

# Reentrant classicality of a damped system

BENJAMIN SPRENG<sup>1</sup>, GERT-LUDWIG INGOLD<sup>1(a)</sup> and ULRICH WEISS<sup>2</sup>

<sup>1</sup> *Institut für Physik, Universität Augsburg - D-86135 Augsburg, Germany, EU*

<sup>2</sup> *II. Institut für Theoretische Physik, Universität Stuttgart - D-70550 Stuttgart, Germany, EU*

**Introduction.** – Physical systems are expected to develop quantum effects as temperature is lowered. This expectation can be motivated by considering the dimensionless quantity  $k_B T / \hbar \omega$ , where  $T$  is the temperature and  $\omega$  is a characteristic system frequency. Decreasing the temperature is thus equivalent to an effective increase of Planck’s constant  $\hbar$  and consequently to more pronounced quantum effects. A very particular case in this respect is the free particle where the system does not possess a frequency which in our argument could play the role of  $\omega$ .

Confining a free particle of mass  $M$  to a finite region of size  $L$  supplies it with a frequency scale  $\hbar/2ML^2$ . In the following, we assume that  $L$  is sufficiently large so that finite-size effects are irrelevant at the temperatures of interest. However, finite-size effects become important as the strict zero-temperature limit is taken.

Another frequency scale, the one of interest here, arises if we account for the fact that a physical system is never completely decoupled from its environment. The coupling to the environment gives rise to damping of the free particle and provides the damping strength  $\gamma$  as a frequency scale. Interestingly, the dimensionless quantity  $k_B T / \hbar \gamma$  implies that the environmental coupling drives the particle into the quantum regime, quite in contrast to the usual

expectation that the environment should suppress quantum effects. The latter would be the case, *e.g.*, for the harmonic oscillator where the oscillator frequency provides a temperature scale. Our reasoning might make the free particle appear as very particular. However, there exist applications where the low-temperature scale is dominated by the dissipation, as is the case, *e.g.*, for the Casimir effect in the presence of Drude-type metal boundaries [1].

In order to quantify the properties of the damped free particle, we will focus on the specific heat. As other thermodynamic quantities, the specific heat of a system in the presence of a finite coupling to the environment is not uniquely defined except in the classical limit of very high temperatures [2]. Here, we will define the specific heat on the basis of the reduced partition function

$$\mathcal{Z} = \mathcal{Z}_{S+B} / \mathcal{Z}_B, \quad (1)$$

*i.e.* the ratio of the partition function of system and bath  $\mathcal{Z}_{S+B}$  and the partition function of the bath alone  $\mathcal{Z}_B$ . Based on (1) the specific heat of the damped system is given by the difference [3]

$$C = C_{S+B} - C_B \quad (2)$$

of the specific heats of system and bath, on the one hand, and of the bath alone on the other hand. Such a definition is employed, *e.g.*, in the study of impurity systems [4,5].

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<sup>(a)</sup>E-mail: gert.ingold@physik.uni-augsburg.de

For an ohmic environment, the specific heat defined in this way can become negative if the coupling between system and bath is sufficiently strong [6]. What may appear as an artifact of the definition (2) has a clear physical meaning. It can be shown that the coupling between the free particle and its environment leads to a shift of environmental modes towards higher frequencies. The resulting suppression of low-frequency bath modes is responsible for negative values of the specific heat at low temperatures [7].

In the following, we will generalize previous studies of the free particle coupled to an ohmic environment to the non-ohmic case. An example of a super-ohmic bath is given by a phononic environment. The specific heat for non-ohmic baths has been studied for systems with an internal frequency scale like the two-level system or the harmonic oscillator [8–12]. But as we have seen above, such systems can behave quite differently from the damped free particle even in their thermodynamic equilibrium properties. In fact, the specific heat of the damped free particle at low temperatures has some surprises to offer.

