Quantum vortex dynamics in granular superconducting films

Ulrich Eckern and Albert Schmid
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, Postfach 6980, D-7500 Karlsruhe, Federal Republic of Germany
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We study the quantum (zero-temperature) dynamics of vortices in a granular film, which we model as a square lattice of superconducting grains, weakly coupled by Josephson junctions. The model takes into account the capacitance to the ground and between nearest-neighbor grains, as well as the special features related to the tunneling of quasiparticles. Starting from the general model, we derive and discuss different approximate models. We investigate the dissipation in the motion of the vortices, which results from excitation of acoustic modes and particularly, of quasiparticles.

I. INTRODUCTION

Recent experimental studies on granular superconducting films[1] and arrays of Josephson junctions[2] have stimulated renewed interest in the quantum mechanics of a system of Josephson-coupled superconducting grains.[3] While it was suggested[4] long ago that in the case in which the capacitance of a Josphson junction is very small, quantum fluctuations of the phase difference become important, an interesting additional feature has emerged only recently, namely that the capacitance may be strongly modified by virtual tunneling of quasiparticles.[5–7] In fact, based on this idea, a mean-field analysis[3] supports the experimental findings[1,2] that the low-temperature behavior of the film depends crucially on the parameter $\alpha=R_0/R_N$, where $R_0=\pi\hbar/2e^2\simeq 6.45$ kΩ is the quantum of resistance, and R_N the normal-state resistance.

Motivated by the above-mentioned experiments, we investigate in this article various aspects of the dynamics of vortices in the quantum regime, with emphasis on dissipation which is due to coupling to acoustic vibrations, and due to the creation of quasiparticles. We study the simplified model, in which the grains form a two-dimensional square lattice, and are only weakly coupled such that the magnitude of the order parameter can be assumed fixed. In this case, the phases $\{\phi_l\}$ are the relevant variables, where $l=(l_x,l_y)$, l_α being integers, labels the lattice site. Taking into account, as a matter of introduction, for a moment only the charging energy and the Josephson coupling, the quantum mechanics of the system is determined through the following Hamiltonian:

$$\hat{H}_0=\frac{1}{2}\sum_{l,l'}\hat{Q}_l(\mathcal{C}^{-1})_{l,l'}\hat{Q}_{l'}-E_J\sum_{l,\mu}\cos(\hat{\phi}_{l+\mu}-\hat{\phi}_l)\ , \qquad (1.1)$$

where E_J is the Josephson coupling energy and $\mu=(1,0)$ and (0,1). Furthermore, $\mathcal{C}^{-1}$ is the inverse of the capacitance matrix. In the following, we take into account only the capacitance to the ground (c) and between nearest neighbors (C_0); thus, the only nonvanishing elements are $\mathcal{C}_{l,l}=c+4C_0$ and $\mathcal{C}_{l,l\pm\mu}=-C_0$. [Note, however, that $(\mathcal{C}^{-1})_{l,l'}$ is of long range.]

Clearly, the eigenvectors of $\mathcal{C}$ are plane waves, $\sim\exp(i\mathbf{q}\cdot\boldsymbol{l})$, and the eigenvalues are given by

$$\mathcal{C}(\mathbf{q})=c+2C_0\sum_{\mu}[1-\cos(\mathbf{q}\cdot\boldsymbol{\mu})]\ . \qquad (1.2)$$

Note that $\mathcal{C}(\mathbf{q})\simeq c+C_0q^2$ for small $\mathbf{q}$. In addition, the charge and phase operators are conjugate variables, such that

$$(2e)^{-1}[\hat{Q}_l,\hat{\phi}_l]=-i\delta_{l,l'}\ . \qquad (1.3)$$

As a direct consequence of this relation, it follows that

$$\hbar\dot{\hat{\phi}}_l=2e\sum_{l'}(\mathcal{C}^{-1})_{l,l'}\hat{Q}_{l'}\equiv 2e\hat{V}_l \qquad (1.4)$$

corresponding to Josephson's relation.

However, a description in terms of a Hamilton operator is not convenient for the problems we wish to discuss, but rather the formulation which employs path-integral methods.[8] In particular, it has been found that quasiparticle tunneling can be included in the (Euclidean) effective action in a compact way.[5–7] Especially, note that for zero temperature and in the adiabatic limit ($\hbar\omega\ll\Delta$, see below), an important consequence of quasiparticle processes is the modification of the nearest-neighbor capacitance, such that $C_0\to C_0+C_{qp}$ in (1.1); also, for small junctions, we may have the inequality $C_0\ll C_{qp}$. Some of our results explicitly rely on this limit.

In Sec. II we describe the model which is based on the results of Refs. 5–7 for single junctions. Different approximations are discussed: The adiabatic limit of low frequencies, the continuum limit in which in addition the phase differences are assumed to be small, and the linear medium approximation, in which the junctions *not* traversed by a moving vortex are considered a linear medium. Accordingly, the remaining sections are organized as follows. In Sec. III we present the results for the continuum limit including, in particular, the notion of the mass of a vortex, its damping due to the coupling to the acoustic vibrations of the system, and the dynamic modification of the vortex-vortex interaction. In Sec. IV the linear medium approximation is investigated in detail, revealing in particular that the quantum dynamics of a single vortex for small velocities is close to the dynamics of a free particle for the parameters under consideration ($c \sim C_0 \ll C_{qp}$, $\alpha \sim 1$). On the other hand, for velocities comparable to the gap frequency, or in the presence of a finite subgap conductance,[9] the creation of quasiparticles leads to strong dissipation, which is considered in Sec. V. Finally, the results are summarized and discussed in Sec. VI. We remark that earlier works describing some aspects of vortex motion are Refs. 10–12; also, the limit $\alpha \gg 1$ has been treated in Ref. 13, which is based on similar methods as employed here.

II. DESCRIPTION OF THE MODELS

The models to be considered are based on the assumption that the magnitude of the order parameter of the single grains Δ is a fixed quantity, independent of any supercurrent flow. In addition, we assume that the grains form a two-dimensional square lattice with lattice constant $a=1$, where each lattice point is occupied by a grain, and the links between lattice points correspond to (identical) Josephson junctions. Each grain has a capacitance c to the ground, and the capacitance between neighboring grains is C, which is in general the sum of a geometrical and a contribution due to quasiparticle tunneling $C=C_0+C_{qp}$. In order to introduce the parameters of the model, we note that the Josephson coupling energy is (at $T=0$) related to the gap by the relation (I_J is the critical current of a single junction)

$$E_J=\hbar I_J/2e=\alpha\Delta/2 \ , \tag{2.1}$$

where $\alpha=R_0/R_N$, R_N is the normal-state resistance of a single junction, and R_0 the unit of resistance. In addition, we have the relation $C_{qp}=3e^2\alpha/16\Delta$ for zero temperature. Note that C_{qp} can be much larger than $C_0(\simeq c)$ for sufficiently small junctions.

The relevant variables of the model are the phases of the order parameter $\{\phi_l\}$, where l labels the lattice site. The model in its general form is given in terms of the Euclidean (imaginary time) action as follows:

$$S=S_0+S_A+S_B+S_T \ , \tag{2.2}$$

where

$$S_0=\frac{1}{2}\sum_l \int_0^{\hbar\beta} d\tau \left[m\dot{\phi}_l^2+M_0\sum_\mu(\dot{\phi}_{l+\mu}-\dot{\phi}_l)^2 + 2E_J\sum_\mu[1-\cos(\phi_{l+\mu}-\phi_l)] \right] . \tag{2.3}$$

Here, we introduced $m=\hbar^2c/4e^2$, $M_0=\hbar^2C_0/4e^2$, and $\beta=(kT)^{-1}$. The contributions to the action denoted by S_A and S_B, which are related to the tunneling of quasiparticles between neighboring grains, are given by

$$\begin{bmatrix} S_A \\ S_B \end{bmatrix} = -\sum_{l,\mu}\int_0^{\hbar\beta}d\tau\int_0^{\hbar\beta}\begin{bmatrix} A(\tau-\tau') \\ B(\tau-\tau')\end{bmatrix}\cos\left[\frac{\Delta\phi_{l,\mu}(\tau)-\sigma\Delta\phi_{l,\mu}(\tau')}{2}\right] , \tag{2.4}$$

where $\Delta\phi_{l,\mu}(\tau)=\phi_{l+\mu}(\tau)-\phi_l(\tau)$, and $\sigma=+1(-1)$ in $S_A(S_B)$. In the following, the limits of integration, which become $(-\infty,+\infty)$ in the limit of zero temperature, will be omitted for simplicity. The kernels $A(\tau), B(\tau)$, for an ideal junction[14] and for $T=0$, can be expressed in terms of modified Bessel functions as follows:

$$\begin{aligned} A(\tau)&=\frac{\alpha\Delta^2}{\pi^2\hbar}K_1^2\left[\frac{\Delta|\tau|}{\hbar}\right] , \\ B(\tau)&=\frac{\alpha\Delta^2}{\pi^2\hbar}K_0^2\left[\frac{\Delta|\tau|}{\hbar}\right]-E_J\delta(\tau) . \end{aligned} \tag{2.5}$$

In particular, the Fourier transform of $A(\tau)$ is related to the normal current $I_n(\omega)$ by the following relation:

$$A(\omega)=\int\frac{d\omega'}{2\pi}\frac{2\omega'}{\omega'^2+\omega^2}\frac{\hbar I_n(\omega')}{2e} . \tag{2.6}$$

Since for an ideal junction $I_n(\omega)=0$ for $|\hbar\omega|<2\Delta$, we have for small frequencies

$$A(\omega)-A(\omega=0)=-2M_{qp}\omega^2, \quad B(\omega)=2M_{qp}\omega^2/3 \ ,$$

where the quasiparticle mass M_{qp} is given by

$$M_{qp}=\left[\frac{\hbar}{2e}\right]^2 C_{qp}=\frac{3\alpha\hbar^2}{64\Delta} . \tag{2.7}$$

Finally, S_T is the contribution due to a uniform transport current (say, along the $\hat{y}$ direction) I_T, given by

$$S_T=-f\sum_l\int d\tau(\phi_{l+(0,1)}-\phi_l) \ , \tag{2.8}$$

where $f=\hbar I_T/2e$. In order to clarify different features of the general model, it suffices to study various approximations, which are introduced in the remainder of this section.

