratic growth conditions with boundary data of class C^1. In dimensions $n \in \{2, 3\}$ we prove that u is locally Hölder continuous for every exponent $\lambda \in (0, 1 - \frac{n-2}{p})$ outside a singular set of Hausdorff dimension less than $n - p$. This result holds up to the boundary both for non-degenerate and degenerate systems. In the proof we apply the direct method and classical Morrey-type estimates introduced by Campanato.

1 Introduction and result

In this paper we consider weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of a general inhomogeneous system of second order elliptic equations in divergence form

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here $n, N \geq 2$, $p \in (1, 2)$, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain of class C^1, and we assume boundary values $g \in C^1(\overline{\Omega}, \mathbb{R}^N)$. As usual this boundary condition is to be understood in the sense of traces. For the coefficients $a \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ we impose standard boundedness, differentiability, growth and ellipticity conditions: $z \mapsto a(\cdot, \cdot, z)$ is of class $C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus \{0\}, \mathbb{R}^{nN})$, and for fixed $0 < \nu \leq L$ and all $x, \bar{x} \in \overline{\Omega}$, $u, \bar{u} \in \mathbb{R}^N$, and $z, \bar{z}, \lambda \in \mathbb{R}^{nN}$ we have

$$\begin{cases} \big|a(x,u,z)\big| + \big|D_z a(x,u,z)\big| \left(\mu^2 + |z|^2\right)^{\frac{1}{2}} \;\leq\; L \left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}}, \\ D_z a(x,u,z)\,\lambda \cdot \lambda \;\geq\; \nu \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2\,, \\ \big|a(x,u,z) - a(\bar{x},\bar{u},z)\big| \;\leq\; L \left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}} \omega\big(|x - \bar{x}| + |u - \bar{u}|\big)\,, \end{cases} \tag{1.2}$$

where $\omega \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity, i.e. bounded by 1 (without loss of generality), concave and non-decreasing with $\lim_{\rho \to 0} \omega(\rho) = 0$. The parameter $\mu \in [0, 1]$ specifies whether the system is non-degenerate, $\mu \neq 0$, or degenerate, $\mu = 0$. We have excluded $z = 0$ in conditions $(1.2)_1$ and $(1.2)_2$ in order to deal also with degenerate systems. Condition $(1.2)_3$ means that the coefficients $a(x, u, z)$ are continuous with respect to (x, u), uniformly for fixed z. Moreover, we assume the inhomogeneity $b \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ to be a Carathéodory map (that is, it is continuous with respect to (u, z) and measurable with respect to x) and that $b(\cdot, \cdot, \cdot)$ satisfies one of the following growth conditions:

(B1) Controllable growth condition: for all $(x, u, z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ we have

$$|b(x,u,z)| \;\leq\; L \left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}},$$

(B2) Natural growth condition: there exists a constant L_2 (possibly depending on $M_2 > 0$) such that for all $(x, u, z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ with $|u| \leq M_2$ we have

$$|b(x,u,z)| \leq L_2\, |z|^p + L\,.$$

In this work we are interested in obtaining Morrey-type estimates up to the boundary, and the question of partial regularity of the weak solution u in low dimensions where $n \in (p, p+2]$. For this purpose we

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define the set of regular and singular points of u via

$$\begin{aligned} \mathrm{Reg}_u(\overline{\Omega}) &:= \big\{x \in \overline{\Omega} : u \in C^0(\overline{\Omega} \cap A, \mathbb{R}^N) \text{ for some neighbourhood } A \text{ of } x\big\}\,, \\ \mathrm{Sing}_u(\overline{\Omega}) &:= \overline{\Omega} \setminus \mathrm{Reg}_u(\overline{\Omega})\,. \end{aligned}$$

In this setting of low-dimensional analysis various results have been proved: under a controllable growth assumption, Campanato [7] obtained local Hölder continuity of the weak solution on the regular set in the interior of Ω, and he gave an upper bound on the Hausdorff dimension of the singular set. He further achieved similar results for systems of higher order [8]. Moreover, Campanato [9, 10] presented global Morrey-estimates for the weak solution of systems with coefficients not depending explicitly on u, i. e., $a(x,u,z) \equiv a(x,z)$, for $p \geq 2$ (further higher order Morrey-type and Hölder estimates can be found e. g. in [32]). Under a natural growth condition, Arkhipova [2, 3] proved a partial regularity result up to the boundary for non-degenerate systems in the superquadratic case.

In this paper we are concerned with low order regularity in the subquadratic case: we prove that the weak solution u to the nonlinear system (1.1) is locally Hölder continuous on $\mathrm{Reg}_u(\overline{\Omega})$ for some Hölder exponent $\lambda > 0$ under the assumption that the inhomogeneity obeys either a controllable or a natural growth condition (in the latter case we require additionally that u is bounded and that a standard smallness assumption on $\|u\|_{L^\infty}$ holds). Moreover, we show that the set of singular points is of Hausdorff dimension strictly less than $n-p$, which implies immediately that $\mathcal{H}^{n-1}$-almost every boundary point is regular. For arbitrary dimension n, under such a mild continuity assumption on the coefficients, this property has only been proved for quasilinear systems, see for example [12, 31, 21, 26], whereas in the general setting partial Hölder regularity of Du (as opposed to the regularity of u) can be proved outside a set of Lebesgue measure zero (for subquadratic growth problems in the interior we refer to [11]); the related problem of dimension reduction of the singular set $\mathrm{Sing}_{Du}(\Omega)$ was a long-standing issue which was recently tackled by Mingione [28, 29] under additional assumptions on $\omega(\cdot)$, and by Duzaar, Kristensen and Mingione [16] for the dimension reduction up to the boundary. To return to the low-dimensional case we now state our main theorem:

Theorem 1.1: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 and $g \in C^1(\overline{\Omega}, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1,2)$, be a weak solution of* (1.1) *with coefficients $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfying the assumptions* (1.2)*, and inhomogeneity $b : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$. If one of the following assumptions is fulfilled:*

1. *$b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition* (B1)*,*
2. *$b(\cdot,\cdot,\cdot)$ obeys a natural growth condition* (B2)*; additionally, we assume $u \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq M$ and $2L_2M < \nu$,*

then there exists $\delta > 0$ depending only on n, N, p and $\frac{L}{\nu}$ such that for $n \in [2, p+2+\delta)$ there hold

$$\dim_{\mathcal{H}}\big(\overline{\Omega} \setminus \mathrm{Reg}_u(\overline{\Omega})\big) \,<\, n-p \qquad \text{and} \qquad u \in C^{0,\lambda}_{\mathrm{loc}}\big(\mathrm{Reg}_u(\overline{\Omega}), \mathbb{R}^N\big)$$

for all $\lambda \in \big(0, \min\{1 - \frac{n-2-\delta}{p}, 1\}\big)$. Moreover, the singular set $\mathrm{Sing}_u(\overline{\Omega})$ of u is contained in

$$\Sigma \,:=\, \Big\{x_0 \in \overline{\Omega} : \liminf_{R \searrow 0} \, R^{p-n} \int_{B_R(x_0)\cap\Omega} \big(1 + |Du|^p\big)\, dx \,>\, 0\Big\}\,.$$

We mention that the number δ arises from the application of Gehring's lemma on higher integrability and depends only on the structure constants (see e. g. [5, Remark 3] for an explicit possible choice of the higher integrability exponent). Therefore, the condition $n \in [2, p+2+\delta)$ mostly means $n \in \{2,3\}$ unless p is close to 2 or δ happens to be large.

Taking into account the general form of the coefficients (i. e., their u-dependency) and the counterexamples given in [13, 24, 19, 30] (for $n \geq 3$), it is well known that we cannot expect full Hölder continuity. In contrast, due to the global higher integrability of the weak solution and the Sobolev embedding theorem, we see that full Hölder regularity up to the boundary holds true provided that p is close to n. However, since the literature lacks appropriate counterexamples in the two-dimensional case (all the counterexamples mentioned above are for codimension ≥ 3), it is still an open question whether there might exist a singular point in dimension $n = 2$ and arbitrary $p \in (1,2)$.

We note that we have included partial Hölder continuity of weak solutions to degenerate systems. A model case of the degenerate situation is given by the p-Laplacian

$$\operatorname{div}\left(|Du|^{p-2}\, Du\right) \;=\; 0 \qquad \text{in } \Omega\,.$$

Finally we briefly comment on the techniques used within this paper: the strategy for the proof of the partial regularity result stated in Theorem 1.1 above relies on the direct method and the application of classical techniques pioneered by Campanato, see e. g. [7, 8, 9, 10]. For the examination of both the boundary situation and the interior, we define adequate comparison maps which are solutions of a frozen homogeneous system and for which we provide good a priori estimates. The major difficulty here lies in establishing an appropriate Caccioppoli-type inequality up to the boundary. The decay estimates for the comparison map then allow us to deduce Morrey-type estimates for the gradient Du, namely that Du belongs to a suitable Morrey space $L^{p,\gamma}(\Omega,\mathbb{R}^{nN})$, which yields the desired Hölder continuity of u (in view of the Campanato-Meyer embedding theorem). In the case of natural growth of the inhomogeneity these techniques require some modifications for which we adapt Arkhipova's cut-off procedure from [2, 3, Proof of Theorem 1]. The upper bound for the Hausdorff dimension of the singular set then follows immediately from the characterization of the singular set and a measure density result due to Giusti.

Lastly we mention that, for the sake of brevity, we only sketch some of the proofs or refer to other papers, but the proofs of all statements can be found in detail in the author's PhD thesis [4].

2 Notation and preliminaries

We start with some remarks on the notation used below: we write $B_\rho(y) = \{x \in \mathbb{R}^n : |x-y| < \rho\}$ and $B_\rho^+(y) = \{x \in \mathbb{R}^n : x_n > 0,\, |x-y| < \rho\}$ for a ball or the intersection of a ball with the upper half-space $\mathbb{R}^{n-1}\times\mathbb{R}^+$, centred at a point $y \in \mathbb{R}^n$ (respectively $\in \mathbb{R}^{n-1}\times\mathbb{R}_0^+$ in the latter case) with radius $\rho > 0$. Furthermore, we write

$$\Gamma_\rho(y) \;=\; \left\{x \in \mathbb{R}^n : |x-y| < \rho,\, x_n = 0\right\},$$

for $y \in \mathbb{R}^{n-1}\times\{0\}$. In the case $y = 0$ we set $B_\rho := B_\rho(0)$, $B := B_1$ as well as $B_\rho^+ := B_\rho^+(0)$, $B^+ := B_1^+$ with $\Gamma_\rho := \Gamma_\rho(0)$, $\Gamma := \Gamma_1$. We introduce the following notation for $W^{1,p}$-functions defined on a half-ball $B_\rho^+(y)$ and which vanish on the flat part of the boundary (in the sense of traces):

$$W_\Gamma^{1,p}(B_\rho^+(y),\mathbb{R}^N) := \left\{u \in W^{1,p}(B_\rho^+(y),\mathbb{R}^N) : u = 0 \text{ on } \Gamma_{\sqrt{\rho^2-(y)_n^2}}(x_0'')\right\}.$$

where $y_n < \rho$ is satisfied and where $y'' := (y_1,\dots,y_{n-1},0)$ denotes the projection of y onto $\mathbb{R}^{n-1}\times\{0\}$. Sometimes, it will be convenient to treat the tangential derivative $D'u := (D_1u,\dots,D_{n-1}u)$ and the normal derivative D_nu of a function $u \in W^{1,p}(B_\rho^+(y),\mathbb{R}^N)$ separately.

For a given set $X \subset \mathbb{R}^n$ we write $\mathcal{L}^n(X) = |X|$ and $\dim_{\mathcal{H}}(X)$ for its n-dimensional Lebesgue-measure and its Hausdorff dimension, respectively. Furthermore, if $h \in L^1(X,\mathbb{R}^N)$ and $0 < |X| < \infty$, we denote the average of h by $(h)_X = \fint_X h\, dx$. The constants c appearing in the different estimates will all be chosen greater than or equal to 1, and they may vary from line to line. For ease of notation, some of the constants are labelled by the superscript (i) and refer to the growth condition (Bi) for $i = 1,2$.