In the next section we review those aspects of damped systems which will be required in our study of the specific heat of the damped free particle. After providing an expression for the partition function and the specific heat, we present a qualitative discussion of those features of the dependence of the specific heat on temperature and damping which are in the focus of the present paper. Analytical results are then given for the high-temperature quantum corrections to the specific heat and for its low-temperature behavior. Finally, we draw our conclusions.

**Description of the damped system.** – The free damped particle will be modelled by a Hamiltonian in which the system described by its position  $Q$  and momentum  $P$  is coupled bilinearly to a set of harmonic oscillators with masses  $m_n$  and frequencies  $\omega_n$  described by their positions  $q_n$  and momenta  $p_n$  [13],

$$H = \frac{P^2}{2M} + \sum_{n=1}^{\infty} \left[ \frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} (q_n - Q)^2 \right]. \quad (3)$$

This Hamiltonian contains a potential renormalization term proportional to  $Q^2$  in order to ensure that even in the presence of damping a zero mode exists, as is manifested by the translational invariance of (3). The parameters in the Hamiltonian are sufficiently general [14,15] to describe general linear damping because the properties of the damped system depend on the microscopic parameters of the bath exclusively via the spectral density

$$J(\omega) = \frac{\pi}{2} \sum_{n=1}^{\infty} m_n \omega_n^3 \delta(\omega - \omega_n). \quad (4)$$

The total mass  $M_{\text{bath}}$  of the bath oscillators reads

$$M_{\text{bath}} \equiv \sum_{n=1}^{\infty} m_n = \frac{2}{\pi} \int_0^{\infty} d\omega \frac{J(\omega)}{\omega^3}. \quad (5)$$

The dynamics of the free Brownian particle is specified by the causal velocity response function  $\mathcal{R}(t)$ . In the denominator, its Laplace transform

$$\hat{\mathcal{R}}(z) = \frac{1}{z + \hat{\gamma}(z)} \quad (6)$$

contains the inertial term as well as the spectral damping function. The latter is determined by the spectral density  $J(\omega)$  through the relation

$$\hat{\gamma}(z) = \frac{2}{\pi M} \int_0^{\infty} d\omega \frac{J(\omega)}{\omega} \frac{z}{\omega^2 + z^2}. \quad (7)$$

To be specific, we assume a continuous distribution of bath oscillators with a spectral density of the form

$$J(\omega) = M \gamma \omega^s \frac{\omega_c^{2p-s+1}}{(\omega_c^2 + \omega^2)^p}. \quad (8)$$

For small frequencies, the spectral density (8) increases with the power law  $\omega^s$ . The regimes  $s < 1$  and  $s > 1$  correspond to sub-ohmic and super-ohmic damping, respectively, and  $s = 1$  is the special case of ohmic damping. The last factor in (8) constitutes an algebraic high-frequency cutoff. The frequency integral (7) with (8) is convergent for  $s$  in the range

$$0 < s < 2p + 2, \quad (9)$$

and yields

$$\hat{\gamma}(z) = \frac{\gamma}{\pi} \left( \frac{z}{\omega_c} \right)^{s-1} B\left(\frac{s}{2}, p+1 - \frac{s}{2}\right) \times {}_2F_1\left[p, \frac{s}{2}; p+1; 1 - \left(\frac{z}{\omega_c}\right)^2\right]. \quad (10)$$

Here,  ${}_2F_1$  denotes the hypergeometric function and the beta function can be expressed in terms of gamma functions as  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ .

In the sequel we use units where  $\omega_c = k_B = \hbar = 1$ . Furthermore, we will occasionally restrict the range (9) of exponents  $s$  to

$$0 < s < 2p. \quad (11)$$

This tighter constraint offers the advantage that in the integral (7) the last factor can be replaced by  $1/z$  if  $z$  is much bigger than the high-frequency cutoff.

