Public Consumption, Optimal Capital Accumulation and the Social Rate of Time Preference

by

ALFRED MAUSSNER*

1. Introduction

In the Western industrialized countries, the nineteen seventies witnessed both a decline in productivity growth and an increase in public expenditures as a proportion of GNP. In a report published in the spring of 1985, the German Council of Economic Advisers at the Economics Ministry attributed this phenomenon to an increase in the social rate of time preference. Yet while the inverse relation between per capita income (growth) and the rate of time preference is well established in neoclassical growth theory, there is as yet no theorem relating public expenditures to time preference.

This paper is an attempt to close this gap. It presents a two sector growth model of a mixed economy. An important property of this model is the indeterminacy of an optimal path towards a well-determined stationary equilibrium. On a more general level, this can be regarded as a stylized fact of democratic societies. In such societies, (almost) unanimous consent with respect to constitutional objectives can be observed. But at the same time, political debates demonstrate that no commonly agreed-upon ways for realizing these very objectives exist.

Within the analytical framework of this model, sufficient conditions are derived for a positive relation between the social rate of time preference and an appropriate measure of government activity.

The approach adopted in the model is to consider the private and the public sector of a mixed economy as production processes of private and public goods, respectively. The inputs into both processes are labor and

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1 Cf. Wissenschaftlicher Beirat beim Bundesministerium für Wirtschaft [1985], Tz.9.

2 Public goods are distinguished from private goods by the well known criterion of joint consumability. See e.g. Blümel, Pethig and von dem Hagen [1986], p. 245.
capital. The crucial supposition in the model is that the production of public goods is more labor intensive than that of private goods. This can be justified on two grounds. First, as is well known, the physical production of pure public consumption goods, such as social security, requires relatively little capital. Second, and what is more relevant for the succeeding argument, the public (or political) process of choice which results in the physical production of public goods necessitates above all personal engagement; voting, bargaining, lobbying or participating in parliamentary decision processes are obvious examples.

Of course, this highly abstract point of view neglects most of the differences between political and private or market allocation which actually exist. Specifically, by focusing on a representative individual I eliminate the problem of aggregating preferences and the free rider problem. One justification for this device may be that, in the model presented below, market failures are excluded, as is usually the case; thus political and market allocation are modelled under likewise ideal conditions. But on the other hand this procedure draws attention to a simple but fundamental mechanism relating public consumption to the social rate of time preference: in as much as time preference increases, the relative price of capital rises and thus reduces the relative production costs of that type of goods being produced with relatively little capital. Consequently, public consumption becomes more favorable.

Generally speaking, if more emphasis is put on present than on future welfare, political allocation appears to be more attractive than market allocation.

The rest of the paper is organized in the following way. In the next section the formal model of a mixed economy is developed. The economy’s time path of optimal capital accumulation is analyzed in section three. Section four covers the comparative-statics of the model with respect to time preference. The paper concludes with some conclusions and hints as to possible extensions of the model.

2. The Model

Consider an economy with $N$ identical agents and two sectors of production. Variables referring to the private sector are distinguished from those referring to the public sector by superscripts $p$ and $g$, respectively. Since it is the choice of a representative agent that is studied, all quantities must be regarded as per capita magnitudes. All variables except total labor supply are time-dependent. For convenience, this is not explicitly noted.

Each sector produces one single good. An agent’s share $y^j$ of sector’s $j, j = p, g$, production is a function of the amount of labor $n^j$ and capital $k^j$ devoted to this sector. Let

$$y^p = f(n^p, k^p)$$
$$y^g = h(n^g, k^g)$$
be these functions, each being homogeneous of degree one, twice continuously differentiable, satisfying

\[ f_1, f_2 > 0; \quad f_{11}, f_{22} < 0; \quad f_{12} = f_{21} > 0; \]
\[ h_1, h_2 > 0; \quad h_{11}, h_{22} < 0; \quad h_{12} = h_{21} > 0. \]

Private goods can be used either for consumption or capital formation. Public goods are pure consumption goods. Therefore the yields of private production are the sole source for financing investment. Hence public investment may also be regarded as a tax levied on the private sector. Investment of both sectors consists of reinvestment (a fixed proportion \(\delta^i\) of the existing capital stock) and of expansion investment. Thus,

\[ i^p = k^p + \delta^p k^p \]
\[ i^g = k^g + \delta^g k^g \]

define private and public investment, respectively. As a consequence, private consumption is restricted to

\[ c^p \leq y^p - i^p - i^g \]

whereas public consumption obeys

\[ N c^g \leq N y^g. \]

This last equation results from the fact that, once it has been produced, a unit of a public good can be consumed jointly by all members of society.