In what follows, we shall use the following definitions of Morrey and Campanato spaces:

Definition: *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $1 \le p < \infty$. By $L^{p,\varsigma}(\Omega,\mathbb{R}^N)$, $\varsigma \ge 0$, we denote the Morrey space of all functions $u \in L^p(\Omega,\mathbb{R}^N)$ such that*

$$\|u\|^p_{L^{p,\varsigma}(\Omega,\mathbb{R}^N)} := \sup_{y\in\Omega,0<\rho\le\operatorname{diam}\Omega} \rho^{-\varsigma}\int_{B_\rho(y)\,\cap\,\Omega} |u|^p\, dx \;<\; \infty\,.$$

By $\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)$, $0 \le \varsigma \le n+p$, we denote the Campanato space of all functions $u \in L^p(\Omega,\mathbb{R}^N)$ such that

$$[u]^p_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)} := \sup_{y\in\Omega,0<\rho\le\operatorname{diam}\Omega} \rho^{-\varsigma}\int_{B_\rho(y)\,\cap\,\Omega} \left|u-(u)_{B_\rho(y)\cap\Omega}\right|^p dx \;<\; \infty\,.$$

To handle the subquadratic case the V-function is very useful. For $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$, $\mu \in [0,1]$ and $p > 1$ it is defined by

$$V_\mu(\xi) = \big(\mu^2 + |\xi|^2\big)^{\frac{p-2}{4}}\, \xi\,,$$

which is a locally bi-Lipschitz bijection on $\mathbb{R}^k$. When we deal with the V_μ-function, we will need some technical lemmas:

Lemma 2.1: *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0,1]$ and $q > -1$. Then there exist constants $c_1(q), c_2(q) \geq 1$ not depending on μ such that*

$$c_1^{-1}\,\big(\mu + |\xi| + |\eta|\big)^q \;\leq\; \int_0^1 \big(\mu + |\xi + t\eta|\big)^q\,dt \;\leq\; c_2\,\big(\mu + |\xi| + |\eta|\big)^q\,.$$

A proof of the latter statement can be found in [1, Lemma 2.1] and for the case $\mu = 1$ also in [6]. The second lemma collects some basic inequalities:

Lemma 2.2: *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0,1]$ and $p \in (1,2)$. Then there exist constants $c_1(k,p)$ and $c_2(p)$ such that the following inequalities hold true:*

(i) $c_1^{-1}\,|\xi - \eta|\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}} \;\leq\; |V_\mu(\xi) - V_\mu(\eta)| \;\leq\; c_1\,|\xi - \eta|\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}$

(ii) $(\mu^2 + |\xi|^2)^{\frac{p}{2}} \;\leq\; c_2\,(\mu^2 + |\eta|^2)^{\frac{p}{2}} + c_2\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}\,|\xi - \eta|^2\,,$

(iii) $(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}\,|\xi|\,|\eta| \;\leq\; \varepsilon\,(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}\,|\xi|^2 + \varepsilon^{1-p}(\mu^2 + |\eta|^2)^{\frac{p}{2}}$ *for* $\varepsilon \in (0,1)$.

Proof: The inequality in (i) is proved in [1, Lemma 2.2], while the other inequalities are easily obtained by distinguishing cases: for (ii) we consider $\max\{\mu, |\eta|\} > \frac{1}{2}|\xi|$ and $\max\{\mu, |\eta|\} \leq \frac{1}{2}|\xi|$, and for (iii) we study the cases $|\eta| > \varepsilon|\xi|$ and $|\eta| \leq \varepsilon|\xi|$. □

3 Comparison estimates

In this section we provide some up-to-the-boundary comparison estimates concerning degenerate and non-degenerate homogeneous elliptic system which do not depend on (x,u). We here restrict ourselves to the model case of a half-ball and we thus turn our attention to weak solutions $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, $R < 1$ and $p \in (1,2)$, of the system

$$\operatorname{div} a_0(Dv) \;=\; 0 \qquad \text{in } B^+_R(x_0)\,, \tag{3.1}$$

where the coefficients $a_0 \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are class $C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus \{0\}, \mathbb{R}^{nN})$ and satisfy boundedness, differentiability, growth and ellipticity conditions corresponding to the assumptions $(1.2)_1$ and $(1.2)_2$ above. We now prove the existence of second order derivatives for the solution v of (3.1) using a difference quotients method. Furthermore, we derive a Caccioppoli-type estimate for second order derivatives, where a certain integral involving second derivatives is bounded by only the tangential part of $V(Dv)$. Then, via a global version of Gehring's lemma, this improved representation of the Caccioppoli inequality allows to obtain a higher integrability result up to the boundary which in turn yields a decay estimate for the weak derivative Dv.

Theorem 3.1: *Let $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$ be a weak solution to the system (3.1), whose coefficients $a_0(\cdot)$ satisfy the conditions $(1.2)_1$ and $(1.2)_2$, and let $\mu \in [0,1]$ be arbitrary. Then v is twice differentiable in the weak sense, more precisely $v \in W^{2,p}(B^+_{R'}(x_0), \mathbb{R}^N)$ for all $R' < R$, and there exists a constant c depending only on n, N, p and $\frac{L}{\nu}$ such that there hold*

a) (close to the boundary) for all $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n \leq \frac{3}{4}r$

$$\int_{B^+_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2\,dx \;\leq\; c\,r^{-2} \int_{B^+_r(y)} \big|V'_\mu(Dv)\big|^2\,dx\,, \tag{3.2}$$

where $V'_\mu(Dv) := \big(V_{\mu,1}(Dv), \dots, V_{\mu,n-1}(Dv)\big)$ is the tangential part of $V_\mu(Dv)$, *b) (in the interior) for all $y \in B^+_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n > \frac{3}{4}r$*

$$\int_{B_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2 \, dx \;\le\; c\, r^{-2} \int_{B_{3r/4}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B_{3r/4}(y)}\big|^2 \, dx\,. \tag{3.3}$$

Remark 3.2: We note that in statement a) the normal derivative of v is not involved in the quadratic term on the right-hand side. If we pass to systems with coefficients which additionally depend explicitly on x (as in the original formulation), this result can no longer be expected because a dependence only on the x_n-variable of the solution might occur: consider for example the coefficients $a(x,z)$ defined by

$$a(x,z) \;=\; \frac{\big(1+|z|^2\big)^{\frac{p-2}{2}}\, z}{\big(1+(1+x_n^\alpha)^2\big)^{\frac{p-2}{2}}\,(1+x_n^\alpha)}$$

for some number $\alpha \in (0,1)$. Then, $v(x) = \frac{1}{1+\alpha}\, x_n^{1+\alpha} + x_n$ is a weak solution of $\operatorname{div} a(x,Dv) = 0$ in $B^+ \subset \mathbb{R}^n, n \ge 2$, but the statement of the theorem obviously does not hold on any (half-)ball $B^+_r(y) \subset B^+$, and even $v \in W^{2,p}(B^+_\rho, \mathbb{R}^N)$ does not hold for some $0 < \rho < 1$ (in fact, v only belongs to a suitable fractional Sobolev space).

Proof: We proceed in several steps, concentrating on the estimates close to the boundary:

Step 1: A preliminary estimate. We begin by deriving the following Caccioppoli-type inequalities: close to the boundary, we have for all $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n \le \frac{3}{4}r$

$$\int_{B^+_\rho(y)} \big|D'(V_\mu(Dv))\big|^2 \, dx \;\le\; c(n,p,\tfrac{L}{\nu})\,(r-\rho)^{-2} \int_{B^+_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}} \, dx \tag{3.4}$$

for all $\rho < r$, whereas in the interior, we obtain for all $y \in B^+_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n > \frac{3}{4}r$

$$\int_{B_\rho(y)} \big|D(V_\mu(Dv))\big|^2 \, dx \;\le\; c(n,p,\tfrac{L}{\nu})\,(r-\rho)^{-2} \int_{B^+_{3r/4}(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}} \, dx$$

for all $\rho < \frac{3}{4}r$. In order to prove (3.4) we proceed similarly to the proof of [23, Theorem 8.1], merely adjusting it to the boundary situation and $\mu \in [0,1]$. We fix y, r, ρ as above and consider a standard cut-off function $\eta \in C^\infty_0(B_{(r+\rho)/2}(y), [0,1])$ satisfying $\eta \equiv 1$ on $B_\rho(y)$ and $|D\eta|^2 + |D^2\eta| \le c\,(r-\rho)^{-2}$. In the sequel, we abbreviate the usual difference quotient of v with respect to x_s and stepsize h by $\triangle_{s,h} v$, i.e., $\triangle_{s,h} v(x) := (v(x + he_s) - v(x))/h$, where e_s, $s = 1, ..., n$, denotes the standard basis of $\mathbb{R}^n$. Let $|h| < \frac{r-\rho}{2}$. Then, since tangential difference quotients preserve the zero boundary values of v on $\Gamma_r(y)$, we observe that $\eta^2 \triangle_{s,h} v \in W^{1,p}_0(B_{(r+\rho)/2}(y), \mathbb{R}^N)$ for all tangential directions $s = 1, \dots, n-1$. Hence, the function

$$\varphi \;=\; \triangle_{s,-h}\big(\eta^2 \triangle_{s,h} v\big) \qquad \in W^{1,p}_0(B^+_r(y), \mathbb{R}^N)$$

is admissible for testing the system (3.1). Integration by parts for finite differences yields

$$\int_{B^+_r(y)} \triangle_{s,h}\, a_0(Dv) \cdot D\triangle_{s,h} v\, \eta^2 \, dx \;=\; -2 \int_{B^+_r(y)} \triangle_{s,h}\, a_0(Dv) \cdot (\triangle_{s,h} v \otimes D\eta)\, \eta \, dx\,. \tag{3.5}$$

The difference quotient $\triangle_{s,h}\, a_0(Dv(x)) = [a_0\big(Dv(x+he_s)\big) - a_0\big(Dv(x)\big)]/h$ can be rewritten as follows:

$$\triangle_{s,h}\, a_0\big(Dv(x)\big) = \int_0^1 D_z a_0\big(Dv(x) + th\triangle_{s,h} Dv(x)\big)\, dt\, \triangle_{s,h} Dv(x)\,. \tag{3.6}$$

Here, the term involving the derivative $D_z a_0(\cdot)$ might not be well defined for some $\tilde{t} \in [0,1]$ for degenerate systems ($\mu = 0$), but the integral in (3.6) exists, see the justification in [17, p. 749]. Using the ellipticity condition $(1.2)_2$, Young's inequality and $p < 2$, we deduce the following inequality for the right-hand side of the previous identity (3.6):

$$\begin{aligned} &\int_0^1 D_z a_0\big(Dv(x) + th\triangle_{s,h} Dv(x)\big)\, dt\, \triangle_{s,h} Dv(x) \cdot \triangle_{s,h} Dv(x) \\ &\qquad \ge\; 2^{\frac{p-2}{2}}\, \nu\, \big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\, |\triangle_{s,h} Dv(x)|^2 \\ &\qquad =:\; 2^{\frac{p-2}{2}}\, \nu\, Z_\mu(x)^{p-2}\, |\triangle_{s,h} Dv(x)|^2 \end{aligned} \tag{3.7}$$

with the obvious abbreviation of $Z_\mu(x)$. Combining (3.7) with the identities (3.6) and (3.5), we find

$$2^{\frac{p-2}{2}}\nu\int_{B_r^+(y)} Z_\mu^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx \le \int_{B_r^+(y)}\int_0^1 D_z a_0(Dv+th\triangle_{s,h}Dv)\,dt\,\triangle_{s,h}Dv\cdot\triangle_{s,h}Dv\,\eta^2\,dx$$
$$= -2\int_{B_r^+(y)} \triangle_{s,h}\,a_0(Dv)\cdot(\triangle_{s,h}v\otimes D\eta)\,\eta\,dx\,. \tag{3.8}$$