For integer  $p$ , the  ${}_2F_1$  function can be transformed into a hypergeometric series which terminates. We then have

$$\hat{\gamma}(z) = \frac{\gamma}{(1-z^2)^p} \left[ \frac{z^{s-1}}{\sin(\frac{\pi s}{2})} + \frac{1}{\pi} \sum_{n=0}^{p-1} \frac{(-1)^{n-1} B(\frac{s}{2}, p+1 - \frac{s}{2})}{n+1 - \frac{s}{2} B(n+1, p-n)} z^{2n+1} \right]. \quad (12)$$

This form allows to easily read off the behavior of  $\hat{\gamma}(z)$  for small and large arguments which will be needed below to determine the specific heat at low and high temperatures.

The low-temperature properties of the free damped particle, which is the main focus of this paper, are determined by the low-frequency characteristics of  $\hat{\gamma}(z)$ . The leading terms in the hypergeometric series in eq. (12) yield

$$\hat{\gamma}(z) = \frac{\gamma}{\sin(\pi \frac{s}{2})} z^{s-1} + \frac{\Delta M}{M} z + \frac{2p+2-s}{4-s} \frac{\Delta M}{M} z^3 + \mathcal{O}(z^5) \quad (13)$$

with

$$\frac{\Delta M}{M} = \frac{2\gamma p B(\frac{s}{2}, p+1-\frac{s}{2})}{\pi (s-2)}. \quad (14)$$

The first term in eq. (13) is responsible for frequency-dependent damping. The second term adds to the kinetic term  $z$  in  $\hat{\mathcal{R}}(z)$ . Its prefactor can therefore be interpreted as an effective change  $\Delta M$  of the particle's mass  $M$  due to the coupling to the environment. This mass renormalization will be discussed in more detail below. Finally, the third term only becomes relevant for low temperatures in the regime  $0 < s < 2$  at the particular point  $\Delta M/M = -1$ .

In the limit  $s \rightarrow 2$ , both the first term and the second term in eq. (13) become singular. The singularities cancel each other, however, and a logarithmic term accrues,

$$\hat{\gamma}(z) = -\frac{\gamma}{\pi} z [2\ln(z) + \psi(p) - \psi(1)]. \quad (15)$$

The relative mass renormalization  $\Delta M/M$  is depicted in fig. 1 as a function of the exponent  $s$  for a cutoff function characterized by  $p = 2$ . For  $s > 2$ , the dressed mass  $M + \Delta M$  is larger than the particle's bare mass, since  $\Delta M$  is always positive. From (5) and (7) one finds that

$$\Delta M = M\hat{\gamma}'(0) = M_{\text{bath}}, \quad (16)$$

where the prime denotes the derivative with respect to the argument. For  $s > 2$ , we can therefore view the free damped particle as a free particle carrying the bath oscillators as an extra load around with it. This fact is known from the long-time behavior of the position autocorrelation function which turns out to be ballistic [16]. It will also play a role for the specific heat at low temperatures.

For  $s \leq 2$ , the mass (5) diverges and the interpretation just given ceases to hold. However, one can easily convince oneself, *e.g.* by means of (10) or (12), that for  $0 < s < 2$  the leading term in (13) can be obtained as  $\hat{\gamma}(z, p = 0)$ . The mass renormalization then becomes

$$\frac{\Delta M}{M} = \lim_{z \rightarrow 0} \frac{d}{dz} (\hat{\gamma}(z) - \hat{\gamma}(z, p = 0)) \quad (17)$$

The interpretation in terms of the mass of bath oscillators again follows from (5) and (7). For  $0 < s < 2$ , the term

$$\Delta M = M_{\text{bath}}(p) - M_{\text{bath}}(p = 0) \quad (18)$$

represents the total mass of oscillators which is missing in the actual bath relative to the reference bath without spectral cutoff. Hence  $\Delta M$  is negative in this range as can also be seen from fig. 1. Indeed, (18) describes the analytical continuation of (14) to the regime  $0 < s < 2$ . It