(i) The adiabatic limit (AL). In this limit, it is assumed that the temporal variation of the phases is slow compared to the gap frequency $\Delta/\hbar$, such that we may expand the difference in (2.4) according to

$$\Delta\phi_{l,\mu}(\tau)-\Delta\phi_{l,\mu}(\tau')\simeq\Delta\dot{\phi}_{l,\mu}(\tau-\tau')\ll 1 . \tag{2.9}$$

As a result, if we in addition also neglect S_B (see below) the action in this limit is given by S_0, however, with the replacement $M_0\to M=M_0+M_{qp}$,

$$S^{AL}=S_0(M_0\to M) . \tag{2.10}$$

(ii) The continuum limit (CL). In addition to (*i*), we assume that the spatial variation of the phases is small, such that differences can be replaced by gradients according to

$$\phi_{l+\mu}-\phi_l \simeq (\boldsymbol{\mu}\cdot\nabla)\phi_l \ll 1 \; . \tag{2.11}$$

Thus we obtain the following expression:

$$S^{\mathrm{CL}}=\tfrac{1}{2}\int d\tau \int d^2l\,[m\dot{\phi}_l^2+M(\nabla\dot{\phi}_l)^2+E_J(\nabla\phi_l)^2] \; . \tag{2.12}$$

(iii) The linear medium approximation (LM). This approximation, which is an extension of the one discussed in Ref. 13, is based on the assumption that the only nonlinear elements in the system are the junctions which the vortex under consideration is supposed to traverse in its motion. We define the phase differences relating to the nonlinear elements (see Fig. 1) as

$$\phi_n=\phi_{(n,1)}-\phi_{(n,0)} \tag{2.13}$$

and approximate the action of (2.2) by $S\rightarrow S_0^{\mathrm{LM}}+S'$, where S_0^{LM} is the action given in (2.2) but excluding the linear elements. Thus

$$S_0^{\mathrm{LM}}=\sum_n \int d\tau[\tfrac{1}{2}M_0\dot{\phi}_n^2+E_J(1-\cos\phi_n)-f\phi_n] - \sum_n \int d\tau\, d\tau' \left[A(\tau-\tau')\cos\left[\frac{\phi_n(\tau)-\phi_n(\tau')}{2}\right] + B(\tau-\tau')\cos\left[\frac{\phi_n(\tau)+\phi_n(\tau')}{2}\right]\right] \; . \tag{2.14}$$

The Josephson elements of the medium are taken into account approximately in S', which is chosen as follows:

$$S'=\tfrac{1}{2}\sum_l \int d\tau \left[m\dot{\phi}_l^2+\sum_\mu{}' [M_0(\Delta\dot{\phi}_{l,\mu})^2+\tilde{E}_J(\Delta\phi_{l,\mu})^2]\right] - \tfrac{1}{2}\sum_{l,\mu}{}' \int d\tau\, d\tau' (A-B)_{\tau-\tau'}\Delta\phi_{l,\mu}(\tau)\Delta\phi_{l,\mu}(\tau') \; . \tag{2.15}$$

Here the prime in the summation indicates that for $l_y=0$, we have to omit $\boldsymbol{\mu}=(0,1)$. Note that we introduced the parameter $\tilde{E}_J$, which will be determined below. Furthermore, the transport current can be omitted in S', as shown by the subsequent analysis, at least as long as it is constant in space and independent of time.

Following closely the method described in Ref. 13, we derive an effective action which contains only the phase differences $\{\phi_n\}$ of the nonlinear elements, with the result

$$S^{\mathrm{LM}}=S_0^{\mathrm{LM}}+S_1^{\mathrm{LM}} \; , \tag{2.16}$$

where S_0^{LM} is given in (2.14), and

$$S_1^{\mathrm{LM}}=\tfrac{1}{2}\sum_{n,n'} \int d\tau\, d\tau'\, \phi_n(\tau)H_{n-n'}(\tau-\tau')\phi_{n'}(\tau') \; . \tag{2.17}$$

The kernel $H_{n-n'}(\tau-\tau')$ is most easily given in its Fourier representation,

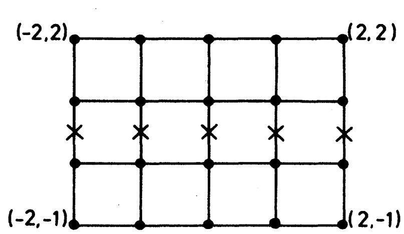

FIG. 1. Small portion of the square lattice of superconducting grains; the links are to be identified with Josephson junctions. In the linear medium approximation, the links indicated by a cross, i.e., the phase differences $\phi_n=\phi_{(n,1)}-\phi_{(n,0)}$ across these links are taken into account exactly. All other links are considered a linear medium.

$$H_n(\tau)=\beta^{-1}\sum_\omega \int_{-\pi}^{\pi}\frac{dq}{2\pi}H(q,\omega)e^{-i\omega\tau+iqn} \tag{2.18}$$

with the following result:

$$H(q,\omega)=1/G(q,\omega)-(M^\omega\omega^2+\tilde{E}_J) \; , \tag{2.19}$$

where

$$M^\omega\omega^2=M_0\omega^2-(A_\omega-A_{\omega=0}-B_\omega)/2 \; ,$$

and

$$G(q,\omega)=\int_{-\pi}^{\pi}\frac{dq'}{2\pi}\sin^2(q'/2)/\phi(q,q',\omega) \tag{2.20}$$

with

$$\phi(q,q',\omega)=\frac{m\omega^2}{4} +(M^\omega\omega^2+\tilde{E}_J)\left[\sin^2\left[\frac{q}{2}\right]+\sin^2\left[\frac{q'}{2}\right]\right] \; . \tag{2.21}$$

A considerable simplification is achieved in the limit $m=0$, in which case $(H_0\equiv H_{m=0})$ we obtain the following expression:

$$H_0(q,\omega)=h(q)(M^\omega\omega^2+\tilde{E}_J) \; , \tag{2.22}$$

$$h(q)=\sin\left[\frac{|q|}{2}\right]\left[1+\sin^2\left[\frac{q}{2}\right]\right]^{1/2}+\sin^2\left[\frac{q}{2}\right] \; . \tag{2.23}$$

Note that in the adiabatic limit $(\hbar\omega\ll\Delta)$, we have $M^\omega\simeq M_0+2M_{qp}/3$.

Finally, for finite m, we write $H=H_0+\Delta H$, where ΔH in the limit of small frequencies,

$$\omega^2 \ll \tilde{c}^2 \equiv \tilde{E}_J/m, \quad \omega^2 \ll \tilde{E}_J/M ,$$

is given by

$$\Delta H(q,\omega) = -m\tilde{c}^4(1+h_q)^2 \int_{-\pi}^{\pi} \frac{dq'}{2\pi} \frac{4\sin^2(q'/2)}{\omega^2+\tilde{\omega}_q^2} , \tag{2.24}$$

where

$$\tilde{\omega}_q^2 = 4\tilde{c}^2[\sin^2(q/2)+\sin^2(q'/2)]$$

is the spectrum of the medium, and we omitted a constant which ensures that $\Delta H(q,\omega=0)=0$.

III. THE CONTINUUM LIMIT

In this section we explore the consequences that follow from the action S^{CL} of the continuum limit as given in (2.12). Let us first note that the spectrum of small amplitude oscillations (which correspond to spin waves in a planar model) is given by

$$\omega_q^2 = (cq)^2/[1+\Omega_0^{-2}(cq)^2] , \tag{3.1}$$

where $\Omega_0^2 = E_J/M$, and $c^2 = E_J/m$. Thus ω_q is linear in q for small q (acoustic part), and it approaches Ω_0 for large q (optical part); note, however, that only the optical part survives the limit $m=0$. An interesting feature of the vortex dynamics to be discussed below is the dissipation (even at $T=0$) which results from the coupling of the vortices to the acoustic modes.