In view of $\operatorname{spt}\eta\subset B_{(r+\rho)/2}(y)$ and the restriction $|h|<\frac{r-\rho}{2}$ we rewrite the right-hand side of the latter inequality using partial integration for finite differences, and we then apply the growth condition $(1.2)_1$, Young's inequality and standard properties of difference quotients (see e. g. [22, Chapter 7.11]) to find

$$\begin{aligned}
-2\int_{B_r^+(y)} \triangle_{s,h}\,a_0(Dv)\cdot(\triangle_{s,h}v\otimes D\eta)\,\eta\,dx \,&=\, 2\int_{B_r^+(y)} a_0(Dv)\cdot\triangle_{s,-h}\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\,dx\\
&\le\, 2\,L\,(r-\rho)^{-2}\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + 2\,L\,(r-\rho)^{2p-2}\int_{B_r^+(y)}\big|D_s\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\big|^p\,dx\\
&\le\, 2\,L\,(r-\rho)^{-2}\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + c\,L\,(r-\rho)^{-2}\int_{B^+_{(r+\rho)/2}(y)}|\triangle_{s,h}v|^p\,dx\\
&\quad + c\,L\,(r-\rho)^{p-2}\int_{B_r^+(y)}|\triangle_{s,h}D_sv|^p\,\eta^p\,dx\,,
\end{aligned}$$

where we have applied Young's inequality and the properties of the cut-off function η in the last line. We now observe from Young's inequality that we have

$$|\triangle_{s,h}Dv(x)|^p \,\le\, Z_\mu(x)^p + Z_\mu(x)^{p-2}|\triangle_{s,h}Dv(x)|^2 \tag{3.9}$$

(note: if $Z_\mu(x)=0$ then both sides vanish and the inequality trivially holds true). Using adequate modifications of inequality (3.9), we thus infer from (3.8):

$$\begin{aligned}
&2^{\frac{p-2}{2}}\,\nu\int_{B_r^+(y)} Z_\mu^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx\\
&\quad\le\, c(\tfrac{L}{\varepsilon})\,L\,(r-\rho)^{-2}\Big(\int_{B^+_{(r+\rho)/2}(y)}\big(Z_\mu^p+|\triangle_{s,h}v|^p\big)\,dx + \int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\Big)\\
&\qquad + \varepsilon\int_{B_r^+(y)} Z_\mu^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx\,.
\end{aligned}\tag{3.10}$$

Keeping in mind the definition of the function Z_μ, $B^+_{(r+\rho)/2}(y)\supset\operatorname{spt}(\eta)$ and $|h|\le\frac{r-\rho}{2}$, we observe

$$\int_{B^+_{(r+\rho)/2}(y)}\big(Z_\mu^p+|\triangle_{s,h}v|^p\big)\,dx \,\le\, 3\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\,. \tag{3.11}$$

Therefore, choosing $\varepsilon=2^{\frac{p-4}{2}}\nu$ in (3.10), dividing through by $2^{\frac{p-4}{2}}\nu$, recalling that $\eta=1$ on $B_\rho(y)$, we finally arrive at

$$\int_{B_\rho^+(y)} Z_\mu^{p-2}|\triangle_{s,h}Dv|^2\,dx \,\le\, c\,(r-\rho)^{-2}\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\,, \tag{3.12}$$

and the constant c depends only on $\frac{L}{\nu}$. We mention here: in order to conclude that the tangential derivatives belong to the space L^p, we deduce analogously to [23, Proof of Theorem 8.1] from inequality (3.9): the family $\big(\triangle_{s,h}Dv\big)_h$, $h\in\mathbb{R}$ with $|h|<\frac{r-\rho}{2}$, is bounded in $L^p(B_\rho(y),\mathbb{R}^{nN})$ (see (3.11), (3.12)) and therefore converges in $L^p(B^+_{\rho'}(y),\mathbb{R}^{nN})$ to D_sDv for all $\rho'<\rho$ (see e.g. [18], Chapter 5.8.2, Proof of Theorem 3 and the remark immediately after). Keeping in mind $s\in\{1,\dots,n-1\}$, we end up with $D'v\in W^{1,p}(B^+_{R'}(x_0),\mathbb{R}^{(n-1)N})$ for all $R'<R$.

We now apply Lemma 2.2 (i) and obtain

$$\begin{aligned}
&\int_{B_\rho^+(y)}\big|\triangle_{s,h}V_\mu(Dv)\big|^2\,dx\\
&\quad\le\, c(p)\,h^{-2}\int_{B_\rho^+(y)}\big(\mu^2+|Dv(x)|^2+|Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\,\big|Dv(x+he_s)-Dv(x)\big|^2\,dx\\
&\quad=\, c(p)\int_{B_\rho^+(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv(x)|^2\,dx \,\le\, c\big(p,\tfrac{L}{\nu}\big)\,(r-\rho)^{-2}\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\,.
\end{aligned}$$

As above, the sequence $\big(\triangle_{s,h}V_\mu(Dv)\big)_h$ is uniformly bounded in $L^2(B_\rho(y),\mathbb{R}^{nN})$ and therefore converges strongly to $D_s(V_\mu(Dv))$, $s=1,\ldots,n-1$. Thus we obtain the *tangential estimate*, and summing up yields the desired inequality (3.4) for the boundary situation. Finally we note that the proof of the corresponding inequality in the interior case is achieved in the same way, but we do not need any constraint of the direction, i. e., we can take $s=1,\ldots,n$.

Step 2: An improved estimate. We again start with the boundary situation and consider $y\in B_R^+(x_0)\cup\Gamma_R(x_0)$ and $0<r<R-|y-x_0|$ with $y_n\le\frac34 r$. We first note that inequality in (3.4) is for the tangential derivatives already the desired estimate, apart from the fact that the bound stated in (3.2) is sharper. To prove the inequality in the final form we proceed similarly to the first step, but with the important difference that we already may take advantage of the fact $D_sv\in W^{1,p}_\Gamma(B^+_{R'}(x_0),\mathbb{R}^N)$ for all $0<R'<R$ and for all tangential derivatives ($s=1,\ldots,n-1$). Thus, the function

$$\varphi\,=\,\triangle_{s,-h}(\eta^2D_sv)\qquad\in W^{1,p}_0(B_r^+(y),\mathbb{R}^N)$$

is admissible for testing the system (3.1), where $s\in\{1,\ldots,n-1\}$, $|h|<\frac r4$ and $\eta\in C_0^\infty(B_{3r/4}(y),[0,1])$ is a standard cut-off function satisfying $\eta\equiv1$ on $B_{r/2}(y)$ and $D\eta\le c\,r^{-1}$ (cf. the previous test function). With integration by parts for finite differences we infer the identity

$$\int_{B_r^+(y)}\triangle_{s,h}\,a_0(Dv)\cdot\big(DD_sv\,\eta+2\,D_sv\otimes D\eta\big)\eta\,dx\;=\;0\,.$$

Therefore, instead of inequality (3.8), we now obtain

$$\begin{aligned}\nu\int_{B_r^+(y)}&\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}}|DD_sv|^2\,\eta^2\,dx\;\le\;\int_{B_r^+(y)}D_za_0(Dv)\,DD_sv\cdot DD_sv\,\eta^2\,dx\\&=\int_{B_r^+(y)}\big(D_sa_0(Dv)-\triangle_{s,h}a_0(Dv)\big)\cdot\big(DD_sv\,\eta+2\,D_sv\otimes D\eta\big)\eta\,dx\\&\quad-2\int_{B_r^+(y)}D_sa_0(Dv)\cdot(D_sv\otimes D\eta)\,\eta\,dx\end{aligned}\tag{3.13}$$

(note: all integrands vanish on the set $\{x\in B_r^+(y):Dv(x)=0\}$). We rewrite the first integral on the right-hand side as $\int_{B_r^+(y)}f_h\cdot g\,dx$, where we have abbreviated

$$\begin{aligned}f_h&:=\;\big(D_sa_0(Dv)-\triangle_{s,h}a_0(Dv)\big)\big(\mu^2+|Dv|^2\big)^{\frac{2-p}{4}}\,\eta\,,\\g&:=\;\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}\big(DD_sv\,\eta+2\,D_sv\otimes D\eta\big)\,,\end{aligned}$$

and in what follows, we will show that it vanishes as h tends to zero using a weak convergence argument. Taking into account

$$\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}|D_sDv|\;\le\;2\big|D_s(V_\mu(Dv))\big|\;\le\;4\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}|D_sDv|\,,\tag{3.14}$$

we infer $g\in L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$ from the first step. Furthermore, the sequence $\{f_h\}$ is uniformly bounded in $L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$: to this aim we first employ the identity (3.6), use condition $(1.2)_1$, the technical Lemma 2.1 and a reasoning similar to the justification for (3.6), and we deduce

$$\big|\triangle_{s,h}a_0(Dv(x))\big|\;\le\;L\,c(p)\,\big(\mu^2+|Dv(x)|^2+|Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\,|\triangle_{s,h}Dv(x)|$$

(for $\mu=0$ this inequality is trivially satisfied if $Dv(x)=\triangle_{s,h}Dv(x)=0$). From $(1.2)_1$ we further infer $|D_sa_0(Dv(x))|\le L(\mu^2+|Dv(x)|^2)^{(p-2)/2}|DD_sv(x)|$ for all $x\in B^+_{3r/4}(y)$ (note that if $Dv(x)=0$ then $DD_sv(x)=0$ and hence, this inequality also holds true). Hence, we end up with

$$\begin{aligned}\int_{B^+_{3r/4}(y)}|f_h|^2\,dx&\le2\int_{B^+_{3r/4}(y)}\Big(\big|D_sa_0(Dv(x))\big|^2+\big|\triangle_{s,h}a_0(Dv(x))\big|^2\Big)\big(\mu^2+|Dv(x)|^2\big)^{\frac{2-p}{2}}\,dx\\&\le\;L\,c(p)\int_{B^+_{3r/4}(y)}\Big(\big(\mu^2+|Dv(x)|^2\big)^{\frac{p-2}{2}}|DD_sv(x)|^2\\&\qquad+\big(\mu^2+|Dv(x)|^2+|Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}|\triangle_{s,h}Dv(x)|^2\Big)\,dx\\&\le\;L\,c\big(p,\tfrac L\nu\big)\,r^{-2}\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac p2}\,dx\,,\end{aligned}$$

where we have applied the estimates (3.12), (3.4) with $\rho = \frac{3}{4}r$ and (3.14) in the last line. Thus, we find $f \in L^2(B^+_{3r/4}(y), \mathbb{R}^{nN})$ such that a subsequence of $\{f_h\}$ converges weakly in $L^2(B^+_{3r/4}(y), \mathbb{R}^{nN})$ to f. Furthermore, we estimate via Hölder's inequality for every $\phi \in L^{p/(p-1)}(B^+_{3r/4}(y), \mathbb{R}^{nN})$:

$$\int_{B^+_{3r/4}(y)} |f_h \cdot \phi|\,dx \le \Big(\int_{B^+_{3r/4}(y)} \big|D_s a_0(Dv) - \triangle_{s,h} a_0(Dv)\big|^2\,dx \Big)^{\frac{1}{2}} \cdot \Big(\int_{B^+_{3r/4}(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}}\,dx \Big)^{\frac{2-p}{2p}} \Big(\int_{B^+_{3r/4}(y)} |\phi|^{\frac{p}{p-1}} \Big)^{\frac{p-1}{p}}.$$