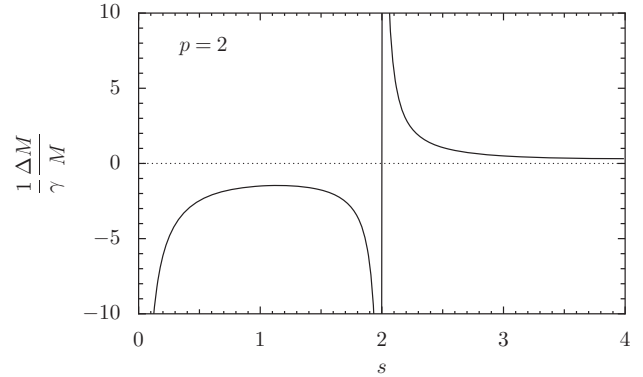


Fig. 1: Mass renormalization (14) for  $p = 2$  as a function of the exponent  $s$  of the spectral density of bath oscillators.

generalizes a result obtained earlier for the special case of ohmic damping [7].

In the regime  $0 < s < 2$ , the contribution  $(\Delta M/M)z$  to  $\hat{\gamma}(z)$  is usually subleading and therefore not of particular interest. For this reason, the fact that the renormalized mass  $M + \Delta M$  is smaller than the bare mass  $M$  largely escaped notice. Most interestingly, the thermodynamic low-energy properties of the present system are crucially influenced just by the second term in (13). More specifically, when  $\gamma$  exceeds a critical value so that  $-\Delta M > M$ , the renormalized mass  $M + \Delta M$  reaches a negative value which manifests itself as anomalous thermodynamic behavior, as we shall see.

**Reduced partition function and specific heat.** – Thermodynamic quantities of the open system are suitably calculated from the reduced partition function. The partition function of a free particle is only defined if the particle is confined to a finite region. For a one-dimensional infinite square well of width  $L$  and inner potential  $V_0 = 0$ , the partition function is given by  $\mathcal{Z}_0 = \sum_{n=1}^{\infty} e^{-(E_g/T)n^2}$  with the ground-state energy  $E_g = \pi^2/2ML^2$ . In the temperature regime

$$T > E_0 = cE_g, \quad (19)$$

where  $c$  is a positive number sufficiently large so that for  $T$  above  $E_0$  the discreteness of the energy eigenstates may be disregarded, the sum in the partition function can be turned into an integral. We thus arrive at the classical partition function of the undamped free particle  $\mathcal{Z}_{0,\text{cl}} = (\pi T/4E_g)^{1/2}$ .

For a damped particle, the partition function is augmented by quantum fluctuations due to the bath coupling. The accessory part may be written as an infinite Matsubara product [13] which, under the condition (19), does not depend on the width  $L$  of the square well. The resulting reduced partition function reads

$$\mathcal{Z} = \sqrt{\frac{\pi T}{4 E_g}} \prod_{n=1}^{\infty} \frac{\nu_n}{\nu_n + \hat{\gamma}(\nu_n)}, \quad T > E_0, \quad (20)$$

in which  $\nu_n = 2\pi Tn$  are the bosonic Matsubara frequencies. The subsequent thermodynamic analysis is based on the expression (20).

The specific heat follows from the reduced partition function by

$$C = \frac{\partial}{\partial T} \left( T^2 \frac{\partial \ln(\mathcal{Z})}{\partial T} \right). \quad (21)$$

Based on the representation (20), one then finds

$$C = \frac{1}{2} + \sum_{n=1}^{\infty} \left[ \frac{[\hat{\gamma}(\nu_n) - \nu_n \hat{\gamma}'(\nu_n)]^2}{[\nu_n + \hat{\gamma}(\nu_n)]^2} - \frac{\nu_n^2 \hat{\gamma}''(\nu_n)}{\nu_n + \hat{\gamma}(\nu_n)} \right]. \quad (22)$$

Here, the prime denotes again the derivative.