In the following we wish to construct an effective action for the vortex positions. In a first step, we investigate the classical equation of motion for the phase $\{\phi_l\}$. Since S^{CL} is a quadratic form, we find that

$$D^{-1}\phi = 0 , \tag{3.2}$$

where

$$D^{-1} = (-m+M\nabla^2)\partial_\tau^2 - E_J\nabla^2 \tag{3.3}$$

which corresponds in Fourier representation to

$$D_{\mathbf{q},\omega}^{-1} = m\omega^2 + M\omega^2 q^2 + E_J q^2 . \tag{3.4}$$

We separate from ϕ_l a contribution which incorporates explicitly the vortex configuration. Thus

$$\phi_l = \phi_l^V + \phi_l^S , \tag{3.5}$$

where ϕ^S is a unique function of space and time, and we choose ϕ^V to be the appropriate solution of the static equation of motion, $\nabla^2\phi^V=0$. This means that

$$\phi_l^V(\tau) = \sum_j e_j \arctan\left[\frac{l_y - y_j(\tau)}{l_x - x_j(\tau)}\right] , \tag{3.6}$$

where $\mathbf{r}_j(\tau) = [x_j(\tau), y_j(\tau)]$ is the time dependent center of the jth vortex. Note that ϕ^V is a multiple-valued function; it increases by $2\pi e_j$, where $e_j = \pm 1$, as one goes around the center of the jth vortex. For reasons of similarity with the Coulomb gas problem, we will call e_j the charge of a vortex; in order to avoid an infinite energy later, we will impose at once the condition $\sum_j e_j = 0$, which means charge neutrality.

Inserting the ansatz (3.5) and (3.6) in Eq. (3.2), we obtain a linear inhomogeneous equation for ϕ^S

$$D^{-1}\phi^S = (m - M\nabla^2)\partial_\tau(\partial_\tau\phi^V) . \tag{3.7}$$

Some care has to be exerted in handling the right-hand side of (3.7) since generally partial derivatives on ϕ^V cannot be interchanged. Nevertheless, one finds that in terms of Fourier transforms, the solution of Eq. (3.7) is given by

$$[\phi^S]_{\mathbf{q},\omega} = -i\omega(m+Mq^2)D_{\mathbf{q},\omega}[\partial_\tau\phi^V]_{\mathbf{q},\omega} , \tag{3.8}$$

where

$$[\partial_\tau\phi^V]_{\mathbf{q}} = \frac{2\pi i}{q^2}\sum_j e_j(q_x\dot{y}_j - q_y\dot{x}_j)e^{-i\mathbf{q}\cdot\mathbf{r}_j} . \tag{3.9}$$

Eventually, we obtain the effective vortex action $A[\mathbf{r}_j(\tau)]$ by evaluating S^{CL} for the ansatz (3.5) and (3.6)

$$A[\mathbf{r}_j(\tau)] \equiv S^{\mathrm{CL}}[\phi_l^V(\tau)+\phi_l^S(\tau)] . \tag{3.10}$$

In order to gain transparency, we will concentrate in the following on two simple cases.

Case (i). $M\neq 0; m=0$. From Eq. (3.7), it follows that

$$[\phi^S]_{\mathbf{q},\omega} = -i\omega M(M\omega^2+E_J)^{-1}[\partial_\tau\phi^V]_{\mathbf{q},\omega} .$$

Evaluating the expression (3.10) in the limit of small frequencies $M\omega^2 \ll E_J$, we obtain

$$A[\mathbf{r}_j(\tau)] = \tfrac{1}{2}\int d\tau \sum_{i,j} e_i e_j[\mathcal{M}_{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j)\dot{r}_i^{\alpha}\dot{r}_j^{\beta} + V(\mathbf{r}_i-\mathbf{r}_j)] , \tag{3.11}$$

where the potential is given by

$$V(\mathbf{r}) = E_J\int\frac{d^2q}{q^2}(e^{i\mathbf{q}\cdot\mathbf{r}}-1)e^{-q/q_c}$$
$$= -2\pi E_J\ln\{\tfrac{1}{2}[1+(1+2\pi r^2)^{1/2}]\} . \tag{3.12}$$

The additional factor in the integrand above provides a large wave-vector cutoff $q_c = (2\pi)^{1/2}$ which is required by the lattice model. Concerning the kinetic energy, we find that

$$\begin{bmatrix}\mathcal{M}_{yy} & \mathcal{M}_{xy}\\ \mathcal{M}_{yx} & \mathcal{M}_{xx}\end{bmatrix} = -\frac{M}{E_J}\begin{bmatrix}\partial_x^2 & \partial_x\partial_y\\ \partial_x\partial_y & \partial_y^2\end{bmatrix} V(\mathbf{r}) . \tag{3.13}$$

This expression decreases rapidly $\sim|\mathbf{r}|^{-2}$ with distance; thus, we retain only the term where $\mathbf{r}=0$, i.e., where $j=i$. This means that

$$A[\mathbf{r}_j(\tau)] = \tfrac{1}{2}\int d\tau\left[\overline{\mathcal{M}}\sum_i \dot{\mathbf{r}}_i^2 + \sum_{i,j} e_i e_j V(\mathbf{r}_i-\mathbf{r}_j)\right] , \tag{3.14}$$

where the expression

$$\overline{\mathcal{M}} = 2\pi^2 M \tag{3.15}$$

can also be obtained by a direct argument[15] if one recalls that the area of a Brillouin zone is equal to

$$\int d^2q \triangleq 2\pi \int_0 dq\, q \exp(-q/q_c) = 2\pi q_c^2 = (2\pi)^2 .$$

Note that the action (3.11) with the mass tensor given by Eq. (3.13) is invariant under Galilei transformations. Thus, there is no background or a medium which exerts a force on a moving vortex.

Case (ii). $m \neq 0; M = 0$. Quite generally, the evaluation of (3.10) is simplified by the fact that $[\nabla\phi^S]_q$ is perpendicular to $[\nabla\phi^V]_q$. Therefore,

$$A[\mathbf{r}_j(\tau)] = A_l[\mathbf{r}_j(\tau)] + A_n[\mathbf{r}_j(\tau)] , \tag{3.16}$$

where the local contribution is given by relation (3.14) with $\mathcal{M}=0$. On the other hand, the nonlocal contribution is of the form

$$A_n[\mathbf{r}_j(\tau)] = \frac{1}{2}\sum_{i,j} e_i e_j \int d\tau\, d\tau'\, F(\dot{\mathbf{r}}_i, \dot{\mathbf{r}}_j, \mathbf{r}_i - \mathbf{r}_j; \tau - \tau') , \tag{3.17}$$

where $\mathbf{r}_i = \mathbf{r}_i(\tau)$ and $\mathbf{r}_j = \mathbf{r}_j(\tau')$. Specifically, the integrand is given by

$$\begin{aligned} F_{ij} = {} & \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j f_x(\Delta_{ij}; \tau - \tau') \\ & + \{\dot{\mathbf{r}}_i \cdot \mathbf{\Delta}_{ij}\}\{\dot{\mathbf{r}}_j \mathbf{\Delta}_{ij}\} \frac{1}{\Delta_{ij}^2} \\ & \times [f_y(\Delta_{ij}; \tau - \tau') - f_x(\Delta_{ij}; \tau - \tau')] , \end{aligned} \tag{3.18}$$

where $\mathbf{\Delta}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Furthermore, the functions f_α can be calculated according to

$$f_x = -\frac{\partial^2}{\partial \Delta^2} G(\Delta; \tau), \quad f_y = -\frac{1}{\Delta}\frac{\partial}{\partial \Delta} G(\Delta, \tau) , \tag{3.19}$$

where G is given by

$$\begin{aligned} G(\Delta, \tau) = \pi \frac{E_J}{c} \{ & c|\tau| \ln[c|\tau| + (\Delta^2 + c^2\tau^2)^{1/2}] \\ & - (\Delta^2 + c^2\tau^2)^{1/2}\} . \end{aligned} \tag{3.20}$$

Suppressing detailed arguments, we only state that in its original form (except for an additive function of τ)

$$G(\Delta, \tau) = mE_J \int \frac{d\omega\, d^2q}{2\pi} D_{\mathbf{q},\omega} e^{-i\omega\tau} \frac{1}{q^2} (e^{i\mathbf{\Delta}\cdot\mathbf{q}} - 1) . \tag{3.21}$$

For the sake of simplicity, a large-q cutoff has been omitted.