Keeping in mind $D_s a_0(Dv) \in L^2(B^+_{R'}(x_0), \mathbb{R}^{nN})$, $s \in \{1, \ldots, n-1\}$, for all $R' < R$ due to Step 1, there holds $\triangle_{s,h} a_0(Dv) \to D_s a_0(Dv)$ strongly in $L^2(B^+_{3r/4}(y), \mathbb{R}^{nN})$ as $h \to 0$, i. e., we have $\{f_h\}_h \rightharpoonup 0$ weakly in $L^p(B^+_{3r/4}(y), \mathbb{R}^{nN})$. Since weak limits are unique, we conclude $f_h \rightharpoonup f \equiv 0$ in $L^2(B^+_{3r/4}(y), \mathbb{R}^{nN})$. Hence, in view of $g \in L^2(B^+_{3r/4}(y), \mathbb{R}^{nN})$ we finally arrive at $\int_{B^+_r(y)} f_h \cdot g\,dx \to 0$ as $h \to 0$, and taking this limit in (3.13), we obtain

$$\begin{aligned}\nu \int_{B^+_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2 \eta^2\,dx &\le -2 \int_{B^+_r(y)} D_s a_0(Dv) \cdot (D_s v \otimes D\eta)\,\eta\,dx \\ &= -2 \int_{B^+_r(y)} D_z a_0(Dv)\, DD_s v \cdot (D_s v \otimes D\eta)\,\eta\,dx\,.\end{aligned}$$

Evaluating the integral on the right-hand side in a standard manner and keeping in mind (3.14) reveals the stronger tangential estimate

$$\int_{B^+_{r/2}(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\,dx \;\le\; c\big(\tfrac{L}{\nu}\big)\, r^{-2} \int_{B^+_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_s v|^2\,dx\,. \tag{3.15}$$

In contrast to inequality (3.4) only the tangential part of $V_\mu(Dv)$ appears on the right-hand side; this will be a crucial point for later applications. In the interior of $B^+_R(x_0)$ we proceed similarly, but we need a modification of the arguments to obtain the mean value version: Step 1 applied in the interior shows that partial derivatives of Dv exist in L^p for every direction. We thus may choose $\triangle_{s,-h}(\eta^2(D_s v - \xi_s))$ as a test function, $s = 1, \ldots, n$. Here, $\eta \in C^\infty_0(B_{5r/8}(y), [0,1])$ is once again an appropriate cut-off function, and $\xi \in \mathbb{R}^{nN}$ is determined via $V_\mu(\xi) = (V_\mu(Dv))_{B_{3r/4}(y)}$ (note that V_μ is surjective). Calculations similar to the boundary situation then yield the Caccioppoli-type inequality (3.3).

Step 3: The normal direction for the boundary estimate. At the boundary it still remains to find an estimate for the normal derivative. To this end we make use of the differentiated system (3.1), see e. g. [10, Section 5]. We first recall that $a_0 \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ consists of N-dimensional vectors $(a_0)_i$, $i = 1, \ldots, n$. In components $\operatorname{div} a_0(Dv) = \sum_{i=1}^n D_i\big((a_0)_i(Dv)\big) = 0$ can be rewritten as

$$\sum_{\beta=1}^{N} \frac{\partial (a_0)^\alpha_n}{\partial z^\beta_n}(Dv)\, D_{nn} v^\beta \;=\; -\sum_{\beta=1}^{N} \sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^{n} \frac{\partial (a_0)^\alpha_i}{\partial z^\beta_j}(Dv)\, D_{ij} v^\beta$$

for $\alpha = 1, \ldots, N$ almost everywhere in $B^+_{r/2}(y) \cap \{x_n > \varepsilon\}$ for every $\varepsilon > 0$. An estimate for $D_{nn} v$ is then derived as follows: since all second derivatives exist in the interior, we may multiply the previous relation by $D_{nn} v^\alpha$ and sum up upon α; using the growth and ellipticity conditions $(1.2)_1$, $(1.2)_2$, we get

$$\begin{aligned}\nu \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_{nn} v|^2 &\le \sum_{\alpha,\beta=1}^{N} \frac{\partial (a_0)^\alpha_n}{\partial z^\beta_n}(Dv)\, D_{nn} v^\beta\, D_{nn} v^\alpha \\ &= -\sum_{\alpha,\beta=1}^{N} \sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^{n} \frac{\partial (a_0)^\alpha_i}{\partial z^\beta_j}(Dv)\, D_{ij} v^\beta\, D_{nn} v^\alpha \\ &\le c(n,N)\, L\, \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD'v|\, |D_{nn} v|\end{aligned}$$

almost everywhere in $B^+_{r/2}(y) \cap \{x_n > \varepsilon\}$ (in order to apply $(1.2)_1$ and $(1.2)_2$ also for degenerate systems, we recall that all integrands above vanish if $Dv(x) = 0$). Then Young's inequality and absorbing the term involving $|D_{nn}v|$ implies

$$\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_{nn}v|^2 \,\leq\, c\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|DD'v|^2$$

for a constant c depending only on n, N and $\frac{L}{\nu}$. Since the right-hand side of the last inequality exists and belongs to $L^1(B^+_{r/2}(y))$, we hence integrate the previous inequality on $B^+_{r/2}(y)\cap\{x_n>\varepsilon\}$. Letting $\varepsilon\to 0$ and employing the tangential estimate (3.15), we gain

$$\int_{B^+_{r/2}(y)}\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_{nn}v|^2\,dx \,\leq\, c\left(\tfrac{L}{\nu}\right)r^{-2}\int_{B^+_r(y)}\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_s v|^2\,dx\,.$$

Combined with (3.14) and (3.15), this is the desired Caccioppoli-type inequality at the boundary. Finally, we note that the decomposition $|D^2v|^p \,\leq\, (\mu^2+|Dv|^2)^{p/2}+(\mu^2+|Dv|^2)^{(p-2)/2}\,|D^2v|^2$, cf. (3.9), gives $v\in W^{2,p}(B^+_{R'}(x_0),\mathbb{R}^N)$ for all $R'<R$. Thus, the proof of the theorem is complete. □

Starting from the Caccioppoli inequalities close to the boundary and in the interior in Theorem 3.1, we next apply the Sobolev-Poincaré inequality, see e. g. [23, Chapter 3.6], to the right-hand side of (3.2) and (3.3), respectively, and obtain a reverse Hölder inequality of the form

$$\fint_{B^+_{r/2}(y)}\left|D(V_\mu(Dv))\right|^2dx \,\leq\, c(n,N,p,\tfrac{L}{\nu})\left(\fint_{B^+_r(y)}|D(V_\mu(Dv))|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}$$

for all points $y\in B^+_R(x_0)\cup\Gamma_R(x_0)$ and radii $0<r<R-|y-x_0|$. If we fix a ball $B_\rho(z)$ with centre $z\in B^+_R(x_0)\cup\Gamma_R(x_0)$ and radius $0<\rho<R-|x_0-z|$, we thus have verified assumption (9) of the up-to-the-boundary version of Gehring's lemma [15, Theorem 2.4] for every ball $B_r(y)\cap\partial B_\rho(z)\cap B^+_R(x_0)=\emptyset$. Applying the latter theorem with

$$g\,=\,\left|DV_\mu(Dv)\right|^{\frac{2n}{n+2}}\,,\quad p\,=\,\tfrac{n+2}{n}\,,\quad \Omega=B_\rho(z)\cap B^+_R(x_0)\quad\text{and}\quad A=\partial B_\rho(z)\cap B^+_R(x_0)\,,$$

we then deduce an appropriate higher integrability result, namely that there exists a number $t_0=t_0(n,N,p,\frac{L}{\nu})>1$ such that for all $z\in B^+_R(x_0)\cup\Gamma_R(x_0)$ and $0<\rho<R-|x_0-z|$ there holds $|D(V_\mu(Dv))|\in L^{2t_0}(B^+_{\rho/2}(z))$ with

$$\left(\fint_{B^+_{\rho/2}(z)}|D(V_\mu(Dv))|^{2t_0}\,dx\right)^{\frac{1}{t_0}} \,\leq\, c\big(n,N,p,\tfrac{L}{\nu}\big)\fint_{B^+_\rho(z)}\left|D(V_\mu(Dv))\right|^2dx\,. \tag{3.16}$$

The previous higher integrability result enables us to bound the L^2-norm of $D(V_\mu(Dv))$ on half-balls of different radii. To this end we argue as follows: for fixed $\tau\in(0,\frac12)$ we estimate via Jensen's inequality and the higher integrability estimate (3.16) for $D(V_\mu(Dv))$:

$$\begin{aligned}\int_{B^+_{\tau\rho}(z)}\left|D(V_\mu(Dv))\right|^2dx &\leq\, c(n)\,(\tau\rho)^n\,(2\tau)^{-\frac{n}{t_0}}\left(\fint_{B^+_{\rho/2}(z)}\left|D(V_\mu(Dv))\right|^{2t_0}dx\right)^{\frac{1}{t_0}}\\ &\leq\, c\big(n,N,p,\tfrac{L}{\nu}\big)\,\tau^{\varepsilon}\int_{B^+_\rho(z)}\left|D(V_\mu(Dv))\right|^2dx\,,\end{aligned} \tag{3.17}$$

where we have defined $\varepsilon:=n(1-1/t_0)>0$ in the last line. We note that inequality (3.17) trivially holds true for $c=\tau^{-\varepsilon}\leq 2^{\varepsilon}\leq 2^n$ if $\tau\in[\frac12,1)$. This result for $D(V_\mu(Dv))$ is now carried over to an estimate for $V_\mu(Dv)$: With some minor modifications to adapt it for the boundary situation, the next estimate is achieved following the line of arguments in the proof of [10, Theorem 3.I], where the corresponding estimate is shown for the interior situation in the superquadratic case (note that our function V is called W in Campanato's paper).

Lemma 3.3: *Let $v\in W^{1,p}_\Gamma(B^+_R(x_0),\mathbb{R}^N)$ be a weak solution of the system (3.1) under the assumptions $(1.2)_1$ and $(1.2)_2$ with $\mu\in[0,1]$. Then for every $B^+_\rho(y)\subset B^+_R(x_0)$ with $y\in B^+_R(x_0)\cup\Gamma_R(x_0)$, $0<\rho<R-|x_0-y|$ and for all $\tau\in(0,1)$ we have*

$$\int_{B^+_{\tau\rho}(y)}\left|V_\mu(Dv)\right|^2dx \,\leq\, c\,\tau^{\gamma_0}\int_{B^+_\rho(y)}\left|V_\mu(Dv)\right|^2dx \tag{3.18}$$

with $\gamma_0=\min\{2+\varepsilon,n\}$ (where $\varepsilon:=n(1-\frac{1}{t_0})>0$ is given above), and the constant c depends only on n,N,p and $\frac{L}{\nu}$.

We close this section by stating two relevant consequences of Lemma 3.3: we obtain a Morrey type decay-estimate for Dv and we further find a fundamental estimate for v which is analogous to [10, Theorem 1.II] for the superquadratic setting:

Corollary 3.4: *Let the assumptions of Lemma 3.3 be satisfied. Then there exists a constant $c = c\big(n, N, p, \frac{L}{\nu}\big)$ independent of v such that for every $B^+_\rho(y) \subset B^+_R(x_0)$ with centre $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and radius $0 < \rho < R - |x_0 - y|$ there holds*

$$\int_{B^+_{\tau\rho}(y)} \big(\mu^p + |Dv|^p\big)\, dx \;\le\; c\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx \qquad \forall\, \tau \in (0,1]\,. \tag{3.19}$$

Furthermore, if $n \in [2, p+\gamma_0)$ is satisfied, we have

$$\int_{B^+_{\tau\rho}(y)} |v|^p\, dx \;\le\; c\,\tau^n \Big[\int_{B^+_\rho(y)} |v|^p\, dx + \rho^p \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx\Big] \qquad \forall\, \tau \in (0,1]\,. \tag{3.20}$$

Proof: Using (3.18), the definition of $V_\mu(Dv)$ and keeping in mind $\gamma_0 \le n$, we infer the decay estimate (3.19) for Dv as follows:

$$\begin{aligned}\int_{B^+_{\tau\rho}(y)} \big(\mu^p + |Dv|^p\big)\, dx &\le 4 \int_{B^+_\rho(y)} \Big[\tau^n \mu^p + c\,\tau^{\gamma_0} \big|V_\mu(Dv)\big|^2\Big]\, dx \\ &\le c\big(n, N, p, \tfrac{L}{\nu}\big)\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx\,.\end{aligned}$$

The decay estimate (3.20) is a straightforward adaptation of the arguments in [10, Chapter 4] where the interior analogue is achieved in the superquadratic case. We mention that the assumption $n \in [2, p+\gamma_0)$ is needed to be in a position to employ the isomorphy between the Campanato $\mathcal{L}^{p,p+\gamma_0}$ and the Hölder space $C^{0,1-(n-\gamma_0)/p}$. □

Remark: For an appropriate reference estimate in the interior we consider a weak solution in $v \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$, $R < 1$ and $p \in (1,2)$, to the homogeneous system $\operatorname{div} a_1(Dv) = 0$ in $B_R(x_0)$. It is easy to see that all estimates achieved above remain true in the interior of $B_R(x_0)$. In particular, the higher integrability estimate (3.16) and the interior estimates analogous to the statements in Lemma 3.3 and Corollary 3.4 still hold if $B^+_R(x_0)$ is replaced by the full ball $B_R(x_0)$.