A more physical interpretation of (22) can be given in terms of the change  $\xi(\omega)$  of the oscillator density caused by the coupling of the system to the heat bath [7]. Under the condition (11) the Matsubara sum (22) can in fact be rewritten as a frequency integral

$$C = \frac{1}{2} + \int_0^{\infty} d\omega \xi(\omega) \left[ \left( \frac{\omega}{2T \sinh(\omega/2T)} \right)^2 - 1 \right]. \quad (23)$$

The first term in the square brackets is the specific heat of a harmonic oscillator with eigenfrequency  $\omega$ . The change of the oscillator density can be given in terms of (6) as

$$\xi(\omega) = \frac{1}{\pi} \text{Im} \frac{\partial \ln[\hat{\mathcal{R}}(-i\omega)]}{\partial \omega}, \quad (24)$$

where Im denotes the imaginary part. The fact that this quantity can take on negative values is at the origin of the peculiar thermodynamic behavior to be discussed below.

The integral in (23) vanishes for very high temperatures so that the specific heat reaches its classical value in this limit. This term thus represents a pure quantum contribution. On the other hand, at zero temperature the integral takes the value

$$\int_0^{\infty} d\omega \xi(\omega) [-1] = \frac{s-2}{2} \Theta(2-s) \quad (25)$$

which can also be obtained by taking one half of the zero-frequency limit of the summand in the Matsubara sum (22). As a consequence, the specific heat tends towards  $(s-1)/2$  for  $s < 2$ , while it reaches  $1/2$  for  $s \geq 2$ .

Before considering the regimes of low- and high- $T$  where analytical expressions for the specific heat are available, we discuss the main features of the temperature dependence of the specific heat obtained from either (22) or (23). Figure 2 displays the specific heat as a function of temperature for different values of the exponent  $s$  with fixed damping strengths  $\gamma/\omega_c = 3$  (solid lines) and  $0.3$  (dashed lines) and a cutoff function characterized by  $p = 2$ .

We first discuss the black curve in fig. 2(b) corresponding to the ohmic case  $s = 1$  which depicts a behavior known from the literature [6]. Decreasing the temperature starting from the high-temperature regime, quantum effects due to the bath coupling lead to a decrease in the

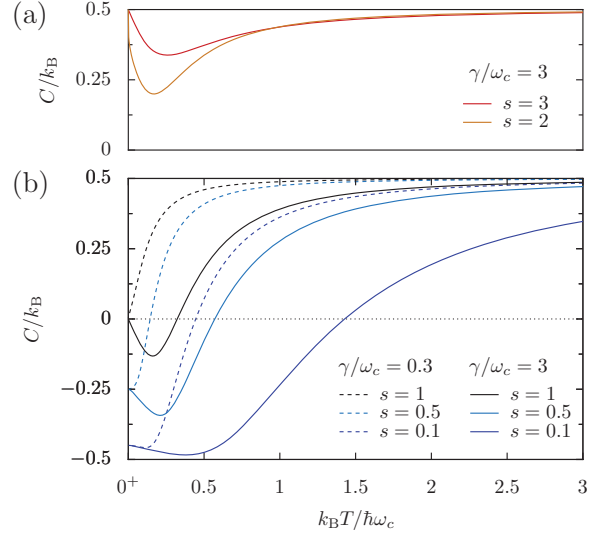


Fig. 2: (Colour on-line) The specific heat (22) of a free damped particle is displayed as a function of temperature for spectral densities of the form (8) with a cutoff characterized by  $p = 2$ . The label  $0^+$  on the temperature axis indicates that the curves are only valid for  $k_B T > E_0$ . (a) For  $s = 2$  and  $3$ , the phenomenon of reentrant classicality is shown for damping strength  $\gamma/\omega_c = 3$ . (b) The appearance of a thermodynamic anomaly in the specific heat at low temperatures is shown for  $s = 0.1, 0.5$ , and  $1$  for  $\gamma/\omega_c = 0.3$  (dashed curves) and  $\gamma/\omega_c = 3$  (solid curves). For  $s = 0.5$  and  $1$ , the dashed curves correspond to a situation where  $M + \Delta M > 0$  so that no thermodynamic anomaly is present (cf. eq. (29)). At very low temperatures, the specific heat approaches values between  $-1/2$  and  $1/2$ .