It is possible to simplify the expression (3.17) considerably if one exploits the fact that terms in the form of a total time derivative do not contribute to the action. Since

$$\frac{d}{d\tau} = \dot{\mathbf{r}}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} + \frac{\partial}{\partial \tau}, \quad \frac{d}{d\tau'} = \dot{\mathbf{r}}_j \cdot \frac{\partial}{\partial \mathbf{r}_j} + \frac{\partial}{\partial \tau'} , \tag{3.22}$$

we have

$$\begin{aligned} & \left[\frac{d^2}{d\tau\, d\tau'} - \frac{d}{d\tau}\frac{\partial}{\partial \tau'} - \frac{d}{d\tau'}\frac{\partial}{\partial \tau} \right] G(\mathbf{r}_i - \mathbf{r}_j; \tau - \tau') \\ & \qquad = \left[\dot{\mathbf{r}}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} \right] \left[\dot{\mathbf{r}}_j \cdot \frac{\partial}{\partial \mathbf{r}_j} \right] G - \frac{\partial^2}{\partial \tau \partial \tau'} G \end{aligned} \tag{3.23}$$

and we may add the right-hand side of Eq. (3.23) to the integrand of Eq. (3.17) without changing the action. This means that we may replace F_{ij} of Eq. (3.18) by

$$\tilde{F}_{ij} = \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j (f_x + f_y) - \frac{\partial^2}{\partial \tau \partial \tau'} G . \tag{3.24}$$

Note that the second time derivative of G contains a local contribution $\propto \delta(\tau - \tau')$ which cancels completely the logarithmic interaction of $A_l[\mathbf{r}_j(\tau)]$. Thus, the action of Eq. (3.16) is found to be equal to

$$A[\mathbf{r}_i(\tau)] = \frac{1}{2}\frac{\pi E_J}{c} \sum_{i,j} e_i e_j \int d\tau\, d\tau' \frac{\dot{\mathbf{r}}_i(\tau) \cdot \dot{\mathbf{r}}_j(\tau') + c^2}{\{[\mathbf{r}_i(\tau) - \mathbf{r}_j(\tau')]^2 + c^2(\tau - \tau')^2\}^{1/2}} . \tag{3.25}$$

For a static configuration $\dot{\mathbf{r}}_i(\tau) = 0$, we may integrate with respect to the time difference $(\tau - \tau')$ and we recover $A_l[\mathbf{r}_i(\tau)]$. More interesting, however, is the fact that the full dynamic form of the action (3.25) includes a contribution which represents dissipation. In particular, we obtain from the region $|\mathbf{r}_i - \mathbf{r}_j| \ll c|\tau - \tau'|$

$$A_D[\mathbf{r}_i(\tau)] = \frac{\pi m}{2} \sum_{i,j} e_i e_j \int d\tau\, d\tau' \left[\frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j}{|\tau - \tau'|} - \frac{(\mathbf{r}_i - \mathbf{r}_j)^2}{2c^2|\tau - \tau'|^3} \right] . \tag{3.26}$$

Considering the fact that the Fourier transform (in the sense of a distribution) of $|\tau|^{-1}$ and $|\tau|^{-3}$ is equal to $2\ln|\omega|^{-1}$ and $-\omega^2 \ln|\omega|^{-1}$, respectively, we may write

$$A_D[\mathbf{r}_i(\tau)] = \frac{1}{2}\sum_{i,j} e_i e_j \int \frac{d\omega}{2\pi} |\omega| \eta(\omega) [\mathbf{r}_i]_\omega [\mathbf{r}_j]_{-\omega} , \tag{3.27}$$

where the quantity

$$\eta(\omega) = \pi m |\omega| \ln \left[\frac{1}{|\omega|} \right] \tag{3.28}$$

means a frequency-dependent friction. According to the classification of Ref. 16, the dissipation implied by the relation (3.28) is subohmic; this feature is connected with the fact that the vortices are coupled to a *two-dimensional* system of acoustic vibrations. Note that the relation (3.28) is correct in logarithmic accuracy for $|\omega|\to 0$. The response of a single vortex to a driving force is considered in detail in the next section.

We emphasize that the action (3.25) is not invariant under Galilei transformations, as it is already implied by the appearance of friction. This may be surprising since the basic form $S^{\rm LM}$ of Eq. (2.12) is even invariant under Lorentz transformations (for the case we are considering here, $M=0$); note, however, that we have imposed implicitly the condition that $\phi_l(\tau)$ should vanish for $|l|\to\infty$ and $|\tau|\to\infty$. In this connection, we wish to mention that the viscosity above may also be considered to represent a time-dependent mass $\pi m \ln(c|\tau|)$. Suppose that there is a single vortex which starts moving at $\mathbf{r}=0$ and $\tau=0$ with velocity $\mathbf{v}$. Then, a disturbance $\delta\phi_l$ will propagate such that roughly

$$\delta\dot{\phi}_l \sim \mathbf{v}\cdot(-l_y,l_x)/l^2 \quad \text{for } l^2\le c^2\tau^2 ,$$

and zero elsewhere. Hence, we obtain a kinetic energy

$$E_{\rm kin}=\frac{m}{2}\int d^2l(\delta\dot{\phi}_l)^2=\frac{\pi m}{2}v^2\ln(c|\tau|) \tag{3.29}$$

which essentially agrees with the contribution A_D as given by Eqs. (3.27) and (3.28).

Eventually, we discuss qualitatively the case where M as well as m are different from zero. As a rule, the acoustic vibrations implied by a finite value of m dominate at large distances in time and space. For instance, one finds that the long-range logarithmic interaction is perfectly canceled at large distances $|\mathbf{r}_i-\mathbf{r}_j|\gg M/m$. Furthermore, for large-time separations $|\tau-\tau'|$, the action is still given by (3.25). This means that the dissipative contribution (3.27) is not changed. Thus, the most important consequence of a finite M seems to be the effective mass $\overline{\mathcal{M}}$—see Eq. (3.15)—of a vortex. Of course, if M/m is very large, say, of the order of the sample dimension, then it seems possible to set $m=0$, and to take the action as given by Eq. (3.14).

IV. LINEAR MEDIUM APPROXIMATION (ADIABATIC LIMIT)

A. Effective action of a single vortex

In order to derive the effective action in terms of the vortex coordinate, we insert an appropriate ansatz $\phi_n(x)$ into $S^{\rm LM}(\{\phi_n\})$ of (2.16), which we consider in the adiabatic limit. For simplicity (compare also Appendix A), we consider the following variational ansatz:

$$\phi_n(x)=\pi-2\arctan\left[\frac{n-z(x)}{b}\right] , \tag{4.1}$$

where b is a variational parameter, and $z(x)$ is determined below. Within this ansatz, $\tilde{E}_J$ can be determined by minimizing the potential Eq. (A2) with respect to b, which gives $b=b(\tilde{E}_J)$ after averaging over a unit cell. Turning the argument around, we choose $\tilde{E}_J$ such that $b=\frac{1}{2}$, with the result $\tilde{E}_J\simeq 0.73E_J$, which is about 10% larger than (A5). After these preliminaries it is clear that, neglecting initially the dissipative contribution related to ΔH, the effective action is given by

$$A_0[x]=\int d\tau[\tfrac{1}{2}\mathcal{M}(z)\dot{z}^2+U(z)-2\pi f z] , \tag{4.2}$$

where

$$\mathcal{M}(z)=\sum_{k=0}^{\infty}\mathcal{M}(k)\cos(2\pi k z) \tag{4.3}$$

and

$$U(z)=\sum_{k=1}^{\infty}u_k[\cos(2\pi k z)-\cos(\pi k)] . \tag{4.4}$$

We find the following results:

$$\mathcal{M}(k=0)\simeq 1.10\times 2\pi^2 M ,$$
$$\mathcal{M}(k=1)/\mathcal{M}(k=0)\simeq 0.45 ,$$
$$u_1\simeq 0.087E_J . \tag{4.5}$$

Furthermore, higher Fourier components are found to be small

$$\mathcal{M}(k=2)/\mathcal{M}(k=1)\simeq 0.08 ,$$
$$u_2/u_1\simeq 0.06 ,$$

and can be neglected. These results differ by about 10% from the results of Appendix A, which we believe is not a significant difference. Strictly speaking, we also find that in (4.2), f should be replaced by $f[\kappa_0+\kappa_1\cos(2\pi z)+\cdots]$; however, since $\kappa_0\simeq 0.997$, and $\kappa_1\simeq 0.09$, we expect this modification to be negligible for all practical purposes. Note also that the critical current, i.e., the current at which the potential barrier vanishes, is given by $I_c\simeq 0.1 I_J$.

In order to define the vortex coordinate x, we choose $z=z(x)$ such that

$$\mathcal{M}(z)\dot{z}^2=\overline{\mathcal{M}}\dot{x}^2 \tag{4.6}$$

where $\overline{\mathcal{M}}$ is independent of x, which leads to the relation given in (A10). From the above results, we find $\overline{\mathcal{M}}\simeq 1.07\times 2\pi^2 M$, and also

$$z(x)\simeq x-0.036\sin(2\pi x)+\cdots . \tag{4.7}$$

Finally, we consider in more detail the dissipative contribution to the action, which is given by inserting (4.1) into $S_1^{\rm LM}$ of (2.17). Thus we obtain

$$A_D[x]\equiv S_1^{\rm LM}(\{\phi_n(x)\})$$
$$=\tfrac{1}{2}\int d\tau\, d\tau'\, W(z,z',\tau-\tau') , \tag{4.8}$$

where $z=z(x_\tau)$, $z'=z(x_{\tau'})$, and W is given by

$$W=\sum_g\int_{-\infty}^{\infty}\frac{dq}{2\pi}\phi_q\phi_{q-g}\Delta H(q,\tau-\tau')$$
$$\times e^{-iqz+i(q-g)z'} , \tag{4.9}$$

where

$$\phi_q=(2\pi/q)\exp(-|q|/2) ,$$

and $g=2\pi\times$ *integer* denotes a reciprocal lattice vector. Neglecting umklapp processes, which enter (4.9) through $g\neq 0$ and $z=z(x)$, and in addition considering small velocities such that $\Delta_x\equiv x_\tau - x_{\tau'} \ll \tilde{c}|\tau-\tau'|$, W is found to be given by (omitting unimportant terms)

$$W(\Delta_x,\tau-\tau')\simeq\pi(\Delta_x)^2\gamma(\tau-\tau')\rightarrow\frac{m\pi}{2}\frac{\dot{x}_\tau\dot{x}_{\tau'}}{|\tau-\tau'|}\ , \tag{4.10}$$

where

$$\gamma(\tau)\simeq-\int dq\,\Delta H(q,\tau)\simeq(m/2)|\tau-\tau'|^{-3}\ ,$$

and we integrated by parts in the last step. In Fourier representation, we thus obtain the following result:

$$A_D[x]=\frac{m\pi}{4}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\ln\left[\frac{\omega^2+\tau_0^{-2}}{\omega^2}\right]|x_\omega|^2 \tag{4.11}$$

which corresponds to regularizing (4.10) for short times through the replacement

$$|\tau|^{-1}\rightarrow|\tau|^{-1}[1-\exp(-|\tau|/\tau_0)]\ ,$$

where $\tau_0^{-2}\simeq 2\pi c^2$ simulates the original lattice structure. Clearly, this result is in agreement with the one obtained in the continuum limit [compare to (3.27)], since only small wave vectors are important.