4 Decay estimate for the solution

We now turn our attention to the model situation of an upper half-ball, i. e., we consider weak solutions $u \in W^{1,p}(B^+, \mathbb{R}^N)$ or $u \in W^{1,p}(B^+, \mathbb{R}^N) \cap L^\infty(B^+, \mathbb{R}^N)$ of the system

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) \;=\; b(\,\cdot\,, u, Du) & \text{in } B^+\,, \\ \qquad\qquad\;\; u \;=\; g & \text{on } \Gamma\,. \end{cases} \tag{4.1}$$

We first state a higher integrability result up to the boundary for Du which is valid in all dimensions:

Lemma 4.1 (Higher integrability): *Let $u \in g + W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$, $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$, be a weak solution of (4.1), where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the growth and ellipticity conditions $(1.2)_1$ and $(1.2)_2$ with $\mu \in [0,1]$. If one of the following assumptions is fulfilled:*

1. *the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition (B1),*
2. *the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2); additionally, there hold $u \in L^\infty(B^+, \mathbb{R}^N)$ with $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$ and $2L_2M < \nu$,*

then there exists an exponent $s > p$ depending only on $n, N, p, \frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, and in case 2 additionally on $\frac{L_2}{\nu}$ and M such that $u \in W^{1,s}(B^+_\rho, \mathbb{R}^N)$ for all $\rho < 1$. Furthermore, for every $y \in B^+ \cup \Gamma$ and all $\rho \in (0, 1-|y|)$ there holds:

$$\Big(\fint_{B^+_{\rho/2}(y)} \big(1 + |Du|\big)^s\, dx\Big)^{\frac{p}{s}} \;\le\; c^{(i)} \fint_{B^+_\rho(y)} \big(1 + |Du|^p\big)\, dx$$

(for $i = 1,2$) with constants $c^{(1)} = c^{(1)}(n, N, p, \frac{L}{\nu}, \|Dg\|_{L^\infty})$ and $c^{(2)} = c^{(2)}(n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M)$.

Proof: The proof is standard; therefore, we only sketch the proof and refer to [4, Chapter 6.2] for detailed calculation. Testing the system (4.1) with $\varphi = (u-g)\eta^2$ for an estimate close to the boundary part Γ, we first deduce a weak version of a Caccioppoli-type inequality. We note that the arguments in the proof of [14, Lemma 4.1] or [25, Lemma 4.3] may be adapted for the treatment of inhomogeneities under a natural growth condition. We thus find

$$-\!\!\!\!\!\!\int_{B^+_{r/2}(z)} \big(1+|Du|^p\big)\,dx \;\le\; c_{cacc} -\!\!\!\!\!\!\int_{B^+_{r}(z)} \Big(1+\Big|\frac{u-g}{r}\Big|^p\Big)\,dx\,, \tag{4.2}$$

and the constant c_{cacc} depends only on $p, \frac{L}{\nu}, \|Dg\|_{L^\infty}$ when considering (B1), and on $n, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$, M when considering (B2), respectively. In the interior the corresponding estimate follows if g is replaced by the mean-value of u in the definition of φ (and hence also on the right-hand side of (4.2)). Via Poincaré's inequality a reverse Hölder inequality follows which in turn allows to apply Gehring's lemma in an up-to-the-boundary version, see [15, Theorem 2.4]. Hence, we finally deduce the higher integrability of Du with the dependencies stated above. □

Keeping in mind $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$, the previous estimate (4.2) immediately allows us to state the following Morrey-type estimate for bounded weak solutions of systems with inhomogeneities under a natural growth condition (cf. [3, Lemma 2] in the superquadratic case):

Corollary 4.2: *Assume $u \in g + W^{1,p}_\Gamma(B^+,\mathbb{R}^N) \cap L^\infty(B^+,\mathbb{R}^N)$ to be a weak solution to (4.1) with $g \in C^1(B^+\cup\Gamma,\mathbb{R}^N)$, $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$, where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the conditions $(1.2)_1$ and $(1.2)_2$ with $\mu\in[0,1]$ and $2L_2M < \nu$, and where the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2). Then for fixed $\sigma \in (0,1)$ we have $Du \in L^{p,n-p}(B^+_{1-\sigma},\mathbb{R}^N)$ with*

$$\|Du\|^p_{L^{p,n-p}(B^+_{1-\sigma},\mathbb{R}^N)} \;\le\; c_\sigma$$

and c_σ depends on σ and the same parameters as the constant $c^{(2)}$ in the previous Lemma 4.1.

In the next step we deduce an appropriate decay estimate for the solution u of the original system (4.1) by comparing u with the solution $v \in W^{1,p}(B^+_R(x_0),\mathbb{R}^N)$ of the frozen system

$$\begin{cases} \operatorname{div} a_0(Dv) = 0 & \text{in } B^+_R(x_0)\,, \\ \qquad v = u-g & \text{on } \partial B^+_R(x_0)\,, \end{cases} \tag{4.3}$$

where $a_0(z) := a(x_0,(u)_{B^+_R(x_0)},z)$, $x_0\in\Gamma$, and $2R < 1-|x_0|$. Testing the latter system with $u-g-v$, which is admissible, since the functions $u-g$ and v have the same boundary values, we obtain

$$0 \;=\; \int_{B^+_R(x_0)}\int_0^1 D_z a_0(tDv)\,Dv\cdot(Du-Dg-Dv)\,dt\,dx\,.$$

Conditions $(1.2)_1$ and $(1.2)_2$ (applied on the set $\{x\in B^+_R(x_0) : Dv(x)\ne 0\}$), Young's inequality, the technical Lemmas 2.1 and 2.2 (iii) now yield

$$\begin{aligned} \nu\int_{B^+_R(x_0)} \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv|^2\,dx \;&\le\; \int_{B^+_R(x_0)}\int_0^1 D_z a_0(tDv)\,Dv\cdot Dv\,dt\,dx \\ &= \int_{B^+_R(x_0)}\int_0^1 D_z a_0(tDv)\,Dv\cdot(Du-Dg)\,dt\,dx \\ &\le \varepsilon\int_{B^+_R(x_0)} \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv|^2\,dx + c(p)\,\varepsilon^{1-p}\,L^p\int_{B^+_R(x_0)}\big(\mu^p+|Du-Dg|^p\big)\,dx\,. \end{aligned}$$

Choosing $\varepsilon = \frac{\nu}{2}$, absorbing the first integral on the right-hand side and keeping in mind the inequality $\mu^p + |Du|^p \le 2\,(\mu^p+|V_\mu(Du)|^2)$, we end up with an estimate for the p-Dirichlet functional of Dv:

$$\int_{B^+_R(x_0)}|Dv|^p\,dx \;\le\; c\int_{B^+_R(x_0)}\big(\mu^p+|Du-Dg|^p\big)\,dx \;\le\; c\int_{B^+_R(x_0)}\big(1+|Du|^p\big)\,dx \tag{4.4}$$

with $c = c(p, \frac{L}{\nu}, \|Dg\|_{L^\infty})$. Since $\operatorname{div}(-a_0(Dv) + a(\,\cdot\,, u, Du)) + b(\,\cdot\,, u, Du) = 0$ holds in the weak sense in $B_R^+(x_0)$, we also have

$$\begin{aligned} &\operatorname{div}\big(a_0(Dv+Dg) - a_0(Du)\big) \\ &\qquad = \operatorname{div}\big(a_0(Dv+Dg) - a_0(Dv)\big) + \operatorname{div}\big(a(\,\cdot\,,u,Du) - a_0(Du)\big) + b(\,\cdot\,,u,Du) \end{aligned} \tag{4.5}$$

in $B_R^+(x_0)$ in the weak sense. To go on we distinguish the different growth conditions concerning the inhomogeneity.

4.1 Controllable growth of $\mathbf{b}(\cdot,\cdot,\cdot)$

The procedure is quite similar to the one established in [7, Section 4], where (partial) Hölder continuity of the solution in the interior is discussed in low dimensions under similar assumptions concerning the coefficients. By Young's inequality combined with the ellipticity condition $(1.2)_2$ (applied on the set where $Dv + Dg - Du \neq 0$, otherwise all the relevant integrals vanish) we first infer

$$\begin{aligned} 2^{\frac{p-2}{2}}\nu \int_{B_R^+(x_0)} &\big(\mu^2 + |Du|^2 + |Dv+Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\,dx \\ \le& \int_{B_R^+(x_0)} \big(a_0(Dv+Dg) - a_0(Du)\big)\cdot\big(Dv+Dg-Du\big)\,dx\,. \\ =& \int_{B_R^+(x_0)} \big(a_0(Dv+Dg) - a_0(Dv)\big)\cdot\big(Dv+Dg-Du\big)\,dx \\ &+ \int_{B_R^+(x_0)} \big(a(\,\cdot\,,u,Du) - a_0(Du)\big)\cdot\big(Dv+Dg-Du\big)\,dx \\ &- \int_{B_R^+(x_0)} b(\,\cdot\,,u,Du)\cdot(v+g-u)\,dx \;=:\; I + II + III\,. \end{aligned} \tag{4.6}$$

where in the last inequality we have used $u - g - v \in W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N)$ as a test function in relation (4.5). The terms on the right-hand side are bounded from above separately: via the growth condition $(1.2)_1$ on the set $\{x \in B_R^+(x_0) : Dg(x) \neq 0\}$, Lemma 2.1, Young's inequality and the energy estimate (4.4), we estimate term I and, in view of $p < 2$, we obtain

$$I \;\le\; c(p, \tfrac{L}{\nu}, \|Dg\|_{L^\infty})\, L\Big(\delta \int_{B_R^+(x_0)} \big(1 + |Du|^p\big)\,dx + R^n\,\delta^{1-p}\Big) \tag{4.7}$$

for every $\delta \in (0,1)$. For the second term we use assumption $(1.2)_3$ (recalling the definition $a_0(\cdot)$ of the frozen coefficients) and Hölder's inequality (note $\omega(\cdot) \le 1$) with $\frac{p-1}{p}\,\frac{s-p}{s}$, $\frac{p-1}{p}\,\frac{p}{s}$ and $\frac{1}{p}$ where $s > p$ denotes the (up-to-the-boundary) higher integrability exponent of the gradient Du from Lemma 4.1 depending only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. In view of Young's inequality we then obtain

$$\begin{aligned} II \le\; & L \int_{B_R^+(x_0)} \omega\big(|x-x_0| + |u - (u)_{B_R^+(x_0)}|\big)\big(\mu^2 + |Du|^2\big)^{\frac{p-1}{2}} |Du - Dg - Dv|\,dx \\ \le\; & \big|B_R^+(x_0)\big|\, L\,\Big(\fint_{B_R^+(x_0)} \omega\big(R + |u - (u)_{B_R^+(x_0)}|\big)\,dx\Big)^{\frac{p-1}{p}\,\frac{s-p}{s}} \\ &\times \Big(\fint_{B_R^+(x_0)} \big(\mu^p + |Du|^p\big)^{\frac{s}{p}}\,dx\Big)^{\frac{p-1}{p}\,\frac{p}{s}} \Big(3^{p-1} \fint_{B_R^+(x_0)} \big(|Du|^p + \|Dg\|_{L^\infty}^p + |Dv|^p\big)\,dx\Big)^{\frac{1}{p}}. \end{aligned}$$