specific heat. It is a particularity of the ohmic case that at low temperatures the specific heat tends towards zero. In contrast to the dashed line the full line corresponds to sufficiently strong damping to allow for a negative specific heat at low temperatures. This behavior can be traced back to a suppression of the density of bath oscillators at low frequencies due to the coupling to the free particle [7].

Let us now turn to non-ohmic damping. With the caveat that zero temperature actually means  $T > E_0$  as indicated by the label  $0^+$ , we see from fig. 2(b) that for  $0 < s < 2$  the specific heat tends to  $(s-1)/2$  in that limit. For sub-ohmic damping,  $s < 1$ , we thus find negative values for the specific heat down to the lowest temperatures where the form (20) is still valid.

On the other hand, for  $s \geq 2$  depicted in fig. 2(a), the specific heat approaches its classical value  $k_B/2$  for very low temperatures. We thus find the surprising phenomenon of reentrant classicality. For decreasing temperatures, quantum effects due to the environmental coupling set in and tend to decrease the specific heat. At very low temperatures, only the low-frequency oscillators of the environment are relevant but these are suppressed for  $s \geq 2$ . As a consequence the environmental influence and its associated quantum effects become less pronounced. Ultimately, the classical specific heat is restored. This

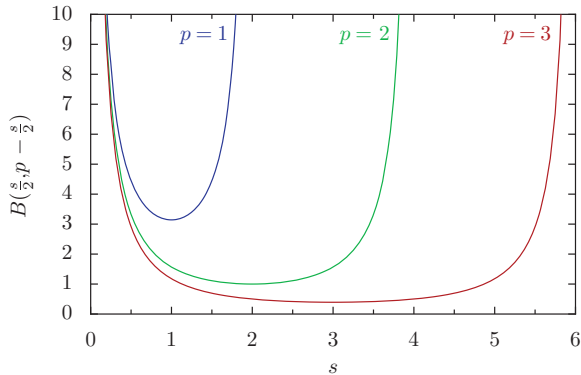


Fig. 3: (Colour on-line) Beta function  $B(\frac{s}{2}, p - \frac{s}{2})$  appearing in the large-argument expansion (26) of the Laplace transform of the damping kernel as a function of the exponent  $s$  for cutoffs characterized by  $p = 1, 2$ , and  $3$ .

reentrant behavior is consistent with the fact mentioned above that for  $s > 2$  the damped particle behaves in many respects like an undamped free particle with a renormalized mass. As we had mentioned in the introduction, a free particle does not possess an intrinsic energy scale and thus has to behave classically. It should be noted though, that care has to be taken when measuring the specific heat because the damped free particle is not ergodic for  $s \geq 2$  [17,18].

**Quantum corrections at high temperatures.** – The specific heat at temperatures  $T \gg \gamma$ ,  $\omega_c$  is determined by the behavior of the Laplace transform  $\hat{\gamma}(z)$  of the damping kernel at large arguments. Provided the exponent  $s$  satisfies (11) one finds from (7) and (8)

$$\hat{\gamma}(z) = \frac{\gamma}{\pi} \frac{B(\frac{s}{2}, p - \frac{s}{2})}{z} \text{ for } z \gg 1. \quad (26)$$