B. Quantum dynamics of a Vortex

In order to discuss the quantum mechanics of a vortex in the presence of an external current, we start by considering $m=0$. For this case, the Hamilton operator corresponding to the effective action discussed above is clearly given by

$$\hat{H}=\hat{p}^2/2\overline{\mathcal{M}}+U(\hat{x})-2\pi f\hat{x} \tag{4.12}$$

where $\overline{\mathcal{M}}\simeq 2\pi^2 M$, and

$$U(\hat{x})\simeq u_1\cos(2\pi\hat{x}),\quad u_1\simeq 0.1E_J\ .$$

(The precise numerical factors are not important.) Ignoring in a first approximation the potential, the spectrum is that of a free particle, $\epsilon_p=p^2/2\overline{\mathcal{M}}$. Furthermore, the periodic potential induces a gap in the spectrum near the zone boundaries, which in our units are at $\pm\pi$, given by $E_G=u_1$, which has to be compared with $\epsilon(p=\hbar\pi)$; thus

$$E_G/\epsilon(\hbar\pi)\simeq 0.4ME_J/\hbar^2\simeq 10^{-2}\alpha^2 \tag{4.13}$$

which is very small[17] for $\alpha\sim 1$; here and in the following, we also assume $M\simeq M_{qp}=3\alpha\hbar^2/64\Delta$. Thus we find the surprising result that the periodic potential almost does not affect the motion of a vortex; in particular, the dynamics for small momenta (and $m=0$) is analogous to the dynamics of a free particle, i.e., the vortex, more precisely, a wave packet constructed out of the plane wave eigenstates, is freely accelerated by a constant force. The irrelevance of the lattice structure reflects the quantum nature of the vortex as an extended object: In fact, comparing as an estimate the localization energy $\epsilon(p\sim\hbar/\delta x)$ with the potential in (4.12), we find $\delta x\simeq 4/\alpha$. Note also that, in terms of the charging energy $E_C=e^2/2C$, we have $\epsilon(\hbar\pi)\simeq 2E_C$, such that (4.13) can be expressed as follows:

$$E_G/\epsilon(\hbar\pi)\simeq 0.05E_J/E_C\ . \tag{4.14}$$

However, a finite m introduces the possibility of dissipation due to decay into acoustic vibrations, which is described (for small frequencies) by the contribution to the action given in (3.27) and (4.11). Since this contribution is quadratic in the coordinate, the (real-time) equation of motion for $\hat{x}(t)$ or $v(t)\equiv\langle\hat{p}(t)\rangle/\overline{\mathcal{M}}$ is immediately found to be given by

$$\overline{\mathcal{M}}\dot{v}(t)+\int dt'\,\eta^R(t-t')v(t')=2\pi f(t)\ . \tag{4.15}$$

As is evident from (4.11), the long-time behavior is dominated by the dissipative contribution, related to $\eta^R(t-t')$; note that

$$\eta^R(\omega)=\eta(\omega\rightarrow -i\omega+0)\ .$$

For example, considering a constant force which is switched on at $t=0$, i.e., $f(t)=f_0\Theta(t)$, the leading behavior for large times is found to be given by

$$v(t)\sim\frac{2f_0}{m}\frac{t}{\ln t}\ . \tag{4.16}$$

This result follows by noting that for η^R_ω, we may use the following expression:

$$\eta^R_\omega\simeq i\omega\pi m\frac{\ln[(-i\omega+0)\tau_0]}{1-i\omega\tau_0} \tag{4.17}$$

where the denominator is introduced to remove the unphysical zero of $\ln(-i\omega\tau_0)$ in the upper half of the ω plane. Then one finds ($\overline{\mathcal{M}}=0$)

$$v(t)=(2f_0\tau_0/m)[\nu(t/\tau_0,0)-\nu(t/\tau_0,1)]\ , \tag{4.18}$$

where ν is the function defined in Ref. 18 (see also Appendix B).

On the other hand, for $m=0$, we have

$$v(t)=(2\pi f_0/\overline{\mathcal{M}})t\ ;$$

thus we conclude, for $\pi m\ll\overline{\mathcal{M}}$, that the velocity increases linearly in time for

$$t\lesssim\tau_1=\tau_0\exp(\overline{\mathcal{M}}/\pi m)\ ,$$

and slightly weaker for larger times. Note that from $m<M$ follows the inequality $\tau_1\gtrsim 10^3\tau_0$, and also that $\tau_0^{-2}\simeq(2\Delta)^2M/m$. Of course, the above results differ significantly from the so-called Ohmic dissipation, where η^R_ω is independent of frequency, in which case a constant force results in a constant velocity; in contrast, note that from (4.17) we have $\mathrm{Re}\,\eta^R_\omega=|\omega|m\pi^2/2$ for $\omega\tau_0\ll 1$.

C. Static vortex-vortex interaction

Finally, in order to make further contact with the results of the continuum limit, we determine the static vortex-vortex interaction within the linear medium approximation. Using the ansatz

$$\phi_n(\{x_i\})=\sum_i\{\pi-2e_i\arctan[2(n-x_i)]\} \tag{4.19}$$

we have to evaluate [see (A2)]

$$V(\{x_i\})=U(\{\phi_n(x_1,x_2,\ldots)\}) \tag{4.20}$$

which has two contributions, $V=V_1+V_2$, related to the two terms on the rhs of (A2). Specializing to a vortex-antivortex configuration, i.e., taking $i=1,2$ in (4.19) only, and (for example) $e_1=+1$, $e_2=-1$, V_1 and V_2 are easily evaluated provided we replace the sum over lattice points n by an integration (which corresponds to neglecting Umklapp processes). In this approximation, V depends only on the difference $\Delta_{12}=x_1-x_2$, with the following results:

$$V_1(\Delta_{12})=2\pi E_J\frac{\Delta_{12}^2}{1+\Delta_{12}^2} \tag{4.21}$$

and

$$V_2(\Delta_{12})=4\pi\tilde{E}_J\int_{-\infty}^{\infty}\frac{dq}{q^2}h(q)e^{-|q|}\sin^2\left[\frac{q\Delta_{12}}{2}\right], \tag{4.22}$$

where $h(q)$ is given in (2.23), and we added a constant to ensure that $V_1(0)=V_2(0)=0$. In particular, we find the following limiting results:

$$\Delta_{12}\ll 1:\quad V(\Delta_{12})\simeq 9.61E_J\Delta_{12}^2\ , \tag{4.23}$$

where the factor 9.61 is close to the π^2 obtained in the continuum limit [see (3.12)], and

$$\Delta_{12}\gg 1:\quad V(\Delta_{12})\simeq 2\pi\tilde{E}_J\ln|\Delta_{12}|\ . \tag{4.24}$$

Note that for large Δ_{12}, V_1 is negligible, and also that $\tilde{E}_J$ (instead or E_J) appears in (4.24).

V. BEYOND THE ADIABATIC LIMIT

In this section we wish to investigate the special features of the quasiparticle dynamics, which are reflected by the appearance of the trigonometric functions in S_A and S_B [see (2.4)], as well as by the characteristic time dependence of the kernels $A(\tau)$ and $B(\tau)$. For simplicity, we take $m=M_0=0$.