To continue estimating term II we define

$$\beta := \frac{p-1}{p}\,\frac{s-p}{s}\,, \tag{4.8}$$

and recall that $\omega(\cdot)$ is concave and monotone non-decreasing. Making use of the higher integrability estimate for $1 + |Du|^p$ from Lemma 4.1, the energy estimate (4.4), Jensen's inequality and Poincaré's inequality we then find

$$II \le L\, c\,\omega^\beta\Big(\Big(\fint_{B_R^+(x_0)} \big(R^p + |u-(u)_{B_R^+(x_0)}|^p\big)\,dx\Big)^{\frac1p}\Big)\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx$$

$$\le L\, c\big(n,N,p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,\omega^\beta\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac1p}\Big)\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx\,. \quad (4.9)$$

Finally, we estimate the remaining term III appearing on the right-hand side in inequality (4.6): we first note that, since the functions $u-g$ and v have the same values on the boundary $\partial B_R^+(x_0)$, we obtain via the Poincaré inequality and then (4.4):

$$\int_{B_R^+(x_0)} |v+g-u|^p\,dx \le c\big(n,N,p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,R^p\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\,.$$

Therefore, due to the growth condition imposed on $b(x,u,Du)$ in (B1) and Hölder's inequality, we conclude

$$III \le L\Big(\int_{B_R^+(x_0)}\big(\mu^p+|Du|^p\big)\,dx\Big)^{\frac{p-1}{p}}\Big(\int_{B_R^+(x_0)}|v+g-u|^p\,dx\Big)^{\frac1p}$$

$$\le L\, c\big(n,N,p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,R\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\,. \quad (4.10)$$

Merging the estimates for I, II and III, i. e., (4.7), (4.9) and (4.10), with (4.6), we find the comparison estimate

$$\int_{B_R^+(x_0)}\big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dx$$

$$\le c\Big[\omega^\beta\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac1p}\Big)+R+\delta\Big]\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx + c\,R^n\,\delta^{1-p} \quad (4.11)$$

for every $\delta\in(0,1)$, and the constant c depends only on $n,N,p,\frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. We next transfer the decay properties of v to the weak solution u of the original Dirichlet problem (4.1) in a standard way. We recall the exponent γ_0 defined by

$$\gamma_0 = \min\{2+\varepsilon, n\} \quad (4.12)$$

for some $\varepsilon>0$ depending only on n,N,p and $\frac{L}{\nu}$ (for the precise derivation of γ_0 we refer to Lemma 3.3). Corollary 3.4 then provides the decay estimate

$$\int_{B_\rho^+(x_0)}|Dv|^p\,dx \le c\big(n,N,p,\tfrac{L}{\nu}\big)\Big(\frac{\rho}{R}\Big)^{\gamma_0}\int_{B_R^+(x_0)}\big(1+|Dv|^p\big)\,dx$$

for all radii $\rho\in(0,R]$ where v is the solution of the comparison problem (4.3) with constant coefficients (keep in mind $v=0$ on $\Gamma_\rho(x_0)$ by definition). In view of $\gamma_0\le n$ we further note that

$$\int_{B_\rho^+(x_0)}\big(1+|Dg|^p\big)\,dx \le c\big(\|Dg\|_{L^\infty}\big)\Big(\frac{\rho}{R}\Big)^{\gamma_0}\int_{B_R^+(x_0)}1\,dx$$

for all $\rho\in(0,R]$. We now observe from Lemma 2.2 (ii) that the inequality

$$1+|Du|^p \le c(n,N,p)\Big[\big(1+|Dv+Dg|^p\big)+\big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\Big]$$

holds true. Thus, combining the last three inequalities and taking advantage of the energy inequality (4.4) and the comparison estimate (4.11), we finally arrive at a decay estimate for the gradient Du:

$$\int_{B_\rho^+(x_0)}\big(1+|Du|^p\big)\,dx \le c\Big(\frac{\rho}{R}\Big)^{\gamma_0}\int_{B_R^+(x_0)}\big(1+|Dv|^p\big)\,dx$$

$$+c\int_{B_R^+(x_0)}\big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dx$$

$$\le c\Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0}+\omega^\beta\Big(\Big((2R)^{p-n}\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac1p}\Big)+R+\delta\Big]$$

$$\times\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx + c\,R^n\,\delta^{1-p}$$

for all $x_0 \in \Gamma$, $2R < 1 - |x_0|$ and every $\rho \in (0, R]$. The constant c depends only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$, and the same inequality trivially holds if $\rho \in (R, 2R]$. We mention that this estimate is similar to inequality (4.23) in [7], where regularity up to the boundary of weak solutions is considered in the low-dimensional (non-degenerate) case with $p > 2$. We emphasize that the latter estimate also follows in the interior, i. e., for balls $B_R(x_0)$ contained in B^+ (or for general problems in Ω). In this case we do not need to take into account the function g which specifies the boundary values of u on Γ, and hence term I does not appear in the calculations corresponding to (4.6). All other estimates above as well as the conclusion of (4.13) below remain valid. Replacing $2R$ by R and introducing the *excess functional*

$$\Phi(x_0, r) := \int_{B_r^+(x_0)\cap B^+} \big(1 + |Du|^p\big)\, dx\,,$$

(for $x_0 \in B^+ \cup \Gamma$) we thus conclude altogether

Lemma 4.3: *Let β, γ_0 be chosen as above in (4.8), (4.12), and let $\delta \in (0,1)$. Furthermore, let $u \in g + W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$, $1 < p < 2$, be a weak solution of the system (4.1) under the assumptions (1.2) with $\mu \in [0,1]$, (B1), and $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$. Then, if $x_0 \in \Gamma$, $R < 1 - |x_0|$ or if $x_0 \in B^+$, $R < \min\{1 - |x_0|, (x_0)_n\}$, there holds*

$$\Phi(x_0, \rho) \le c_{ex}^{(1)} \Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\big(R^{p-n}\Phi(x_0,R)\big)^{\frac{1}{p}}\Big) + R + \delta\Big]\, \Phi(x_0, R) + c_{ex}^{(1)}\, R^n\, \delta^{1-p} \tag{4.13}$$

for every $\rho \in (0, R]$, and the constant $c_{ex}^{(1)}$ depends only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$.

4.2 Natural growth of $\mathbf{b}(\cdot,\cdot,\cdot)$

In what follows, we proceed analogously to the situation of the controllable growth condition (B1). For the modifications necessary for natural growth we adapt the techniques used in [3, Proof of Theorem 1]. For fixed $\sigma \in (0,1)$ we consider the unique solution $v \in W^{1,p}(B_R^+(x_0), \mathbb{R}^N)$, $x_0 \in \Gamma_{1-\sigma}$, $2R < 1 - \sigma - |x_0|$, to the Dirichlet problem (4.3), and we again aim for a comparison of the functions u and v. Furthermore, let $n < p + \gamma_0$. System (4.5) still holds in $B_R^+(x_0)$ in the weak sense, but we may now test only with *bounded* functions in $W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N) \cap L^\infty(B_R^+(x_0), \mathbb{R}^N)$ according to the growth condition (B2). Hence, in order to be allowed to test with the function $u - v - g$ as above, we start by proving an L^∞-estimate for v on $B_{R/2}^+(x_0)$: Consider a ball $B_\rho(y)$ with centre $y \in \overline{B_{R/2}^+}(x_0)$ and radius $\rho < \frac{R}{2}$. According to Corollary 3.4 we have

$$\fint_{B_\rho^+(y)} |v|^p\, dx \le c\big(n, N, p, \tfrac{L}{\nu}\big) \Big[R^{-n} \int_{B_{R/2}^+(y)} |v|^p\, dx + R^{p-n} \int_{B_{R/2}^+(y)} \big(\mu^p + |Dv|^p\big)\, dx\Big]$$

(it is obvious that we may allow $|y - x_0| = R/2$). Thus, taking advantage of $B_{R/2}^+(y) \subset B_R^+(x_0)$, the Poincaré inequality (keeping in mind $v = 0$ on $\Gamma_R(x_0)$ by definition), and the estimate (4.4) for the p-Dirichlet functional of Dv, we estimate the mean values of $|v|^p$ as follows:

$$\begin{aligned}\sup_{\substack{y\in\overline{B_{R/2}^+}(x_0)\\ \rho\in(0,R/2)}} \fint_{B_\rho^+(y)} |v|^p\, dx &\le c\, R^{p-n} \int_{B_R^+(x_0)} \big(1 + |Du|^p\big)\, dx \\ &\le c\big(n, N, p, \tfrac{L}{\nu}, \tfrac{L_2}{\nu}, \|Dg\|_{L^\infty}, M, \sigma\big) =: m_0^p\,,\end{aligned}$$

where we have used Corollary 4.2 in the last line. According to Lebesgue's differentiation theorem this yields $v \in L^\infty(B_{R/2}^+(x_0), \mathbb{R}^N)$, see also [23, Proposition 2.2], with

$$\|v\|_{L^\infty(B_{R/2}^+(x_0), \mathbb{R}^N)} \le m_0^p\,. \tag{4.14}$$

Therefore, taking into account $|g(x_0)| = |u(x_0)| \le M$, we have $u - v - g \in W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N) \cap L^\infty(B_{R/2}^+(x_0), \mathbb{R}^N)$ with

$$\|u - v - g\|_{L^\infty(B_{R/2}^+(x_0), \mathbb{R}^N)} \le 2M + \|Dg\|_{L^\infty} + m_0 =: m > 0\,.$$

To obtain an admissible test-function for the system (4.5), we next modify the function $u - v - g$ on $B_R^+(x_0)$ (for which we cannot expect an L^∞-estimate) as follows: we set

$$h := (v + g - u)\big(T^\delta - (|v + g - u| + m)^\delta\big)_+$$

for some exponent $\delta > 0$ to be specified later and a number $T = T(\delta, m) > 0$ determined by the condition

$$T^\delta - (2m)^\delta = \tfrac{1}{2}T^\delta \quad \Leftrightarrow \quad T = 2^{1+\frac{1}{\delta}}\, m\,.$$

In particular, $\delta \to 0$ implies $T \to \infty$, and via the estimate $|u - v - g| \le m$ on $B_{R/2}^+(x_0)$ found above we have

$$\big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \ge \tfrac{1}{2}T^\delta \qquad \text{on } B_{R/2}^+(x_0)\,.$$

Keeping in mind that h vanishes outside of the set $\theta_+ := \{x \in B_R^+(x_0) : |(v + g - u)(x)| < T - m\}$, we observe that the weak differentiability of $v + g - u$ is transferred to h, and hence, by construction we have $h \in W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N) \cap L^\infty(B_R^+(x_0), \mathbb{R}^N)$. We next proceed similarly to (4.6), but we have to take into account a new term which arises by this modification:

$$\begin{aligned}
2^{\frac{p-4}{2}} T^\delta \nu & \int_{B_{R/2}^+(x_0)} \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\, dx \\
&\le \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot \big(Dv + Dg - Du\big)\big(T^\delta - (|v + g - u| + m)^\delta\big)_+ dx \\
&= \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot Dh\, dx \\
&\quad + \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot (v + g - u) \otimes \frac{(Dv + Dg - Du) \cdot (v + g - u)}{|v + g - u|} \\
&\qquad\qquad \times \delta\,(|v + g - u| + m)^{\delta - 1}\, \mathbb{1}_{\theta_+}\, dx\,.
\end{aligned}$$

