

# Casimir effect from a scattering approach

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## I. INTRODUCTION

The non-vanishing ground-state energy of the harmonic oscillator within a quantum description provides a prominent example of the consequences of the Heisenberg uncertainty relation. However, this ground-state energy is not directly accessible. In the quantum theory of the electromagnetic field, where the field modes can be thought of as harmonic oscillators, the infinite number of modes even leads to an infinite ground-state energy. This infinite energy is usually removed from mathematical expressions by so-called normal ordering, where in an operator product the creation operators  $a^\dagger$  are moved to the left and the annihilation operators  $a$  are moved to the right, without accounting for the commutation relation between  $a^\dagger$  and  $a$ . On the level of a single harmonic oscillator of frequency  $\omega$ , normal ordering amounts to replacing the Hamiltonian

$$H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) = \frac{\hbar\omega}{2} (a^\dagger + a)(a + a^\dagger) \quad (1)$$

by

$$H = \hbar\omega a^\dagger a. \quad (2)$$

The expectation value of the normal-ordered Hamiltonian in the ground state is zero.

The situation changes when boundary conditions are imposed. For the electromagnetic field, one can imagine placing mirrors into space. Then, the ground-state energy or vacuum energy, as it is usually called in the context of a field theory, will take on a different, albeit still infinite, value. However, one can ask how the vacuum energy changes when the boundaries are modified. A well-defined question is how this energy changes when two infinite parallel mirrors are put into space. Changing their distance will lead to a change in vacuum energy, or equivalently, the presence of the electromagnetic field even in its ground state will result in a nonzero force between the two mirrors. In 1948, Casimir showed that the force  $F$  between two parallel ideal mirrors at distance  $L$  at zero temperature is given by<sup>1</sup>

$$F_{3D} = \frac{\pi^2 \hbar c}{240 L^4}. \quad (3)$$

Remarkably, apart from the distance between the mirrors and their surface area  $A$ , the Casimir force, as it has been called since then, depends only on physical constants, namely the Planck constant  $\hbar$  and the speed of light  $c$ . For a distance  $L = 1 \mu\text{m}$ , the resulting pressure on the mirrors is  $p = 1.3 \text{ mPa}$ .

A few years after the prediction by Casimir, the first experiments aimed at its experimental verification took place.<sup>2,3</sup> During the next three decades, more Casimir force measurements between two parallel plates or a plate and a spherical surface followed, employing different methods and materials.<sup>4–13</sup> Further insights into the historical context are given in Refs. 14 and 15 as well as in Ref. 16, where the first sixty years of the Casimir effect have been concisely reviewed.

During the past two decades, modern measurement techniques such as precise torsion pendula, atomic force microscopy, and micro-electro-mechanical oscillators have been developed, which allow for new precise Casimir force measurements. Starting with the experiments by Lamoreaux<sup>17</sup> and Mohideen,<sup>18</sup> the Casimir effect has experienced an enormous increase in experimental activities<sup>19–33</sup> as well as in theoretical developments. Taking into account material properties, geometry, temperature, and the surface state in modern calculations is essential for obtaining reliable theoretical predictions to be compared with Casimir force measurements. For a detailed discussion of recent developments, we refer the reader to the collection of papers in Ref. 34 and the textbook by Bordag *et al.*,<sup>35</sup> as well as to the resource letters by Lamoreaux<sup>36</sup> and Milton,<sup>37</sup> which can serve as a guide to the literature.