A. Linear medium approximation

1. Single vortex

The effective action of a single vortex follows by inserting the ansatz (4.1) into $S^{\rm LM}$ given in (2.16). In addition, we neglect Umklapp processes such that $z\to x$ immediately. As a result, the action contains a static contribution—the tilted periodic potential—which has been investigated in Sec. IV A and a quasiparticle contribution, which is of the following form:[19]

$$A_{qp}[x]=-\int d\tau\,d\tau'[A(\tau-\tau')\mathcal{G}_a(x_\tau-x_{\tau'}) +B(\tau-\tau')\mathcal{G}_b(x_\tau-x_{\tau'})]\ , \tag{5.1}$$

where

$$\mathcal{G}_{a,b}(\Delta_x)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}(e^{iq\Delta_x}-1)\mathcal{G}_{a,b}(q) \tag{5.2}$$

and

$$\mathcal{G}_{a,b}(q)=K_1^2(q/2)\pm K_0^2(q/2)\pm\pi^2h(q)e^{-|q|}/q^2\ . \tag{5.3}$$

The adiabatic limit results are recovered from (5.1) by putting $x_\tau-x_{\tau'}\simeq\dot{x}(\tau-\tau')$, and expanding for small $\dot{x}$, which leads to the following result for the mass of a vortex:

$$\overline{\mathcal{M}}=\overline{\mathcal{M}}_d+\overline{\mathcal{M}}_m\simeq 1.05\times 2\pi^2M\ , \tag{5.4}$$

where $\overline{\mathcal{M}}_d=(4\pi+2\pi/3)M$ and $\overline{\mathcal{M}}_m\simeq 6.10M$ are the direct and the medium contribution, respectively. This is very close to the above adiabatic-limit result, though we have taken into account presently S_B, but neglected Umklapp processes. Note that the contribution from S_B is negative and contributes only 5% to (5.4).

The direct contribution to $\mathcal{G}_{a,b}$, related to K_1^2 and K_0^2, is easily evaluated for all values of Δ_x, with the following result:

$$\mathcal{G}_a^d+\mathcal{G}_b^d=\pi-2|\Delta_x|\frac{1+z}{z}E\left[\frac{2z^{1/2}}{1+z}\right]\ , \tag{5.5}$$

$$\mathcal{G}_a^d-\mathcal{G}_b^d=(2z/|\Delta_x|)K(z)-\pi\ , \tag{5.6}$$

where

$$z^2=(\Delta_x)^2/[1+(\Delta_x)^2]\ ,$$

and K and E denote the complete elliptic integral of the first and second kind, respectively. In particular, note that for $|\Delta_x|\gg 1$

$$\mathcal{G}_a^d\simeq\mathcal{G}_b^d\simeq -2|\Delta_x|\ . \tag{5.7}$$

In this limit, the medium contribution is negligible, in contrast to the adiabatic limit. We emphasize that (5.7) is a consequence of the nonlinear dependence on the phase differences, and easily follows by noting (for example) that

$$\mathcal{G}_a^d=-\int dn\left[1-\cos\left[\frac{\phi_n(x_\tau)-\phi_n(x_{\tau'})}{2}\right]\right]\ . \tag{5.8}$$

Furthermore, for $\Delta_x\gg 1$, the argument of the cosine, considered as a function of n, is approximately zero for $n\lesssim x_{\tau'}$ and $n\gtrsim x_\tau$, and $\simeq -\pi$ in between; thus $\mathcal{G}_a^d\simeq -2(x_\tau-x_{\tau'})$. Obviously, this argument also applies to $\mathcal{G}_b^d$.

Concerning the physical interpretation of the kernel $A(\tau-\tau')$, we remark that [see (2.6)]

$$\operatorname{Im}A^R(\omega)=(\hbar/2e)I_n(\omega)\ , \tag{5.9}$$

where $A^R(\omega)=A(\omega\to -i\omega+0)$. As an illustration, we consider the case where a finite-subgap conductance is present in all junctions, such that

$$\operatorname{Im}A^R(\omega)=\hbar\alpha_s\omega/\pi$$

for small frequencies $\hbar\omega\ll\Delta$, where $\alpha_s=R_0/R_s$; R_s denotes the subgap resistance. In this case, it is sufficient to consider the *classical* equation of motion (in real time), which follows by standard analytic continuation from (5.1). Quite generally, and including the external force, we find the following equation:

$$-2\int_{-\infty}^{\infty} dt'[A^R(t-t')\mathcal{G}_a'(x_t-x_{t'}) + B^R(t-t')\mathcal{G}_b'(x_t-x_{t'})]=2\pi f \ , \tag{5.10}$$

where the prime denotes differentiation with respect to the argument. In particular, assuming f to be time independent, we may choose $x_t - x_{t'} = v(t-t')$. Also, since B^R is found to be negligible (which even holds for an ideal junction), the mobility $\mu = v/f$ is easily calculated from the relation

$$-2i\int_{-\infty}^{\infty} \frac{dq}{2\pi} q A^R(qv)\mathcal{G}_a(q)=2\pi f \tag{5.11}$$

with the result $\mu^{-1} \simeq 0.55 \times \hbar \alpha_s$, as long as $\hbar v \ll 2\Delta$. Note that the medium contributes almost 50% to the rhs of this expression and that

$$0.55=\mathcal{M}(k=0)/4\pi^2 M \ ,$$

where $\mathcal{M}(k=0)$ is given in (4.5).

On the other hand, for ideal junctions,[5-7,20] the quasiparticle current is zero for $\hbar\omega < 2\Delta$ (for $T=0$), leading to exponentially small dissipation if the classical equation (5.11) is applied,

$$f \simeq (4\alpha\Delta/\pi^2)\exp(-2\Delta/\hbar v) \ .$$

Thus—classically—strong dissipation sets in when the voltage in the junction just being traversed by the vortex is of the order of 2Δ. However, the classical result is, at best, only qualitatively correct. In fact, we expect dissipation actually to be zero as long as the vortex energy is smaller than 2Δ (see also Appendix C), and strong dissipation to set in at a velocity given by $\mathcal{M}v^2/2 = 2\Delta$, i.e., $\hbar v \simeq 2\Delta/\sqrt{\alpha}$ (see also Sec. VI).

2. *Multivortex configuration*

Some of the above results are easily extended to a multivortex configuration by inserting the ansatz (4.19), generalized to time-dependent coordinates, into $S^{\rm LM}$. Concentrating on the quasiparticle contribution and on the term derived from S_A, we find the following expression:

$$A_{qp}[\{x_i\}] = -\int d\tau d\tau' \ A(\tau-\tau')\mathcal{G}_a(\{x_i\},\{x_i'\}) \ , \tag{5.12}$$

where $x_i \equiv x_i(\tau)$, $x_i' \equiv x_i(\tau')$ on the rhs of this equation. Clearly, $\mathcal{G}_a$ is a (translational invariant) function of the $2N$ coordinates $x_1, \ldots, x_N, x_1', \ldots, x_N'$, where N is the number of vortices plus antivortices. From the above argument, $\mathcal{G}_a$ is easily determined in the limit where *all differences*, $x_i - x_j$, $x_i - x_j'$, and $x_i' - x_j'$ are much larger than the lattice spacing. Define the variables $y_1, \ldots, y_{2N}$ such that y_1 denotes the smallest, y_2 the next to smallest, etc., of the coordinates $x_1, \ldots, x_N'$. Then we find, in the indicated limit,

$$\mathcal{G}_a \simeq -2\sum_{j=1}^{N} L_j \equiv -2L^* \ , \tag{5.13}$$

where $L_j = y_{2j} - y_{2j-1}$ [see Fig. 2(a)]; note that this expression is independent of the charges $\{e_i\}$.

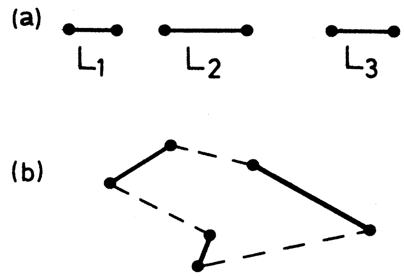

FIG. 2. (a) Three vortices and/or antivortices along a line; the dots indicate the six coordinates $x_1, x_2, x_3, x_1', x_2', x_3'$. The total length of the strings connecting pairs of points is given by $L^* = L_1 + L_2 + L_3$. (b) Generalization of (a) to the two-dimensional case. Clearly, the total length of the bold lines L^* is smaller than the length of (for example) the string configuration given by the dashed lines.

B. Many vortices in the general model

The above results, in particular (5.7) and (5.13), reflect a remarkable feature of the general (nonadiabatic) case, which is qualitatively different from the adiabatic limit. Formally, this is related to the appearance of the factor 2 in the trigonometric functions in S_A and S_B [see (2.4)]; thus it is essential to keep in mind that a "string" is attached to each vortex coordinate, such that the phase difference changes by 2π when crossing the string. Considering the times τ and τ' to be fixed, and returning briefly to a single-vortex configuration with $\mathbf{r}_1 \equiv \mathbf{r}_1(\tau)$, $\mathbf{r}_1' \equiv \mathbf{r}_1(\tau')$, it seems clear that the string can be chosen to be the straight line connecting $\mathbf{r}_1$ and $\mathbf{r}_1'$, and that $\mathcal{G}_{a,b} \simeq -2|\mathbf{r}_1 - \mathbf{r}_1'|$ is the appropriate generalization of (5.7), resulting from integrating along the string ($|\mathbf{r}_1 - \mathbf{r}_1'| \gg 1$). In addition, it appears natural to extend this result to N vortices, in which case we have to consider the $2N$ coordinates $\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{r}_1', \ldots, \mathbf{r}_N'$, with the following result:

$$A_{qp}[\{\mathbf{r}_i\}] = 2\int d\tau d\tau' \ A(\tau-\tau')L^*(\mathbf{r}_1, \ldots, \mathbf{r}_N') \ . \tag{5.14}$$

Here L^* is the total length of the strings, determined by connecting pairs of the $2N$ coordinates by straight lines, and taking the configuration of minimal length [see Fig. 2(b)]. Note that (5.14) holds in the limit where all differences are large compared to the lattice spacing and again is independent of the charges.