The  $1/z$  dependence in (26) implies that for all exponents  $s$  in the range (11), the leading quantum correction at high temperatures goes like the square of the inverse temperature

$$C = \frac{1}{2} - \frac{B(\frac{s}{2}, p - \frac{s}{2})}{12\pi} \frac{\gamma}{T^2}. \quad (27)$$

The universal  $1/T^2$  tail is proportional to the damping strength  $\gamma$  and, after reinserting the constants previously set to one, the cutoff frequency  $\omega_c$ . The parameters  $s$  and  $p$  characterizing the form of the spectral density of bath oscillators enter via a beta function. Its U-shaped form displayed in fig. 3 possesses a minimum at  $s = p$  and divergencies at the edges  $s = 0$  and  $s = 2p$ . For a given cutoff function characterized by  $p$ , quantum effects can depend significantly on the value of the exponent  $s$ . For example, in fig. 2(b) quantum effects at a fixed temperature are significantly larger for  $s = 0.1$  as compared to the other values of  $s$  for which the specific heat is shown. The quantum corrections for  $p = 2$  are weakest for  $s = 2$ .

When, on the other hand,  $s$  is kept fixed and the parameter  $p$  is increased, the beta function gets smaller. Thus,

sharpening the cutoff function in the spectral density of bath oscillators (4) reduces the quantum effects while the temperature is kept fixed at a large value.

**Low-temperature behavior.** – We now study the low-temperature regime  $E_0 < T \ll 1, \gamma$ . For  $L$  sufficiently large, this domain almost extends down to zero temperature. The leading low-temperature correction to the specific heat is obtained from the leading low-frequency term in the change of the oscillator density  $\xi(\omega)$  defined in (24). The latter, in turn, is obtained from the leading small-argument terms of  $\hat{\gamma}(z)$  according to (13) and (15).

In the regime  $s < 2$ , the leading contribution to the density change is found as  $\xi(\omega) = a(s, p)\omega^{1-s}$  with

$$a(s, p) = \frac{(2-s) \sin(\pi \frac{s}{2})^2}{\pi \gamma} \frac{M + \Delta M}{M}. \quad (28)$$

This coefficient increases with decreasing damping strength  $\gamma$ . Consequently, a reduction of the environmental coupling leads to a more rapid approach to the classical regime as temperature is increased. We thus recover the scenario discussed in the introduction according to which damping renders a free particle more quantum mechanical.

From (23) and the sum rule (25), we obtain the leading behavior of the specific heat as

$$C = \frac{1}{2}(s-1) + a(s, p)\Gamma(4-s)\zeta(3-s)T^{2-s}, \quad (29)$$

where  $\zeta$  denotes the Riemann zeta function. The temperature-independent contribution  $\frac{1}{2}(s-1)$  runs from the negative value  $-\frac{1}{2}$  reached in the extreme sub-ohmic limit  $s \rightarrow 0$  to the classical value  $C_{\text{cl}} = \frac{1}{2}$  as  $s$  is raised to the super-ohmic value  $s = 2$ .

Interestingly, only for an ohmic bath,  $s = 1$ , the specific heat expression (29) is zero at zero temperature, and hence in agreement with the Third Law, regardless of whether  $L$  is finite or infinite. For  $s \neq 1$ , the specific heat reaches the non-zero plateau value  $\frac{1}{2}(s-1)$  as  $T$  approaches the lower bound  $E_0$ . The drop to zero,  $C(T \rightarrow 0) \rightarrow 0$ , and hence accordance with the Third Law, takes place only in the ultra-narrow regime  $0 \leq T < E_0$ , in which the discreteness of the energy spectrum is relevant.

The prefactor of the leading power of  $T$  shows a striking anomaly. It changes sign when the dressed mass  $M + \Delta M$  goes through zero, *i.e.* when the damping strength goes through a critical value determined by (14). Hence the decrease of the specific heat for small temperatures increasing from zero is a sign that the dressed mass is negative. The change from a negative slope at the origin in fig. 2(b) (solid lines) to a positive slope in fig. 2(b) (dashed lines) is clearly visible for the cases  $s = 0.5$  and  $1$ .