The traditional way to calculate the Casimir force (3) consists in calculating the ground-state energy for all modes of the electromagnetic field between two parallel ideal plane mirrors. In order to handle the divergence of the vacuum energy, an appropriate high-frequency cutoff procedure is employed. Despite its formal character, this approach following Casimir's original paper<sup>1</sup> is mostly taught and described in textbooks, e.g., Ref. 38. Of course, there are many other methods. Those already presented in this journal are based, e.g., on the calculation of the vacuum radiation pressure<sup>39</sup> or on mode spectrum calculations for the force in one dimension.<sup>40</sup>

In research, various complementary methods have been developed, such as the image method,<sup>41</sup> which can also be used to evaluate van der Waals forces,<sup>42,43</sup> vacuum radiation pressure calculations,<sup>44</sup> or, more recently, the worldline approach,<sup>45</sup> while multiple scattering techniques have been a valuable tool in the context of Casimir physics for many years. For example, Balian and Duplantier used the technique to derive the Casimir energy for a sphere.<sup>46</sup> The general formula to calculate the Casimir effect that we will derive here involves the logarithm of the determinant of the scattering matrix. Such formulas appear already in early calculations of van der Waals and Casimir interactions.<sup>47,48</sup> Their appearance can be traced back to the Lippmann-Schwinger formulation of scattering theory.<sup>49</sup> More recently, they have been rediscovered using quantum optical scattering methods,<sup>44,50</sup> the  $T$ -operator approach,<sup>51</sup> the Krein formula,<sup>52</sup> and fluctuating-current scattering theory;<sup>53</sup> see also the short review by Milton in Ref. 54.

Here, we want to propose the use of the scattering theory applied to a one-dimensional field theory<sup>44</sup> as an alternative to the traditional way of teaching the Casimir effect. Reducing the dimension of the problem presents the advantage of avoiding the necessity to analyze all electromagnetic modes between the two mirrors, which unnecessarily complicates the problem. One-dimensional models have already been used in the past to highlight certain aspects of the Casimir effect.<sup>55,56</sup>

Of course, in one dimension, the distance dependence will differ from the  $A/L^4$  behavior of the Casimir force (3) in three dimensions. A simple dimensional argument allows us to deduce the distance dependence of the force if only one space dimension is considered. As one cannot define a surface in one dimension, the surface drops out of the numerator and the Casimir force must scale as  $1/L^2$ . In the prefactor,  $\hbar c$  will be retained for dimensional reasons, while the numerical prefactor will turn out to be different.

It should be realized that the three-dimensional infinitely large plate-plate configuration underlying expression (3) for the Casimir force refers to a very particular geometry that is never realized in experiments. In fact, in modern experiments, the Casimir force is usually measured between a sphere and a plate, thereby avoiding misalignment.<sup>6</sup> A notable exception is the experiment described in Ref. 57, where the force was measured between two finite parallel plates.

Deriving the Casimir effect within a scattering theory, as we will do in the following, presents several pedagogical advantages.

First, students who have taken a first course in quantum mechanics are acquainted with one-dimensional scattering problems. The scattering at a potential barrier, to name but one example, is a standard exercise. While the formal scattering approach is often not taught, it represents a natural extension of such standard problems. The techniques acquired in this context can be useful in other areas of modern physics such as mesoscopic physics,<sup>58</sup> where the Landauer-Büttiker theory<sup>59</sup> of the conductance constitutes one example.

Second, the formal high-frequency regularization mentioned above is avoided. Already Casimir had remarked in this respect: “The physical meaning is obvious: for very short waves (X-rays, e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate.”<sup>1</sup> A method taking into account the physics at high frequencies is

certainly preferable. Furthermore, the scattering approach allows one to identify the contribution to the vacuum energy depending on the distance between the two mirrors in a natural way. It thereby clarifies the meaning of the Casimir energy.

Third, the scattering approach has proven to be of great value in the theoretical treatment of the Casimir effect. It can be used to deal, for example, with real mirrors described by a dielectric function and non-planar geometries. The calculation presented here therefore gives insight into methods used in present-day research on the Casimir effect.

With this motivation for a scattering approach to the Casimir effect in mind, the following section reviews some basic aspects of scattering theory that will be needed in what follows. In Sec. III, we apply this theory to obtain the change of the vacuum energy due to the presence of scatterers in one dimension. For two scatterers, the vacuum energy shift can be decomposed into terms due to the individual scatterers and a term depending on the distance between the two scatterers. The latter term is the Casimir energy, which is determined in Sec. IV. The distance dependence of the Casimir energy implies a force, which we derive in Sec. V for the one-dimensional case. We conclude in Sec. VI by sketching how the approach can be generalized to three dimensions and geometries of practical interest. In Sec. VII, we have added three problems, which might be instructive for students.