The agent supplies one unit of labor inelastically, so that his labor input must satisfy

\[ n^p + n^g \leq 1. \]

His aim is to maximize the discounted utility of his consumption stream over the time interval \([0, \infty)\). Utility at time \(t \in [0, \infty)\) depends on the consumption of both types of goods according to

\[ u = u(c^p, N c^g). \]

Assume that the instantaneous utility function \(u\) is strictly concave, twice continuously differentiable, satisfying

\[ u_1, u_2 > 0; \quad u_{11}, u_{22} < 0; \quad u_{12} = u_{21} > 0. \]

\[ ^3\] Partial derivatives are denoted by indices and time derivatives by a dot.
Finally, denote by $q > 0$ the social rate of time preference used in discounting instantaneous utility.

Now the stage is set to state the agent's intertemporal choice problem. It reads

$$\max_{(c^p, c^s, n^p, n^s, i^p, i^s)} \int_0^\infty u(c^p, N c^g) e^{-\delta t} dt$$

s.t. $k^p = i^p - \delta^p k^p$

$$k^g = i^g - \delta^g k^g$$

$$0 \leq f(n^p, k^p) - c^p - i^p - i^g$$

$$0 \leq h(n^g, k^g) - c^g$$

$$1 \geq n^p + n^g$$

$$k^p(0) = k_0^p$$

$$k^g(0) = k_0^g$$

(P)

where the non-negativity conditions are not explicitly stated.

The solution of this optimal control problem will be studied in the next section.

3. Optimal Capital Accumulation

Necessary and sufficient conditions for an interior solution to problem (1) are given by the following set of equations:

(1a) $f_1(n^p, k^p) / h_1(n^g, k^g) = N \frac{u_2(c^p, N c^g)}{u_1(c^p, N c^g)}$

(1b) $0 = f(n^p, k^p) - c^p - i^p - i^g$

(1c) $0 = h(n^g, k^g) - c^g$

(1d) $1 = n^p + n^g$

(1e) $\psi^p = u_1(c^p, N c^g)$

(1f) $\psi^p = \psi^g$

(2a) $k^p = i^p - \delta^p k^p$

(2b) $k^g = i^g - \delta^g k^g$

(2c) $\psi^p = (q + \delta^p - f_2(n^p, k^p)) \psi^p$

(2d) $\psi^g = \left(q + \delta^g - N \frac{u_2(c^p, N c^g)}{u_1(c^p, N c^g)} h_2(n^g, k^g)\right) \psi^g$

(3a) $\lim_{t \to \infty} e^{-\delta t} \psi^j(t) \geq 0$

(3b) $\lim_{t \to \infty} e^{-\delta t} \psi^j(t) k^j(t) = 0$

where $j = p, g$.

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4 These conditions are derived using the by now familiar methods of optimal control theory. See e.g. Arrow and Kurz [1971], Proposition 7 and 8 on pages 48–49.
Conditions (1a)–(1f) are designed to determine the optimal values of the control variables $c^p, c^g, n^p, n^g, i^p$ and $i^g$ for given values of the state variables $k^p$ and $k^g$ and the costate variables $\psi^p$ and $\psi^g$. These variables change between two adjacent points of time according to the equations of motion (2a)–(2d). The transversality conditions are given by (3a) and (3b).

$u_2/u_1$ is the agent’s marginal rate of substitution between the consumption of private and public goods. $f_1/h_1$ is the marginal rate at which public goods can be transformed into private goods via a reallocation of labor. At each moment of time, capital inputs are predetermined by previous investment decisions. Hence condition (1a) states that the marginal rate of transformation equals the sum of the individuals’ marginal rates of substitution. This is the famous Samuelson condition for a Pareto efficient allocation in the presence of public goods.\(^5\) Equations (1b)–(1d) are market-clearing conditions. Condition (1e) relates the marginal utility of private consumption to the shadow price of private capital. According to (1f), private and public capital must have the same shadow price for all $t \in [0, \infty)$.