Testing system (4.5) given above with h, we further estimate the first integral on the right-hand side of the last inequality. Hence, we find exactly as in the calculations leading to (4.6):

$$\begin{aligned}
2^{\frac{p-4}{2}} T^\delta \nu & \int_{B_{R/2}^+(x_0)} \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\, dx \\
&\le \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Dv)\big) \cdot Dh\, dx \\
&\quad + \int_{B_R^+(x_0)} \big(a(\,\cdot\,, u, Du) - a_0(Du)\big) \cdot Dh\, dx - \int_{B_R^+(x_0)} b(\,\cdot\,, u, Du) \cdot h\, dx \\
&\quad + \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot (v + g - u) \otimes \frac{(Dv + Dg - Du) \cdot (v + g - u)}{|v + g - u|} \\
&\qquad\qquad \times \delta\,(|v + g - u| + m)^{\delta - 1}\, \mathbb{1}_{\theta_+}\, dx \\
&= \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Dv)\big) \cdot \big(Dv + Dg - Du\big)\big(T^\delta - (|v + g - u| + m)^\delta\big)_+ dx \\
&\quad + \int_{B_R^+(x_0)} \big(a(\,\cdot\,, u, Du) - a_0(Du)\big) \cdot \big(Dv + Dg - Du\big)\big(T^\delta - (|v + g - u| + m)^\delta\big)_+ dx \\
&\quad - \int_{B_R^+(x_0)} b(\,\cdot\,, u, Du) \cdot (v + g - u)\big(T^\delta - (|v + g - u| + m)^\delta\big)_+ dx \\
&\quad + \delta \int_{B_R^+(x_0)} \big(a_0(Dv) - a(\,\cdot\,, u, Du)\big) \cdot (v + g - u) \otimes \frac{(Dv + Dg - Du) \cdot (v + g - u)}{|v + g - u|} \\
&\qquad\qquad \times (|v + g - u| + m)^{\delta - 1}\, \mathbb{1}_{\theta_+}\, dx \\
&=: I' + II' + III' + IV'
\end{aligned} \tag{4.15}$$

with the obvious abbreviations. We first note $(T^\delta - (|v + g - u| + m)^\delta)_+ \le T^\delta$. Therefore, terms I' and II' are estimated as term I in (4.7) and term II in (4.9), respectively, in the controllable growth situation, and we get

$$
\begin{aligned}
|I'| &\le T^{\delta}\, c\, L\Big(\delta \int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx + R^n\,\delta^{1-p}\Big)\,,\\
|II'| &\le T^{\delta}\, c\, L\,\omega^{\beta}\Big(\Big(R^{p-n}\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\Big)^{\frac{1}{p}}\Big) \int_{B^+_{2R}(x_0)} \big(1+|Du|^p\big)\,dx\,.
\end{aligned}
$$

where the constants c depend only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. In view of the growth condition (B2), Hölder's inequality, Lemma 4.1 on higher integrability (where s denotes the higher integrability exponent depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M), the basic inequality $|v+g-u|\,\mathbb{1}_{\theta_+} < T-m \le T$ and the Poincaré inequality, term III' is estimated by

$$
\begin{aligned}
|III'| \le{}& \int_{B^+_R(x_0)} \big(L_2|Du|^p + L\big)\,|v+g-u|\,\big(T^{\delta} - (|v+g-u|+m)^{\delta}\big)_+\,dx\\
\le{}& T^{\delta}\,(L_2+L)\,|B^+_R(x_0)|\,\Big(\fint_{B^+_R(x_0)} \big(1+|Du|^p\big)^{\frac{s}{p}}\,dx\Big)^{\frac{p}{s}} \Big(\fint_{B^+_R(x_0)} \big(|v+g-u|\,\mathbb{1}_{\theta_+}\big)^{\frac{s}{s-p}}\,dx\Big)^{\frac{s-p}{s}}\\
\le{}& T^{\delta}\,c^{(2)}\,(L_2+L)\int_{B^+_{2R}(x_0)} \big(1+|Du|^p\big)\,dx\\
&\qquad\times\big(|v+g-u|\,\mathbb{1}_{\theta_+}\big)^{1-\frac{p(s-p)}{s}} \Big(\fint_{B^+_R(x_0)} |v+g-u|^p\,dx\Big)^{\frac{s-p}{s}}\\
\le{}& T^{\delta}\,c\,(L_2+L)\,T^{1-\frac{p(s-p)}{s}}\,\Big(R^{p-n}\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\Big)^{\frac{s-p}{s}} \int_{B^+_{2R}(x_0)} \big(1+|Du|^p\big)\,dx
\end{aligned}
$$

for $c = c(n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, M, \|Dg\|_{L^\infty})$. In the last line we have used once again the energy estimate (4.4). For the last integral IV', we obtain via $(1.2)_1$, Young's inequality and (4.4):

$$
\begin{aligned}
|IV'| \le{}& 2\,\delta\,L\int_{B^+_R(x_0)} \big(\mu^{p-1}+|Du|^{p-1}+|Dv|^{p-1}\big)\,\big(|Du|+|Dv|+\|Dg\|_{L^\infty}\big)\\
&\qquad\times|v+g-u|\,(|v+g-u|+m)^{\delta-1}\,\mathbb{1}_{\theta_+}\,dx\\
\le{}& T^{\delta}\,c\big(p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,\delta\,L\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\,.
\end{aligned}
$$

Hence, combining the estimates for the terms $I' - IV'$ with (4.15) we finally arrive at

$$
\begin{aligned}
&\int_{B^+_{R/2}(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}\,|Du-Dv-Dg|^2\,dx\\
&\le c\,\Big[\omega^{\beta}\Big(\Big(R^{p-n}\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\Big)^{\frac{1}{p}}\Big) + T^{1-\frac{p(s-p)}{s}}\,\Big(R^{p-n}\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\Big)^{\frac{s-p}{s}} + \delta\Big]\\
&\qquad\times\int_{B^+_{2R}(x_0)} \big(1+|Du|^p\big)\,dx + c\,R^n\,\delta^{1-p}
\end{aligned}
$$

for a constant c depending only on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M. This estimate corresponds to (4.11) above for systems with inhomogeneities under a controllable growth assumption. For a similar up-to-the-boundary estimate concerning the superquadratic case we refer to [3, inequality (36)]. Furthermore, we note that the reasoning leading to the latter inequality also applies for balls $B_R(x_0) \subset B^+_{1-\sigma}$, and thus, a corresponding estimate holds in the interior (without the function g). Following the arguments of the comparison principle above and recalling the definition $\Phi(x_0, r)$ of the excess function, we deduce the following decay estimate for the gradient Du:

Lemma 4.4: *Let β, γ_0 be chosen as above in* (4.8), (4.12), *and let $\delta \in (0,1)$, $\sigma \in (0,1)$ and $n < p+\gamma_0$. Furthermore, let $u \in g + W^{1,p}_\Gamma(B^+,\mathbb{R}^N) \cap L^\infty(B^+,\mathbb{R}^N)$ with $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$, $1<p<2$, $g \in C^1(B^+\cup\Gamma,\mathbb{R}^N)$, be a weak solution of the system* (4.1) *under the assumptions* (1.2) *with $\mu \in [0,1]$,* (B2) *and $2L_2M < \nu$. Then, if $x_0 \in \Gamma_{1-\sigma}$, $R < 1-\sigma-|x_0|$ or if $x_0 \in B^+$, $R < \min\{1-\sigma-|x_0|, (x_0)_n\}$,* *there holds*

$$\Phi(x_0,\rho) \le c_{ex}^{(2)} \Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\big(R^{p-n}\Phi(x_0,R)\big)^{\frac{1}{p}}\Big) + T^{1-\frac{p(s-p)}{s}} \big(R^{p-n}\Phi(x_0,R)\big)^{\frac{s-p}{s}} + \delta\Big] \Phi(x_0,R) + c_{ex}^{(2)} R^n \delta^{1-p} \tag{4.16}$$

for every $\rho \in (0,R]$. Here, the constant $c_{ex}^{(2)}$ depends only on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M, s is the higher integrability exponent from Lemma 4.1 admitting the same dependencies, and T is a positive number additionally depending on σ and δ.

5 Proof of Theorem 1.1

We consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, of class C^1. This means that for every point $x_0 \in \partial\Omega$ there exist a radius $r > 0$ and a C^1-function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (up to an isometry) Ω is locally represented by $\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > h(x_1, \dots, x_{n-1})\}$. Thus we can locally straighten the boundary $\partial\Omega$ by a C^1-transformation. Via a covering argument, the proof of Theorem 1.1 is hence reduced in a standard way to the proof of a partial regularity result in the model situation of the unit half-ball B^+. Therefore, it is sufficient to consider a weak solution $u \in W^{1,p}(B^+, \mathbb{R}^N)$ of the partial Dirichlet-problem (4.1) where $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$, where the coefficients $a : B^+ \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfy the assumptions (1.2) and where the inhomogeneity $b : B^+ \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ fulfills one of the following assumptions

1. $b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition (B1),
2. $b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2); additionally, we assume $u \in L^\infty(B^+, \mathbb{R}^N)$ with $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$ and $2L_2M < \nu$.

In order to prove Theorem 1.1 the objective is to find a number $\delta_2 = \delta_2(n, N, p, \frac{L}{\nu}) > 0$ such that if $n \in [2, p+2+\delta_2)$, then there hold

$$\dim_{\mathcal{H}} \big((B^+ \cup \Gamma) \setminus \mathrm{Reg}_u(B^+ \cup \Gamma)\big) < n - p \qquad \text{and} \qquad u \in C^{0,\lambda}_{\mathrm{loc}}\big(\mathrm{Reg}_u(B^+ \cup \Gamma), \mathbb{R}^N\big)$$

for all $\lambda \in (0, \min\{1 - \frac{n-2-\delta_2}{p}, 1\})$. Moreover, we shall prove that the singular set $\mathrm{Sing}_u(B^+ \cup \Gamma)$ of u is contained in

$$\widetilde{\Sigma} := \Big\{x_0 \in B^+ \cup \Gamma : \liminf_{R\searrow 0} R^{p-n} \int_{B_R(x_0)\cap B^+} \big(1 + |Du|^p\big)\, dx > 0\Big\}.$$

In the sequel we will discuss only the case of natural growth. The result for the controllable growth condition follows completely analogously. We first fix ε in dependence of n, N, p and $\frac{L}{\nu}$ to be the positive number stemming from the application of Gehring's lemma (see Lemma 3.3) if $n \ge 3$ and $\varepsilon = 2p(1-\lambda)$, $\lambda \in (0,1)$ arbitrary, if $n = 2$. We set $\gamma_0 = \min\{2+\varepsilon, n\}$ admitting the same dependencies and choose $\kappa_0 < 1$ according to [20, Chapter III, Lemma 2.1] in dependency of the exponents γ_0, $\gamma_0 - \frac{\varepsilon}{2}$ instead of α, β and the constant $c_{ex}^{(2)}$ in (4.13) instead of A. Furthermore, let s be the higher integrability exponent from Lemma 4.1 depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M, and $\beta = \frac{p-1}{p}\frac{s-p}{s}$ as above. Furthermore, we fix $\sigma \in (0,1)$, and set $\delta = \frac{\kappa_0}{4}$, which in turn fixes a number $T > 0$ (according to Lemma 4.4) depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M, \sigma$ and δ. Since $\omega(\cdot)$ is a modulus of continuity, we then find a positive number ς such that

$$\omega^\beta\big(\varsigma^{\frac{1}{p}}\big) < \frac{\kappa_0}{4} \qquad \text{and} \qquad T^{1-\frac{p(s-p)}{s}} \varsigma^{\frac{s-p}{s}} < \frac{\kappa_0}{4}.$$

We now consider a point $x_0 \in B^+_{1-\sigma} \setminus \widetilde{\Sigma}$ where the excess quantity $R^{p-n}\Phi(x_0,R)$ becomes arbitrarily small for $R \searrow 0$. Hence there exists a radius $R_0 > 0$ such that $B_{R_0}(x_0) \Subset B_{1-\sigma}$ and

$$R_0^{p-n} \int_{B^+_{R_0}(x_0)} \big(1 + |Du|^p\big)\, dx = R_0^{p-n}\, \Phi(x_0, R_0) < \varsigma.$$