## II. SCATTERING THEORY

We first review the basic properties of scattering theory in one spatial dimension. To this end, we assume that there exists a scattering region of finite extent, depicted in Fig. 1(a) by the gray area. In the regions to the left and to the right of the scattering region, plane waves provide an appropriate solution. Their amplitudes are  $a^\pm$  and  $b^\pm$ , with the index  $\pm$  indicating right- and left-going waves, and  $a$  and  $b$  indicating the region to the left and to the right, respectively.

The scattering matrix  $S$  relates the ingoing waves characterized by the amplitudes  $a^+$  and  $b^-$  to the outgoing waves with amplitudes  $a^-$  and  $b^+$ :

$$\begin{pmatrix} b^+ \\ a^- \end{pmatrix} = S \begin{pmatrix} a^+ \\ b^- \end{pmatrix}. \quad (4)$$

The scattering matrix as well as the amplitudes will in general depend on the wave number  $k$ . For the sake of simplicity, we will not make this dependence explicit in most equations.

In the context of the Casimir effect, the scattering matrix will describe a mirror, with the diagonal and non-diagonal

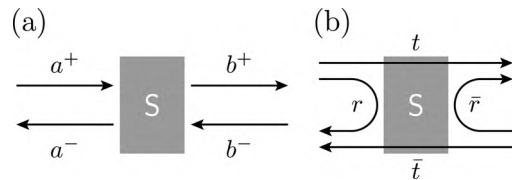


Fig. 1. Notation for the one-dimensional scattering problem where the gray area indicates a finite scattering region. (a) Here  $a$  and  $b$  refer respectively to the regions left and right of the scatterer. The superscripts  $+$  and  $-$  indicate right- and left-going fields, respectively. (b) Reflection and transmission amplitudes are denoted by  $r$  and  $t$  for fields coming from the left and by  $\bar{r}$  and  $\bar{t}$  for fields coming from the right.

matrix elements referring to transmission and reflection processes, respectively. Accordingly, we choose the following notation for the scattering matrix:

$$S = \begin{pmatrix} t & \bar{r} \\ r & \bar{t} \end{pmatrix}, \quad (5)$$

where the scattering amplitudes visualized in Fig. 1(b) are not necessarily the same on both sides of the scatterer. This distinction leaves the possibility open that the mirror behaves differently for modes arriving from the left or the right. It should be kept in mind that  $r, \bar{r}$  and  $t, \bar{t}$  are reflection and transmission amplitudes, respectively, and therefore in general complex numbers. To avoid confusion, we note that here we follow the convention used in quantum field theory, which differs from the one commonly employed in mesoscopic physics. There, the vector on the left-hand side of Eq. (4) is chosen as  $(a^-, b^+)$ , which has as a consequence that the transmission amplitudes appear as off-diagonal elements and not on the diagonal as in Eq. (5).

Current conservation requires the scattering matrix to be unitary:

$$S^\dagger S = 1. \quad (6)$$

As a consequence, the matrix elements have to obey the three conditions

$$|t|^2 + |r|^2 = |\bar{t}|^2 + |\bar{r}|^2 = 1, \quad (7)$$

$$r\bar{t}^* + t\bar{r}^* = 0, \quad (8)$$

where the star indicates complex conjugation. For later use, we note that because of the unitarity relations, the determinant of the scattering matrix can be expressed solely in terms of the transmission coefficients:

$$\det(S) = \frac{t}{\bar{t}^*} = \frac{\bar{t}}{t^*}. \quad (9)$$

A simple example of a scattering matrix is given by

$$S = \begin{pmatrix} 1+r & r \\ r & 1+r \end{pmatrix}, \quad (10)$$

with the complex reflection coefficient

$$r = \frac{g}{2ik - g}, \quad (11)$$

where  $g$  is a constant. With increasing wave number  $k$ , the scatterer turns from perfectly reflecting into almost perfectly transmitting. (The first two problems given in Sec. VII demonstrate how this specific scattering matrix can be obtained in the contexts of single-particle quantum mechanics and of electromagnetic transmission lines.)