For a further analysis of system (1) it is rewarding to consider next a stationary equilibrium, i.e. an equilibrium satisfying $\dot{\psi}^j = \dot{k}^j = 0 \ \forall j = p, g$. From (2a)–(2d) together with (1a)–(1d) the following set of equations determines the stationary values of $c^p, c^g, n^p, n^g, k^p$ and $k^g$:

\begin{align*}
    (4a) & \quad f_1(n^p, k^p) = N \frac{u_2(c^p, Nc^g)}{u_1(c^p, Nc^g)} \\
    (4b) & \quad \varrho + \delta^p = f_2(n^p, k^p) \\
    (4c) & \quad \varrho + \delta^g = N \frac{u_2(c^p, Nc^g)}{u_1(c^p, Nc^g)} h_2(n^g, k^g) \\
    (4d) & \quad 0 = f(n^p, k^p) - c^p - \delta^p k^p - \delta^g k^g \\
    (4e) & \quad 0 = h(n^g, k^g) - c^g \\
    (4f) & \quad 1 = n^p + n^g .
\end{align*}

Condition (4b) and (4c) concern capital accumulation. (4b) states that private capital must be built up until its marginal product equals its user costs, i.e. $\varrho + \delta^p$. Since $Nu_2/u_1$ is the price of public goods in terms of private goods, equation (4c) is the same condition with respect to public capital.

The question to be studied now is whether there is a unique path starting at $(k^p_0, k^g_0)$, that converges to the allocation implied by (4). A look at (1a)–(1f) immediately reveals that this is not the case. The five equations (1a)–(1e) are not sufficient to determine the six control variables for given values of $k^p$, $k^g$ and $\psi^p$ (or $\psi^g$). This result is the direct outcome of the fact that both private and public capital accumulation are financed from the yields of

\(^5\) Cf. Samuelson [1954].
private production. On an optimal path, public investment contributes at the margin as much to individual welfare as does private investment. Consequently, the representative agent is indifferent as between these two types of investment.6

The following manipulations yield a more convenient statement of this result. Differentiate the identity (1f) with respect to time. This yields \( \dot{\psi}^p = \dot{\psi}^q \). Insert from (2c) and (2d) into this equation to obtain

\[
(5) \quad f_2(n^p, k^p) - \delta^p = N \frac{u_2(c^p, N^q)}{u_1(c^p, N^q)} h_2(n^q, k^q) - \delta^q.
\]

This last equation together with the equations (1a)–(1d), (2a), (2b) and (3) summarizes the conditions for an optimal path. To close the model, assume that capital in one sector is accumulated according to

\[
(6) \quad k^j = \kappa(k^{*j} - k^j), \quad \kappa > 0, \quad j = p \text{ or } g,
\]

\( k^{*j} \) being the stationary value of \( k^j \) obtained from (4). From the two possible systems (7) and (8),

\[
\begin{align*}
(7a) & \quad \dot{k}^p = \kappa(k^{*p} - k^p) \\
(7b) & \quad \dot{k}^q = \dot{p} - \delta^q k^q \\
(8a) & \quad \dot{k}^q = \dot{p} - \delta^p k^p \\
(8b) & \quad \dot{k}^q = \kappa(k^{*q} - k^q)
\end{align*}
\]

only (7) is unambiguously locally stable.7 Thus, the transversality conditions discriminate between (7) and (8) in favor of (7).

For each given \( \kappa \) and each pair \((k^p_0, k^g_0)\) in a neighborhood of \((k^{*p}, k^{*g})\), equations (7) together with the suitable modified equations (1a)–(1d) and (5) define a unique convergent path. Since the model does not determine \( \kappa \), there exists an infinity of optimal paths towards the stationary equilibrium.

Viewed from the perspective of social consent, the members of this fictitious society agree with respect to the final targets but they may well argue about the proper way to approach these targets. Regarded in this way the model incorporates a property which is a kind of stylized fact of Western democratic societies, namely the observed (almost) unanimous consent with respect to constitutional objectives combined with considerable debate as to the proper way of achieving them.

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6 If, instead of my assumption, public goods were used for public capital accumulation, (1f) would be replaced by \( N u_2 = \psi^g \) and the indifference would vanish.

7 The roots of the characteristic equation of (7) are \(- \kappa < 0 \) and \( \partial \delta^p / \partial k^p - \delta^p \equiv 0 \). The roots of (8) are \(- \kappa < 0 \) and \( \partial \delta^q / \partial k^q - \delta^q < 0 \). It is a straightforward exercise to prove these statements by calculating \( \partial \delta^j / \partial k^j - \delta^j, j = p, q \) from the suitably modified system of equations (1a)–(1d), (5). Since the formulas are very lengthy, they are not given here.
4. Time Preference and Public Consumption