VI. DISCUSSION

In order to simplify the discussion, we concentrate on the case $m \ll M$. In a first approximation, neglecting dissipation, the response of a vortex is determined by the Hamiltonian given in (4.12), and thus corresponds to the motion of a particle in a (weak) periodic potential. By analogy, we thus expect Bloch oscillations for small fields $f \ll f^*$ and almost free acceleration for $f \gg f^*$, where

$$f^* = \frac{\pi}{8}\frac{E_G^2}{f\epsilon(\hbar\pi)} \simeq 10^{-3}\alpha^2 E_J \tag{6.1}$$

is defined such that the standard Zener tunneling probability is given by $P=\exp(-f^*/f)$.

However, one has to keep in mind that the adiabatic limit Hamiltonian is, strictly speaking, only applicable in the regime $\overline{\mathcal{M}}v^2 \lesssim 2\Delta$, i.e., for momenta not too close to the zone boundary. While the (subohmic) dissipation due to the decay into acoustic vibrations introduces only small (logarithmic) corrections to the time dependence of the vortex velocity—see (4.16)—the possibility of quasiparticle creation leads to a drastic modification, which can be seen as follows. Consider $\eta^R \equiv 0$ in (4.15), and imagine that a constant force is switched on at, say, time zero. Then the vortex velocity increases linear in time, $v(t)=2\pi f t/\overline{\mathcal{M}}$, with the corresponding increase in vortex energy. Thus, after the characteristic time t_{qp}, defined by the relation

$$(2\pi f t_{qp})^2/2\overline{\mathcal{M}}=2\Delta \tag{6.2}$$

which leads to

$$t_{qp}=(\Delta\overline{\mathcal{M}})^{1/2}/\pi f \simeq (3\alpha/32)^{1/2}\hbar/f \ , \tag{6.3}$$

the vortex energy exceeds 2Δ, which opens the possibility of quasiparticle creation, mainly in the junction just being traversed by the vortex. [Note that $v \simeq v(t_{qp}) \simeq 2(\Delta/\overline{\mathcal{M}})^{1/2}$ implies a local voltage $\gtrsim 2\Delta$ (for $\alpha \sim 1$).] To describe the quasiparticle creation process in detail, which we do not attempt here, one has to employ the full nonlinear action as given in (5.1), or the corresponding Hamiltonian (see Appendix C). We expect that at $t \simeq t_{qp}$ the velocity is almost instantaneously, i.e., within a time interval of the order of $\hbar/\Delta$, reduced to zero, after which the acceleration starts again. Thus the time dependence of the velocity is of a saw-tooth form, with the characteristic frequency $2\pi/t_{qp}$; also, the time average is given by

$$\langle v \rangle_t = \frac{\pi f t_{qp}}{\overline{\mathcal{M}}} = \left[\frac{\Delta}{\overline{\mathcal{M}}}\right]^{1/2} \simeq \left[\frac{16}{3\alpha}\right]^{1/2} \frac{2\Delta}{\hbar} \tag{6.4}$$

which implies that the vortex travels the distance $\simeq \Delta/\pi f$ during one cycle (in units of the lattice constant). Note that the average velocity is independent of the applied field (of course, we have $f \ll E_J$), and consequently the dissipated energy is linear in f, namely

$$\frac{\overline{\mathcal{M}}}{2}\langle \partial_t v^2 \rangle_t = 2\pi f \langle v \rangle_t = 2\Delta/t_{qp} \ . \tag{6.5}$$

In order to determine the corresponding electric field $\mathcal{E}$, we remark that the energy dissipated per unit area and time is given by $I_T\mathcal{E}/a$, where a is the lattice constant. (Note that the total current is $I_{\text{tot}}=I_T\mathcal{L}/a$, where $\mathcal{L}$ is the linear dimension of the film.) On the other hand, assuming that N_v vortices are present, the vortex area density is $n_v=N_v/\mathcal{L}^2$. Thus the electric field, as determined by the relation

$$\mathcal{L}^{-2}\Delta E/\Delta t = I_T\mathcal{E}/a=(2\Delta/t_{qp})n_v \ , \tag{6.6}$$

if found to be given by

$$e\mathcal{E}a=\pi(\hbar^2\Delta/\overline{\mathcal{M}})^{1/2}a^2 n_v \simeq (8/3\alpha)^{1/2}a^2 n_v 2\Delta \ . \tag{6.7}$$

Of course, we assumed that the vortices move independent of each other, in order to arrive at (6.6) and (6.7).

In contrast, in the presence of a finite-subgap conductance, the vortex velocity is proportional to the applied current (for $\hbar v/a \ll 2\Delta$):

$$0.55\alpha_s \hbar v/a = f \ . \tag{6.8}$$

In this case, since $n_v a^2 \ll 1$, moving vortices give only a small correction to the voltage. Note that (6.8) is of the same form as the result one obtains for the motion of a vortex in a *continuous* film, where[21]

$$(a^2/2\pi\xi^2)\alpha\hbar v/a = f \tag{6.9}$$

if one uses the (arbitrary) length scale a, for convenience. As before, $\alpha=R_0/R_N$, where R_N is the normal-state resistance of the continuous film, and ξ denotes the coherence length. Thus the viscosity due to a finite-subgap conductance is of the same magnitude as in a continuous film if $\alpha_s \sim \alpha(a/\xi)^2$.

A more general interpretation of the results obtained in this paper is as follows. The starting point, or the original problem, is the investigation of the model given in (2.2), which involves path integration with respect to all phase variables $\{\phi_l(\tau)\}$ with a weight proportional to $\exp(-S/\hbar)$. The idea then is to rewrite this problem (approximately) in terms of vortex coordinates $\{\mathbf{r}_i(\tau)\}$, such that $\exp(-A/\hbar)$ is the appropriate weight, which is assumed to contain the essential physics of the original model. Though the "new" formulation is again of great complexity, it may open the way for new insights into the "old" problem.

As the simplest example in the present context, recall the considerations concerning the continuum limit (Sec. III), and especially for $M=0$. The form of the action (2.12) for this case—see also Ref. 8—suggests a three-dimensional notation. Thus defining (again, we choose $a=1$) $\tilde{\mathbf{x}}=(l_x,l_y,c\tau)$, the corresponding gradient $\tilde{\nabla}$ and the magnetic field $\tilde{\mathbf{h}}=\tilde{\nabla}\phi$, the action (2.12) is written as follows:

$$S \to \frac{E_J}{2c}\int d^3x\, \tilde{\mathbf{h}}^2 \ . \tag{6.10}$$

Also, the result (3.25) transforms into

$$A \to \frac{\pi E_J}{2c}\int d^3x\, d^3x' \frac{\tilde{\mathbf{j}}(\tilde{\mathbf{x}})\cdot\tilde{\mathbf{j}}(\tilde{\mathbf{x}}')}{|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'|} \ , \tag{6.11}$$

where we defined $\tilde{\mathbf{j}}=(\rho,\mathbf{j}/c)$, with ρ and $\mathbf{j}$ denoting the ordinary charge density and current of the Coulomb plasma. Note that $\tilde{\nabla}\cdot\tilde{\mathbf{j}}=0$ corresponds to the continuity equation in the particle system and that $\tilde{\nabla}\times\tilde{\mathbf{h}}=2\pi\tilde{\mathbf{j}}$, $\tilde{\nabla}\cdot\tilde{\mathbf{h}}=0$. Thus $A[\tilde{\mathbf{j}}]$ is of the same form as the magnetostatic energy of unimodular linear currents (in three dimensions) flowing along the trajectories of the vortex centers,[8] and path integration is to be understood with respect to all possible current configurations.

From the continuum-limit results, it also seems to be evident that the model Hamiltonian (1.1), for $T=0$ and $M=0$, is in the same universality class as the classical three-dimensional XY model. We thus expect a continuous phase transition from an ordered to a disordered state upon increasing $\hbar^2/m$ (note that $E_c = e^2/2c = \hbar^2/8m$), at a critical value $(\hbar^2/2m)^* \sim E_J$, characterized by the well-known exponents. On the other hand, as a function of temperature, the transition is of the Berezinskii-Kosterlitz-Thouless type,[22] with $kT_c \sim E_J$ (at least for large m). In fact, recent Monte Carlo simulations[23] have clarified various aspects of the phase diagram, demonstrating in addition a reentrant behavior (from normal to superconducting to normal) as a function of T and, quite unexpectedly, a first-order transition within the superconducting region.

The zero-temperature phase boundary has been studied within mean-field theory,[24] with the result that the ordered region of the phase diagram is enlarged by increasing M, i.e., the nearest-neighbor capacitance; it is an open question, however, whether there exists a phase transition (for $m = 0$) at a finite value of M. Finally, we wish to mention a recent suggestion[25] that local quantum fluctuations are able to destroy the global phase coherence, such that there is no superconducting region in the phase diagram at all. In particular, it is asserted[25] that in the absence of (Ohmic) dissipation, global phase coherence is destroyed independent of the dimensionality of the system. We doubt, however, the validity of the arguments; rather, we believe that correlation functions of the form $\langle \exp[i(\hat{\phi}_l - \hat{\phi}_{l'})] \rangle$ for large distances $|l - l'|$ are only weakly affected by the above mentioned fluctuations. (These fluctuations, in the vortex picture, can be interpreted as vortex rings.) Also, the Monte Carlo results[23] show a well-developed superfluid density for low temperatures and not too large charging energy, contrary to the claim of Ref. 25.