In order to determine the Casimir force in a one-dimensional field theory, we need to describe a system composed of two scatterers and to account for the propagation of the waves in between them. As a first step, we explain how the combined effect of two scatterers can be obtained for the setup shown in Fig. 2. The propagation between the two scatterers will be incorporated at a later stage without any difficulties.

For the description of a combined scattering process, it is convenient to consider the transfer matrix  $T$ , which relates

the waves on the left-hand side to those on the right-hand side of the scatterer according to

$$\begin{pmatrix} b^+ \\ b^- \end{pmatrix} = T \begin{pmatrix} a^+ \\ a^- \end{pmatrix}. \quad (12)$$

In this way, the combined scattering properties of two or more scatterers in series can easily be calculated.

As both Eq. (4) and Eq. (12) are linear equations for the coefficients  $a^\pm$  and  $b^\pm$ , it is straightforward to convert the scattering matrix into the corresponding transfer matrix and vice versa. One finds

$$T = \frac{1}{S_{22}} \begin{pmatrix} \det(S) & S_{12} \\ -S_{21} & 1 \end{pmatrix} \quad (13)$$

and

$$S = \frac{1}{T_{22}} \begin{pmatrix} \det(T) & T_{12} \\ -T_{21} & 1 \end{pmatrix}. \quad (14)$$

If  $S_1$  and  $S_2$  are given, we can now employ Eq. (13) to obtain the corresponding transfer matrices  $T_1$  and  $T_2$ . The transfer matrices allow us to express the effect of the scatterers in series by means of a single transfer matrix  $T = T_2 T_1$ , from which we obtain the effective scattering matrix  $S$  for the two scatterers seen from the outside. Its matrix elements expressed in terms of the reflection and transmission amplitudes of the single scatterers read

$$S = \begin{pmatrix} \frac{t_1 t_2}{1 - \bar{r}_1 r_2} & \bar{r}_2 + \frac{\bar{r}_1 t_2 \bar{t}_2}{1 - \bar{r}_1 r_2} \\ r_1 + \frac{r_2 t_1 \bar{t}_1}{1 - \bar{r}_1 r_2} & \frac{\bar{t}_1 \bar{t}_2}{1 - \bar{r}_1 r_2} \end{pmatrix}. \quad (15)$$

Expanding each of the denominators as a geometric series, one can convince oneself that all possible scattering processes, including an arbitrary number of back-and-forth scatterings between the two scatterers, are contained in this scattering matrix.

Expressing the determinant of the scattering matrix (15) in terms of the transmission coefficients by means of Eq. (9), one finds

$$\det(S) = \frac{t_1 t_2}{\bar{t}_1^* \bar{t}_2^*} \frac{1 - \bar{r}_1^* r_2^*}{1 - \bar{r}_1 r_2}. \quad (16)$$

The first two factors refer to the transmission through a single scatterer, while the third factor accounts for all multiple reflections between the two scatterers. Making use of Eq. (9) once more yields

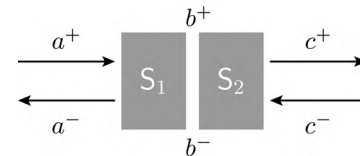


Fig. 2. Setup containing two scatterers with scattering matrices  $S_1$  and  $S_2$ . The wave amplitudes in the left and right regions are denoted by  $a^\pm$  and  $c^\pm$ , respectively, while  $b^\pm$  refer to the wave amplitudes between the two scatterers.

$$\det(S) = \det(S_1)\det(S_2) \frac{1 - \bar{r}_1^* r_2^*}{1 - \bar{r}_1 r_2}. \quad (17)$$

A slightly generalized version of this relation will allow us to isolate the distance-dependent part of the vacuum energy (see Sec. IV).