The relation between the social rate of time preference and the steady state allocation is revealed by a straightforward exercise in comparative-statics. Differentiation of (4) with respect to \( q \) and the endogenous variables \( c_p, c_e, n_p, n_e, k_p \) and \( k_e \) yields a system of linear equations that can be solved for \( \frac{\partial z}{\partial p}, z \in \{c_p, c_e, n_p, n_e, k_p, k_e\} \). A rather tedious calculation yields the following main results:

\[
\begin{align*}
(9a) \quad \frac{\partial (k_p/n_p)}{\partial q} &= (n_p)^{-2} \Delta \left\{ (h_1 h_{12} - h_1 h_{22}) \left[ (\beta f_1 - \alpha h_1) f_1 n_p + q k_p \right] \\
&+ (h_1 h_{12} - h_2 h_{11}) (\beta \delta q - \alpha h_2) f_1 n_p \right\} < 0.
\end{align*}
\]

\[
(9b) \quad \frac{\partial (k_e/n_e)}{\partial q} = (n_p)^{-2} \Delta \left\{ (h_2 f_{12} - h_1 f_{11}) \beta h_1 n_p \\
+ (h_2 f_{12} - h_1 f_{22}) [\beta h_1 (f_1 n_p + \delta q k_e) - \alpha h_1 (h_1 n_p + h_2 k_e)] \right\} < 0
\]

\[
(9c) \quad \frac{\partial (N c_e/n_p)}{\partial q} = N(c_p)^{-2} \Delta \left\{ (\beta c_p + \alpha c_e) \left[ h_2^2 (2 h_2 f_{12} - f_1 h_{22} - h_1 f_{22}) \\
- h_2^2 (h_1 f_{11} + f_1 h_{11}) + 2 f_1 h_1 h_2 h_{12} \\
+ \gamma \left[ f_1^{-1} (h_2 c_p + \delta q c_e) f_{12} h_{12} \left( \frac{n_p k_e}{k_p n_e} - 1 \right) \\
+ (f_1^{-1} h_1 c_p + c_e) f_{12} h_{12} \frac{n_p}{k_p} - \frac{n_e}{k_e} + q f_1^{-1} \left( 1 - \frac{k_p n_e}{k_e} \right) c_e \right] \right\}
\]

with: \( \Delta : = N(u_1 u_{22} - u_2 u_{12}) < 0; \beta : = (u_1 u_{21} - u_2 u_{11}) > 0; \)

\[
\gamma : = u_1 f_1^2 / N > 0 \quad \text{and} \quad \Delta : = \left\{ -\beta f_1 f_{22} (h_2 h_{12} - h_1 h_{22}) + f_1 f_{22} [(h_1 h_{21} - h_2 h_{11}) (\beta \delta q - \alpha h_2)] \right\}^{-1} < 0.
\]

Consider an increase in the social rate of time preference, raising the user costs of capital. Capital intensity in both sectors of production consequently declines as indicated by (9a) and (9b). So labor productivity, being a positive function of capital intensity, is depressed. This induces an income effect, reducing the consumption of both types of goods. If the instantaneous utility function \( u \) is

\[
\frac{\partial (y_p/n_p)}{\partial q} = f_2 \frac{\partial (k_p/n_p)}{\partial q} < 0 \quad \text{and} \quad \frac{\partial (y_e/n_p)}{\partial q} = h_2 \frac{\partial (k_e/n_p)}{\partial q} < 0.
\]

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8 In deriving these formulas I have used (4a)-(4c), \((f_{11} f_{22} - f_{12}^2) = (h_1 h_{22} - h_2 h_{12}) = 0, f_{11} = -f_{22} k_p/n_p, f_{22} = -f_{12} n_p/k_p, h_{11} = -h_2 k_e/n_e, h_{12} = -h_2 n_p/k_e, h_{22} = -h_2 n_e/k_e \) (due to the linear homogeneity of \( f \) and \( h \)) for collecting terms and simplifying expressions.

9 The linear homogeneity of \( f \) and \( h \) together with (9a) and (9b) implies

\[
\frac{\partial (y_p/n_p)}{\partial q} = f_2 \frac{\partial (k_p/n_p)}{\partial q} < 0 \quad \text{and} \quad \frac{\partial (y_e/n_p)}{\partial q} = h_2 \frac{\partial (k_e/n_p)}{\partial q} < 0.
\]
homothetic, i.e. $u = \Omega[N c^p \omega(c^p/N c^p)]$, $\Omega', \omega' > 0$, $\Omega'', \omega'' < 0$, the income effect lowers private and public consumption by the same percentage rate, leaving the ratio $N c^p/c^p$ unaffected. To see this, calculate $u_i, u_{ij}, i, j = 1, 2$ from $u = \Omega[N c^p \omega(c^p/N c^p)]$ and insert the results into $(\beta c^p + \alpha c^p)$: the expression $\varrho(\beta c^p + \alpha c^p)[\cdot]$ in (9c) then vanishes. The remaining term $\gamma[\cdot]$ is a substitution effect originating in changing production costs. Since public production is less capital intensive than private production, $k^g/n^g < k^p/n^p$, increasing user costs of capital raise the relative price of private goods. Thus private consumption is partly replaced by public consumption. This raises $N c^g/c^p$, i.e. the amount of public goods consumed together with each unit of private goods, since $\Delta \gamma[\cdot]$ in (9c) is positive if $k^g/n^g < k^p/n^p$. Due to the fact that the income and substitution effects are opposite in direction, it is an open question whether an increase of $\varrho$ lowers or raises public consumption in absolute terms. If public consumption does not decline, employment in the public sector must be raised in order to compensate for the depressed labor productivity.