Since the function $z \mapsto R_0^{p-n}\,\Phi(z,R_0)$ is continuous, there exists a ball $B_r(x_0)$ such that we have $B_{R_0}(z) \Subset B_{1-\sigma}$ for all $z \in B_r(x_0) \cap (B^+ \cup \Gamma)$ and such that the previous inequality is also satisfied for x_0 replaced by z, i. e., there holds

$$R_0^{p-n}\,\Phi(z,R_0) \,<\, \varsigma \qquad \text{for all } z \in B_r(x_0) \cap (B^+ \cup \Gamma)\,.$$

Our next goal is to show that the gradient Du belongs to an appropriate Morrey space on $B_r(x_0) \cap (B^+ \cup \Gamma)$. To this aim we will show Morrey-type estimates of the form

$$\Phi(z,\rho) \,\le\, c\,\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\Big] \tag{5.1}$$

for all balls $B_\rho^+(z)$ with centre $z \in B_r(x_0) \cap (B^+ \cup \Gamma)$, radius $\rho \le R_0$, and a constant c depending only on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, M$ and $\|Dg\|_{L^\infty}$. For this purpose, we combine the estimates at the boundary and in the interior and need to distinguish several cases:

Case 1: $z \in \Gamma$, $0 < \rho \le R_0$: In view of the choices of $\sigma, \delta, \kappa_0, \varsigma$ and R_0 made above, the boundary version of Lemma 4.4 gives

$$\begin{aligned}\Phi(z,\rho) &\le\, c_{ex}^{(2)}\,\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\kappa_0}{4}\Big]\,\Phi(x_0,R_0) + 4^{p-1}\,c_{ex}^{(2)}\,R_0^n\,\kappa_0^{1-p}\\ &\le\, c\,\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\kappa_0}{4}\Big]\,\Phi(x_0,R_0) + c\,R_0^{\gamma_0-\varepsilon/2}\end{aligned}$$

for all $\rho \le R_0$, and the constant c has the dependencies stated above. Thus we are in a position to apply [20, Chapter III, Lemma 2.1], an iteration scheme to be able to neglect κ_0 by choosing the exponent γ_0 slightly smaller, to deduce the claimed inequality (5.1) for every such centre z.

Case 2: $z \in B^+$, $0 < \rho \le R_0 \le z_n$: There holds $B_{R_0}(z) \subset B^+$, hence we apply the interior version of Lemma 4.4 and inequality (5.1) follows identically to Case 1.

Case 3: $z \in B^+$, $0 < z_n < \rho \le R_0$: Without loss of generality we may assume $\rho \le R_0/4$, otherwise (5.1) is trivially satisfied. Then we have the inclusions

$$B_\rho^+(z) \subset B_{2\rho}^+(z'') \subset B_{R_0/2}^+(z'') \subset B_{R_0}^+(z)$$

where z'' denotes the projection of z onto $\mathbb{R}^{n-1} \times \{0\}$, and the boundary estimate in Case 1 yields the desired inequality

$$\begin{aligned}\Phi(z,\rho) \le\, \Phi(z'',2\rho) \,&\le\, c\,\Big[\Big(\frac{4\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',\tfrac{1}{2}R_0) + (2\rho)^{\gamma_0-\varepsilon/2}\Big]\\ &\le\, c\,\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\Big]\end{aligned}$$

where we have used the monotonicity of Φ with respect to the domain of integration.

Case 4: $z \in B^+$, $0 < \rho \le z_n < R_0$: Without loss of generality we may assume $z_n < R_0/4$, otherwise we apply Case 2 for the inner ball $B_{R_0/4}(z) \subset B^+$. We then take advantage of the inclusions

$$B_\rho(z) \subset B_{z_n}(z) \subset B_{2z_n}^+(z'') \subset B_{R_0/2}^+(z'') \subset B_{R_0}^+(z),$$

the interior estimates in Case 2 and the boundary estimates in Case 1, and we find

$$\begin{aligned}\Phi(z,\rho) &\le c\,\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,z_n) + \rho^{\gamma_0-\varepsilon/2}\Big]\\ &\le c\,\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',2z_n) + \rho^{\gamma_0-\varepsilon/2}\Big]\\ &\le c\,\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2} c\,\Big[\Big(\frac{4z_n}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',\tfrac{1}{2}R_0) + (2z_n)^{\gamma_0-\varepsilon/2}\Big] + \rho^{\gamma_0-\varepsilon/2}\Big]\\ &\le c\,\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\Big]\,.\end{aligned}$$

Combining the estimates above we see that we have covered all cases required to prove inequality (5.1). Recalling the definition of the excess function Φ, this yields

$$Du \in L^{p,\gamma_0-\varepsilon/2}\big(B_r(x_0)\cap(B^+\cup\Gamma),\mathbb{R}^{nN}\big)\,.$$

We define $\delta_2 = \frac{\varepsilon}{2}$ (with exactly the dependencies asserted in the statement of the theorem) and observe that the low-dimensional assumption prescribes

$$n \,<\, p+2+\delta_2 \,=\, p+2+\varepsilon/2\,.$$

We recall $\gamma_0 = 2$ if $n = 2$ and $\gamma_0 = 2+\varepsilon$ if $n > 2$. As a consequence (taking ε smaller if required) we have $\gamma_0 - \varepsilon/2 \in (n-p, n]$, and, according to the Campanato-Meyer embedding, see e. g. [27, Theorem 2.2], we arrive at the conclusion that u is Hölder continuous on $B_r(x_0)\cap(B^+\cup\Gamma)$, more precisely, we have

$$u \in C^{0,\lambda}\big(B_r(x_0)\cap(B^+\cup\Gamma),\mathbb{R}^N\big) \qquad \text{with } \lambda = 1 - \frac{n-\gamma_0+\varepsilon/2}{p}\,.$$

Using a covering argument and the fact that $\sigma \in (0,1)$ is chosen arbitrarily, we conclude immediately the desired regularity result. Furthermore, since we have shown higher integrability of Du in Lemma 4.1, we can improve the condition of x_0 being a regular point via

$$R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx \le c\,\Big(R^{s-n}\int_{B_R^+(x_0)}\big(1+|Du|^s\big)\,dx\Big)^{\frac{p}{s}}$$

for R sufficiently small. As a consequence we get

$$B^+\setminus\widetilde{\Sigma} \supseteq \Big\{x_0 \in B^+\cup\Gamma : \liminf_{R\to 0}\, R^{s-n}\int_{B_R(x_0)\cap B^+}\big(1+|Du|^s\big)\,dx \,=\, 0\Big\}$$

which, in view of Giusti's measure density result [23, Proposition 2.7] applied with $\mu(B_R(x_0)) := \int_{B_R(x_0)\cap B^+}(1+|Du|^s)\,dx$, proves the assertion on the upper bound for the Hausdorff dimension of the singular set. □

References

[1] E. Acerbi and N. Fusco, *Regularity for minimizers of non-quadratic functionals: the case* $1<p<2$, J. Math. Anal. Appl. **140** (1989), 115–135.

[2] A. Arkhipova, *On the Neumann problem for nonlinear elliptic systems with quadratic nonlinearity*, St. Petersbg. Math. J. **8** (1997), no. 5, 845–877.

[3] A. Arkhipova, *Partial regularity up to the boundary of weak solutions of elliptic systems with nonlinearity q greater than two*, J. Math. Sci. (N. Y.) **115** (2003), 2735–2746.

[4] L. Beck, *Boundary regularity results for weak solutions of subquadratic elliptic systems*, Ph.D. thesis, Universität Erlangen-Nürnberg, 2008.

[5] V. Bögelein and A. Zatorska-Goldstein, *Higher integrability of very weak solutions of systems of* $p(x)$*-Laplacean type*, J. Math. Anal. Appl. **336** (2007), no. 1, 480–497.

[6] S. Campanato, *Differentiability of the solutions of nonlinear elliptic systems with natural growth*, Ann. Mat. Pura Appl. Ser. 4 **131** (1982), 75–106.

[7] S. Campanato, *Hölder continuity and partial Hölder continuity results for* $W^{1,q}$*-solutions of non-linear elliptic systems with controlled growth*, Rend. Sem. Mat. Fis. Milano **52** (1982), 435–472.

[8] S. Campanato, *Hölder continuity of the solutions of some nonlinear elliptic systems*, Adv. Math. **48** (1983), 16–43.

[9] S. Campanato, *A maximum principle for nonlinear elliptic systems: Boundary fundamental estimates*, Adv. Math. **66** (1987), 291–317.

[10] S. Campanato, *Elliptic systems with non-linearity q greater or equal to two. Regularity of the solution of the Dirichlet problem*, Ann. Mat. Pura Appl. Ser. 4 **147** (1987), 117–150.

[11] M. Carozza, N. Fusco, and G. Mingione, *Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth*, Ann. Mat. Pura Appl. Ser. 4 **175** (1998), 141–164.

[12] F. Colombini, *Un teorema di regolarità alla frontiera per soluzioni di sistemi ellittici quasi lineari*, Ann. Sc. Norm. Super. Pisa Ser. III **25** (1971), 15–161.

[13] E. De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Unione Mat. Ital., IV. **1** (1968), 135–137.

[14] F. Duzaar and J. F. Grotowski, *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation*, Manuscr. Math. **103** (2000), 267–298.

[15] F. Duzaar, J. F. Grotowski, and M. Kronz, *Partial and full boundary regularity for minimizers of functionals with nonquadratic growth*, J. Convex Anal. **11** (2004), no. 2, 437–476.

[16] F. Duzaar, J. Kristensen, and G. Mingione, *The existence of regular boundary points for non-linear elliptic systems*, J. Reine Angew. Math. **602** (2007), 17–58.

[17] F. Duzaar and G. Mingione, *Regularity for degenerate elliptic problems via p-harmonic approximation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **21** (2004), no. 5, 735–766.

[18] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

[19] M. Giaquinta, *A counterexample to the boundary regularity of solutions to elliptic quasilinear systems*, Manuscr. Math. **24** (1978), 217–220.

[20] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, Princeton, New Jersey, 1983.

[21] M. Giaquinta and E. Giusti, *Nonlinear elliptic systems with quadratic growth*, Manuscr. Math. **24** (1978), 323–349.

[22] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[23] E. Giusti, *Direct Methods in the Calculus of Variation*, World Scientific Publishing, Singapore, 2003.

[24] E. Giusti and M. Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Unione Mat. Ital., IV. Ser. **1** (1968), 219–226.

[25] J. F. Grotowski, *Boundary regularity results for nonlinear elliptic systems in divergence form*, Habilitationsschrift, Erlangen, 2000.

[26] J. F. Grotowski, *Boundary regularity for quasilinear elliptic systems*, Commun. Partial Differ. Equations **27** (2002), no. 11-12, 2491–2512.

[27] J. Kristensen and G. Mingione, *The Singular Set of Minima of Integral Functionals*, Arch. Rational Mech. Anal. **180** (2006), no. 3, 331–398.

[28] G. Mingione, *Bounds for the singular set of solutions to non linear elliptic systems*, Calc. Var. **18** (2003), no. 4, 373–400.

[29] G. Mingione, *The Singular Set of Solutions to Non-Differentiable Elliptic Systems*, Arch. Rational Mech. Anal. **166** (2003), 287–301.

[30] J. Necas, O. John, and J. Stará, *Counterexample to the regularity of weak solution of elliptic systems*, Commentat. Math. Univ. Carol. **21** (1980), 145–154.

[31] L. Pepe, *Risultati di regolarità parziale per le soluzioni* $\mathcal{H}^{1,p}$ $1 < p < 2$ *di sistemi ellittici quasi lineari*, Ann. Univ. Ferrara, N. Ser., Sez. VII **16** (1971), 129–148.

[32] K.-O. Widman, *Hölder continuity of solutions of elliptic systems*, Manuscr. Math. **5** (1971), 299–308.