### III. INFLUENCE OF SCATTERERS ON THE VACUUM ENERGY

So far, we have not addressed the question of why the Casimir effect can be approached by means of scattering matrices at all. Therefore, we now consider the change of the vacuum energy when a scatterer is inserted into our one-dimensional field. Later we will think of this single scatterer in terms of two scatterers representing the two mirrors, as indicated in the lower part of Fig. 3. However, we know already that the two scatterers can be described by a single scattering matrix and thus be viewed from the outside as a single effective scatterer. Hence, for the moment, it is sufficient to consider a single scattering matrix  $S$ .

We imagine the scatterer sitting in the middle of a one-dimensional space of length  $\mathcal{L}$ . Eventually we will take the limit  $\mathcal{L} \rightarrow \infty$ , and then the result will be independent of which boundary conditions are imposed. For convenience, we choose periodic boundary conditions, as indicated in the upper part of Fig. 3. The scatterer is characterized by a scattering matrix  $S$  or, equivalently, a transfer matrix  $T$ . The periodic boundary condition can then be expressed as

$$\mathbb{T}\mathbb{T}_{\mathcal{L}} \begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \quad (18)$$

where

$$\mathbb{T}_{\mathcal{L}} = \begin{pmatrix} e^{ik\mathcal{L}} & 0 \\ 0 & e^{-ik\mathcal{L}} \end{pmatrix} \quad (19)$$

is the transfer matrix describing free propagation of a wave with positive wave number  $k$  moving to the right in the first and to the left in the second component. From Eq. (18), we obtain the eigenvalue condition for the waves in the presence of a scatterer,

$$\exp(-ik^{\pm}\mathcal{L}) = \frac{t + \bar{t}}{2} \pm \left( \frac{(t + \bar{t})^2}{4} - \det(S) \right)^{1/2}, \quad (20)$$

where the scattering matrix is a function of  $k$ .

In view of the linear dispersion relation

$$\omega = ck \quad (21)$$

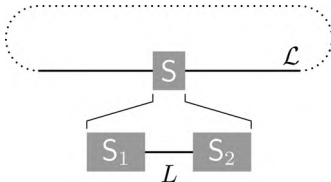


Fig. 3. A scatterer placed in a one-dimensional space of length  $\mathcal{L}$  is discussed in Sec. III. The dotted line indicates the use of periodic boundary conditions. In Sec. IV, the scatterer will be replaced by two scatterers separated by a distance  $L$ , as indicated in the lower part of the figure.

for the field modes of frequency  $\omega$  and wave number  $k$ , the vacuum energy is given by

$$E_{\text{vac}} = \sum_n \frac{\hbar c}{2} (k_n^+ + k_n^-), \quad (22)$$

where the sum runs over all modes. From Eq. (20), we find

$$k_n^+ + k_n^- = 2 \frac{2\pi n}{\mathcal{L}} + \Delta k_n^+ + \Delta k_n^-. \quad (23)$$

The first term refers to the case without scatterers, and corresponds to the sum of the wave numbers  $2\pi n/\mathcal{L}$  of a pair of left- or right-moving modes. The second and third terms are due to the presence of the scatterer and according to Eq. (20) are given by

$$\Delta k_n^+ + \Delta k_n^- = i \frac{1}{\mathcal{L}} \ln[\det(S)]. \quad (24)$$

Note that the determinant of the scattering matrix according to the unitarity condition (6) has modulus 1. Therefore, the logarithm is purely imaginary, and the shift of the wave numbers turns out to be real, as it should.