At the critical point  $M + \Delta M = 0$ , the leading thermal dependence is ruled by the first and the third term in eq. (13). The specific heat then reads

$$C = \frac{s-1}{2} + \frac{(2+2p-s) \sin(\pi \frac{s}{2})^2}{\pi \gamma} \Gamma(6-s)\zeta(5-s)T^{4-s}. \quad (30)$$

Note that the thermal contribution is consistently positive.

For the special case  $s = 2$ , in which the expression (15) for  $\hat{\gamma}(z)$  applies, the fictitious oscillator density takes the form  $\xi(\omega) = -1/[2\omega \ln^2(\omega)]$ , which gives rise to the logarithmic thermal behavior

$$C = \frac{1}{2} + \frac{1}{2 \ln(T)}. \quad (31)$$

Finally, in the regime  $s > 2$ , the second term in the expression (13) is the leading one, yielding for the density change  $\xi(\omega) = -b(s, p)\omega^{s-3}$  with

$$b(s, p) = \frac{\gamma}{\pi}(s-2) \frac{M}{M + \Delta M}. \quad (32)$$

In contrast to the coefficient (28) which was proportional to  $1/\gamma$ , the coefficient  $b(s, p)$  is proportional to the damping strength  $\gamma$ . Nevertheless, since the latter coefficient describes the quantum corrections to the classical value of the specific heat, the physical picture is unchanged: Stronger damping renders the free particle more quantum mechanical.

The leading contributions to the specific heat for  $s > 2$  are now obtained as

$$C = \frac{1}{2} - b(s, p)\Gamma(s)\zeta(s-1)T^{s-2}. \quad (33)$$

The leading thermal contribution in this regime of exponents  $s$  is always negative. Again, accordance with the Third Law in the limit  $T \rightarrow 0$  is achieved only when the discreteness of the energy spectrum is taken into account.

**Conclusions.** – In this letter, we presented a study of thermodynamic properties of a damped free quantum particle confined to a box for a general spectral density (8) of the system-bath coupling. It was found that the specific heat exhibits surprising features.

In the super-ohmic regime  $s > 2$ , the specific heat takes the classical value both at high and at very low temperatures, whereas quantum effects entail reduction in between. Therefore, reentrant classicality can be observed at low temperatures. This phenomenon arises because of the suppressed bath density at low frequencies and the ensuing inability of the bath to drive the free-particle quantum at low temperatures. The appearance of reentrant classicality is therefore expected to only depend on these general properties and not on the precise details of the model (3). Reentrant classicality can be shown to occur also for a particle confined by a harmonic potential if the oscillator frequency is much smaller than the damping strength and the cutoff frequency [19]. Then,  $E_0$  is determined by the oscillator ground-state energy. With an appropriate choice of system parameters, reentrant classicality should thus be amenable to experimental observation.

For  $s \neq 1$ , *i.e.* with the exception of the ohmic case, the Third Law is seemingly violated, if we disregard the finite width of the box. The temperature regime, in which the specific heat eventually goes to zero, becomes arbitrarily

narrow, when the width of the box is chosen correspondingly large. In the range  $0 < s < 2$  mass renormalization usually plays an ancillary role. However, in the present model, in which an intrinsic frequency scale is absent, mass renormalization dictates whether the specific heat at low temperatures decreases or increases with temperature. The threshold to anomalous behavior is associated with a vanishing renormalized mass. Even though several aspects of the low-temperature behavior reappear for a damped harmonic oscillator of very small frequency [19], the possibility that the specific heat can be negative remains a peculiarity of the free damped particle even for non-ohmic damping.

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The authors would like to thank PETER HÄNGGI and PETER TALKNER for stimulating discussions. One of us (UW) has received financial support from the Deutsche Forschungsgemeinschaft through SFB/TRR 21.

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