5. Conclusions

On the preceding pages a model of a mixed economy has been set forth, based on the representation of both the private and the public sector as production processes of private and public goods, respectively.

From the perspective of public choice theory an interesting property of this model is the indeterminancy of an optimal path of capital accumulation in the presence of a well-determined steady state allocation. This is founded in the assumption that both private and public capital accumulation are financed from the yields of private production. In democratic, market-oriented societies, individuals attain welfare gains by effort devoted to both private and public affairs. Thus, on a more general level the conclusion derived may yield one explanation of the fact that in Western industrial societies (almost) unanimous consent with respect to constitutional objectives can be achieved but not about the proper ways to achieve them.

The assumption that private production is more capital intensive than public production together with the assumption of a homothetic instantaneous utility function of the representative decision maker implies a positive relation between public consumption, measured in units of public goods per unit of private goods, and the social rate of time preference. Combined with the hypotheses that time preference is positively related to per capita income this result contributes to the already vast literature on the growth of government, for

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10 This is a special case of the well known Rybczynski-Theorem. Cf. RYBCZYNSKI [1955].

11 See e.g. BUCHANAN and TOLLISON [1972], [1984], MUELLER [1979].

12 Cf. UZAWA [1968]. In NEUMANN [1985] this hypotheses is used to explain long swings in economic development.

13 See e.g. the survey of LARKEY, STOLP and WINER [1981].
it predicts that in the course of growing per capita income, time preference increases and thus reduces private in favor of public consumption. This happens not as a result of shifting preferences between private and public consumption but because of changing relative prices. More generally speaking, the higher the social rate of time preference, the more emphasis is put on political issues than on market forces, because welfare gains from private markets are more difficult to accomplish than those arising from political decision-making when time preference and thus the user costs of capital are high.

Finally some possible extensions of the model may be hinted at. It would be desirable to include the production of capital goods in the public sector and the possible welfare losses brought about by the disincentives of a growing tax burden as well as by public regulation. Furthermore, this model should be extended in order to cover technical progress. Whether these extensions would considerably change the results obtained in this paper is a question left open for further research.

Summary

This paper provides a set of sufficient conditions for a positive relation between the social rate of time preference and a suitable measure of public consumption. The model within which this relation is established allows for infinitely many paths towards a well determined steady state allocation. Thus, from a public choice perspective, the members of the model's society may dispute the proper path but not the steady state allocation. The paper is based on the modelling of both the private and the public sector of a mixed economy as production processes of private and public goods, respectively, the former being more capital intensive than the latter. It is this assumption together with the assumption of a homothetic instantaneous utility function of the representative decision maker that accounts for the positive relation between time preference and public consumption in a steady state.

Zusammenfassung

Öffentlicher Verbrauch, optimale Kapitalakkumulation und gesellschaftliche Zeitpräferenzrate

In diesem Beitrag werden hinreichende Bedingungen dafür abgeleitet, daß ein Anstieg der gesellschaftlichen Zeitpräferenz den Staatskonsum erhöht. Das hierfür entwickelte Modell vermag zwar die stationäre Allokation festzulegen, läßt jedoch Wahlfreiheit bezüglich des Akkumulationspfades. Demgemäß mag in der beschriebenen Gesellschaft zwar Streit um die Wahl des Pfades, jedoch nicht bezüglich des gewünschten Endzustandes ausbrechen. Das Modell beruht darauf, eine duale Wirtschaftsordnung in Form zweier Produktionsprozesse für private bzw. öffentliche Güter darzustellen. Angenommen wird, die Produktion privater Güter sei kapitalintensiver als die öffentlicher Güter. Zusammen mit
der Annahme, die Nutzenfunktion des repräsentativen Individuums sei homothetisch, folgt daraus, daß der staatliche Konsum im stationären Gleichgewicht positiv mit der Zeitpräferenzrate verbunden ist.

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Dr. Alfred Maußner
Volkswirtschaftliches Institut
Universität Erlangen-Nürnberg
Lange Gasse 20
D-8500 Nürnberg 1
Bundesrepublik Deutschland