The scatterer thus induces a shift in the vacuum energy of

$$\begin{aligned} \Delta E_{\text{vac}} &= \frac{\hbar c}{2} \sum_n (\Delta k_n^+ + \Delta k_n^-) \\ &= \frac{\hbar c}{2} \int_0^\infty dk \frac{\mathcal{L}}{2\pi} \frac{i \ln[\det(S)]}{\mathcal{L}} \\ &= \frac{i\hbar c}{4\pi} \int_0^\infty dk \ln[\det(S)]. \end{aligned} \quad (25)$$

In going to the second line, we have made use of the fact that according to Eq. (23) the difference in wave number between subsequent unperturbed modes amounts to  $2\pi/\mathcal{L}$ . In the limit  $\mathcal{L} \rightarrow \infty$  the sum in the first line can then be interpreted as a Riemann sum representing the integral in the second line.

### IV. CASIMIR ENERGY FOR TWO SCATTERERS

If we place two scatterers into the one-dimensional field as depicted in Fig. 3, we can make use of Eq. (25) to obtain the change in the vacuum energy. We only need to determine the determinant of the scattering matrix describing the two scatterers at a distance  $L$  from each other. The total transfer matrix

$$\mathbb{T} = \mathbb{T}_L^{-1} \mathbb{T}_2 \mathbb{T}_L \mathbb{T}_1, \quad (26)$$

which should be read from right to left, then contains four contributions: The transfer matrix  $\mathbb{T}_1$  of the first scatterer, the transfer matrix for free propagation according to Eq. (19) with  $\mathcal{L}$  replaced by  $L$ , the transfer matrix  $\mathbb{T}_2$  of the second scatterer, and finally the inverse of the transfer matrix  $\mathbb{T}_L$ . The role of the last factor becomes clear by inserting the transfer matrix (26) into the eigenvalue condition (18) and realizing that  $\mathbb{T}_L^{-1} \mathbb{T}_{\mathcal{L}} = \mathbb{T}_{\mathcal{L}-L}$ . The original system of length  $\mathcal{L}$  now consists of two parts, of length  $\mathcal{L} - L$  and of length  $L$ . The last transfer matrix in Eq. (26) thus ensures that the overall length of the system remains constant even though a scattering region of length  $L$  has been inserted.



Performing the matrix multiplications, we find

$$\det(S) = \frac{\det(S_1)\det(S_2) - r_1\bar{r}_2 \exp(-2ikL)}{1 - \bar{r}_1 r_2 \exp(2ikL)}. \quad (27)$$

Recalling Eq. (17), we expect that the determinant can be factored. By means of the relations (8) and (9), we obtain from Eq. (27) the factorization of the determinant of the scattering matrix,

$$\det(S) = \det(S_1)\det(S_2) \frac{1 - [\bar{r}_1 r_2 \exp(2ikL)]^*}{1 - \bar{r}_1 r_2 \exp(2ikL)}. \quad (28)$$

This decomposition is physically quite significant because it implies that the change of the vacuum energy (25) due to placing mirrors into the field consists of three parts:

$$\Delta E_{\text{vac}} = \Delta E_{\text{vac}}^{(1)} + \Delta E_{\text{vac}}^{(2)} + \Delta E_{\text{vac}}(L). \quad (29)$$

The first two contributions to the determinant (28) of the scattering matrix and to the vacuum energy (29) arise due to a single mirror, and they are therefore independent of  $L$ . In contrast, the third contribution depends on the distance  $L$  because the coherent field between the two mirrors is sensitive to their distance. Therefore, the third term yields the Casimir energy. The last factor in Eq. (28) is a pure phase factor, so that the Casimir energy can be expressed as

$$\Delta E_{\text{vac}}(L) = \frac{\hbar c}{2\pi} \text{Im} \int_0^\infty dk \ln[1 - \bar{r}_1 r_2 \exp(2ikL)]. \quad (30)$$

The argument of the logarithm has a simple interpretation in terms of one round-trip between the two scatterers consisting of a reflection at scatterer 1, a propagation over a distance  $L$ , a reflection at scatterer 2, and finally another propagation over a distance  $L$  in order to return back to scatterer 1. Recalling the notation introduced in Fig. 1(b), one immediately sees that the force depends only on the inner reflection coefficients  $\bar{r}_1$  and  $r_2$  of the two-scatterer setup.

## V. CASIMIR FORCE IN ONE DIMENSION

The Casimir force is obtained from the distance-dependent part (30) of the change of the vacuum energy induced by the two scatterers, specifically as the derivative with respect to their distance  $L$ :

$$\begin{aligned} F &= - \frac{d\Delta E_{\text{vac}}(L)}{dL} \\ &= \frac{\hbar c}{\pi} \text{Re} \left[ \int_0^\infty dk k \frac{\bar{r}_1 r_2 \exp(2ikL)}{1 - \bar{r}_1 r_2 \exp(2ikL)} \right]. \end{aligned} \quad (31)$$

In order to evaluate the Casimir force, it is convenient to rotate the axis of integration from the positive real to the positive imaginary axis. This step is common in the evaluation of the Casimir force, including the more general cases.<sup>60–62</sup> Excluding the special case of amplifying media, causality ensures that the integrand has no poles in the upper complex half-plane. This can be explicitly verified by considering the scattering matrix (10) with the reflection coefficient (11). Furthermore, because of the exponential function in the numerator, the integrand vanishes at infinity in the upper complex half-plane. Therefore, we can apply the residue theorem to turn the integration contour. Replacing the wave number  $k$  by  $ix/2L$ , one obtains

$$F = - \frac{\hbar c}{4\pi L^2} \int_0^\infty dx x \frac{\bar{r}_1(x)r_2(x)\exp(-x)}{1 - \bar{r}_1(x)r_2(x)\exp(-x)}. \quad (32)$$

Noting that the exponential in the numerator of Eq. (32) cuts off the integrand, we can obtain the limit of perfect reflectors,  $r_1 = \bar{r}_2 = -1$ . The integral can be evaluated by first expressing the integrand in terms of a geometric series and performing a resummation after having integrated each term. We thus arrive at the Casimir force for perfect reflectors in one dimension:

$$F_{1D} = - \frac{\hbar c \pi}{24L^2}. \quad (33)$$

The sign implies an attractive force between the scatterers. The force scales indeed as  $1/L^2$ , as expected from the dimensional argument presented in the introduction, with the same dependence on fundamental constants  $\hbar$  and  $c$  but a different numerical prefactor. This result has been obtained for one-dimensional models in various contexts.<sup>63–66</sup>

## VI. OUTLOOK

The calculations presented here can be generalized to three-dimensional space involving the full electromagnetic field enclosed between two plane parallel mirrors. Then, the scattering on the mirror can still be described by a  $2 \times 2$  scattering matrix relating the two outgoing fields to the two incoming ones. However, now the electromagnetic field is characterized by its frequency, transverse wave vector, and polarization. From the symmetry with respect to time translations and transverse space translations, it follows that these quantities are preserved throughout the scattering process. Therefore, the expression (25) for the shift in the vacuum energy still holds, provided an integration over the transverse wave vector and a summation over the two polarizations is added. In this way, the scattering approach leads to the original result (3) by Casimir.

In an arbitrary static configuration with two scatterers in vacuum, the Casimir energy can still be written in the form of Eq. (25). This includes, e.g., the experimentally relevant cases of a sphere in front of a plate<sup>17,18,20,22,24,26–28,30–32</sup> and structured surfaces.<sup>21,25,29,33</sup> However, in general, plane waves will no longer be adapted to the geometry of the problem. The scattering processes now can lead to changes of the transverse wave vector and to a coupling between polarizations, resulting in high-dimensional scattering matrices of a complex structure.

Besides the geometry, the comparison with experimental results requires one to account also for material properties<sup>61,62,67</sup> and nonzero temperature  $T$ .<sup>61,62,68–70</sup> The former enter into the elements of the scattering matrix,<sup>60</sup> while the latter can be accounted for by replacing the vacuum energy  $\hbar\omega/2$  with the Planck factor<sup>15</sup>  $(\hbar\omega/2)\coth(\hbar\omega/k_B T)$ .

Therefore, a calculation taking all these aspects into account can become rather challenging and numerically demanding.

## VII. SUGGESTED PROBLEMS

1. Derive the scattering matrix (10) with the reflection coefficient (11) for a delta-like scattering potential

$$V_0(x) = \frac{\hbar^2 g}{2m} \delta(x). \quad (34)$$

- Depending on the level of knowledge of quantum mechanics, different approaches can be chosen, e.g.: (1) Start from the textbook expression for the complex reflection coefficient of a rectangular potential barrier of height  $V_0$  and width  $a$ . Take the limit  $V_0 \rightarrow \infty$ ,  $a \rightarrow 0$  while keeping  $V_0 a = \hbar^2 g / 2m$  constant to obtain Eq. (11). Derive the remaining matrix elements of the scattering matrix by exploiting the symmetry of the problem and the unitarity of the scattering matrix. Compare your result with Eq. (10). (2) Start from the boundary conditions at the delta-like potential to obtain relations between the coefficients of the wave functions on both sides of the scatterer (cf. Fig. 1). Rearrange the equations so that you can read off the scattering matrix.
2. Consider an infinite LC transmission line with inductance  $\bar{L}$  and conductance  $\bar{C}$  per unit length. At  $x = 0$ , the two conductors of the transmission line are connected by an inductance  $L$ . Determine the reflection coefficient and prove that it can be brought into the form of Eq. (11). The symmetry of the problem and the requirement of unitarity allow one to arrive at the full scattering matrix (10).
  3. Convince yourself that the scattering matrix (15) accounts for all possible scattering processes involving two scatterers. Hint: Make use of a geometric series.

## ACKNOWLEDGMENTS

The authors thank Serge Reynaud for many insightful discussions. This work has been supported by the DAAD and Égide through the PROCOPE program as well as by the European Science Foundation (ESF) within the activity “New Trends and Applications of the Casimir Effect” ([www.casimir-network.com](http://www.casimir-network.com)).

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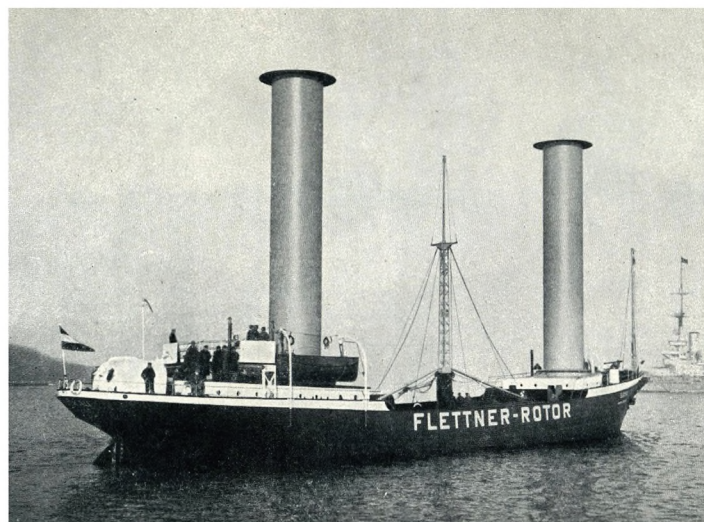
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**Flettner Rotor-Ship**

Anton Flettner (1885-1961) was a German engineer who worked with ship propulsion and control systems in the 1920s. The picture shows the *Baden-Baden*, a sailing ship refitted with two cylinders, each 15 m high and 3 m in diameter that were rotated by a 50 hp motor. The ship utilized the Magnus effect, in which air blowing on a rotating body deflects it to the side. Wind from abeam would drive the ship forward or backward, depending on the rotation direction. In 1926 the vessel traveled from Germany to New York by way of South America. A larger, three rotor ship, the *Barbara*, was used as a freighter in the Mediterranean during the years 1926-1929. The depression and the development of diesel-power for ships put an end to the use of rotor ships, although there has been a recent revival of interest. From Anton Flettner, *The Story of the Rotor* (F.O. Willhofft, New York, 1926), Plate XXV. (Notes by Thomas B. Greenslade, Jr., Kenyon College)