In conclusion, the equilibrium properties as well as the response to an external current of granular films and networks require further investigations. We believe that the vortex picture employed in this paper is a useful concept to investigate (at least some of) the questions discussed above.

APPENDIX A: STATIC VORTEX SOLUTION WITHIN LM

In this Appendix we discuss the static vortex solution within the linear medium approximation, as given by (2.14) and (2.17). The vortex solution is determined from the equations (for all k)

$$\partial U(\{\phi_n\})/\partial\phi_k = 0 , \tag{A1}$$

where

$$U(\{\phi_n\}) = -E_J \sum_n \cos\phi_n + \frac{\tilde{E}_J}{2} \sum_{n,n'} \phi_n h_{n-n'} \phi_{n'} \tag{A2}$$

under appropriate boundary conditions for $|n| \to \infty$, e.g., $\phi_{n\to+\infty} = 0$, $\phi_{n\to-\infty} = 2\pi$. Note that h_n, which is the Fourier transform of $h(q)$ [see (2.23)], has the following properties:

$$h_{n=0} = (\pi+1)/\pi, \quad \sum_n h_n = 0 ,$$

and for $n \to \pm\infty$,

$$h_n \simeq -(2\pi n^2)^{-1} . \tag{A3}$$

Equations (A1) are solved numerically by working with a finite system of N lattice sites, with N ranging from 200 to 3200, though excellent results are obtained by considering $N = 400$ and 800 and extrapolating to infinity. By symmetry, the vortex center is in the middle of the unit cell, to which we attache the coordinate $x = \frac{1}{2}$. Of course, the solution depends on the parameter $\tilde{E}_J$, which is determined by *imposing the geometrical condition*

$$\phi_{n=1}(x = \tfrac{1}{2}) = \frac{\pi}{2}, \quad \phi_{n=0}(x = -\tfrac{1}{2}) = \frac{3\pi}{2} . \tag{A4}$$

From this condition, we find $\tilde{E}_J$ to be given by

$$\tilde{E}_J = (0.6681 \pm 0.0002) E_J \tag{A5}$$

which is slightly larger than the value $(2/\pi)E_J$ of the simpler model of Ref. 13. The other symmetry point, corresponding to a maximum of the potential, i.e., the vortex center being on a link, is identified with $x = 0$, and the solution for this case is calculated by imposing $\phi_{n=0}(x = 0) = \pi$. In particular, the potential barrier

$$\Delta U = U(\{\phi_n(0)\}) - U(\{\phi_n(\tfrac{1}{2})\}) ,$$

is found to be given by

$$\Delta U = (0.2035 \pm 0.0002) E_J \tag{A6}$$

which is very close to the value $0.199E_J$ of Ref. 11, where the two-dimensional static vortex solution was determined. Thus we obtain (up to a constant) the following approximation for the potential:

$$U(x) \simeq 0.102 E_J (1 + \cos(2\pi x)) . \tag{A7}$$

Similarly, we calculate the x-dependent mass of the vortex $\mathcal{M}(x)$, which we define by

$$\mathcal{M}(x) \equiv M \left[\sum_n \left[\frac{\partial\phi_n}{\partial x} \right]^2 + \sum_{n,n'} \frac{\partial\phi_n}{\partial x} h_{n-n'} \frac{\partial\phi_{n'}}{\partial x} \right] , \tag{A8}$$

where

$$\partial\phi_n/\partial x \equiv [\phi_n(x + \delta x) - \phi_n(x)]/\delta x$$

in the limit of small δx, with the following result:

$$\mathcal{M}(x) = \sum_k \mathcal{M}(k) \cos(2\pi k x) , \tag{A9}$$

where $\mathcal{M}(k = 0) \simeq 1.026 \times 2\pi^2 M$, and $\mathcal{M}(k = 1)/\mathcal{M}(k = 0) \simeq 0.583$. Higher Fourier components are found to be negligible. From (A9), the vortex mass can be calculated from the relation

$$\overline{\mathcal{M}} = \left[\int_0^1 dx \sqrt{\mathcal{M}(x)} \right]^2 \simeq 0.98 \times 2\pi^2 M . \tag{A10}$$

The numerical results for the vortex solution can be accurately represented by the following expression:

$$\phi_n(x) = \pi + i \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iq(n-x)} \phi_q , \tag{A11}$$

where

$$\phi_q = \frac{\pi \mathrm{sign} q}{h(q)} e^{-\beta|q|}(1+\gamma|q|) \ . \tag{A12}$$

The parameters β and γ are chosen in order to fit the numerical solution for $x=\frac{1}{2}$, in particular to produce the correct value for $n=1$ and $n\to\infty$, with the result $\beta\simeq 0.63$ and $\gamma\simeq 0.81$. Note that $h(q)\simeq|q|/2+q^2/4+\cdots$ for small q, corresponding to $\phi_n\simeq[1+2(\beta-\gamma)]/n$ for $n\to\infty$.

APPENDIX B: THE FUNCTION $\nu(x,\alpha)$

Here we discuss the asymptotic behavior of the function[18] $\nu(x,\alpha)$, or more precisely of the difference $\Delta\nu(x)\equiv\nu'(x)-\nu(x)$, where $\nu(x)=\nu(x,0)$; note the relation $\nu'(x,\alpha)=\nu(x,\alpha-1)$. From Eq. (10) of Ref. 18, we find the following expression:

$$\Delta\nu(x) = \int_0^\infty \frac{dt}{t} \frac{(1+t)e^{-xt}}{\pi^2+\ln^2 t} \ . \tag{B1}$$

By the substitution $xt\to t$, it becomes apparent that for large x, the dominant contribution is given by

$$\Delta\nu(x) \sim \int_0^\infty \frac{dt}{t} \frac{e^{-t}}{\pi^2+\ln^2(t/x)} \ . \tag{B2}$$

From this expression, the asymptotic behavior $\Delta\nu(x)\sim[\ln x]^{-1}$, $x\to\infty$, easily follows by standard methods, and we also conclude that $\nu(x,0)-\nu(x,1) \sim x/\ln x$ in the same limit.

APPENDIX C: A POLARONIC OBJECT

Adding a kinetic energy, we write the action A_{qp} [see (5.1)] of a vortex interacting with quasiparticle as follows:[26]

$$\begin{aligned}\tilde{A}[x(\tau)] = {} & \tfrac{1}{2}\int d\tau \mathcal{M}_0 \dot{x}^2(\tau) \\ & + \tfrac{1}{2}\int d\tau d\tau' \int \frac{dq}{2\pi} \mathcal{E}(q;\tau-\tau') \\ & \times \exp iq[x(\tau)-x(\tau')] \ .\end{aligned} \tag{C1}$$

Alternatively, this action can be thought of being derived from a Hamiltonian where an object interacts with the vibrational degrees of freedom of an environment,[27]

$$\begin{aligned}\hat{H} = {} & \hat{p}^2/2\mathcal{M}_0 \\ & + \sum_\lambda \int \frac{dq}{2\pi} \{ v_\lambda(q) e^{iq\hat{x}} \hat{\xi}_{q\lambda} + \tfrac{1}{2}[|\hat{\pi}_{q\lambda}|^2 + \omega_{q\lambda}^2 |\hat{\xi}_{q\lambda}|^2] \}\end{aligned} \tag{C2}$$

provided that one choose the coupling $v_\lambda(q)$ and the frequencies $\omega_{q\lambda}$ such that

$$\begin{aligned}\sum_\lambda & |v_\lambda(q)|^2 \langle\langle T_\tau \hat{\xi}_{q\lambda}(\tau) \hat{\xi}_{q'\lambda}(\tau') \rangle\rangle \\ & = -2\pi\delta(q+q')\mathcal{E}(q;\tau-\tau') \ .\end{aligned} \tag{C3}$$

It is instructive to ascertain some properties of the perturbation expansion for the Green's function $\mathcal{D}(k;\tau)$ of the object. In the following we will work in the Matsubara technique,[28] and we assume that the objects are bosons of very low density (so that this assumption is without consequences). Accordingly, we choose the chemical potential $\zeta \ll -kT$, and introducing the Matsubara frequencies $\omega_n=2\pi T n$, the Green's function is in zeroth order given by

$$\mathcal{D}_0(k,\omega_n) = \frac{1}{i\omega_n - \epsilon_k + \zeta}, \quad \epsilon_k = k^2/2\mathcal{M}_0 \ .$$

One may convince oneself without difficulty, that the diagramatic form of the perturbation expansion is the same as in the case of electron-phonon interaction[28] provided that one associates $\mathcal{D}_0$ and $\mathcal{E}$ with a straight and wavy line, respectively.[29]

Let us consider again the action A_{qp} where, for sake of simplicity, we retain only the most important term. Then, $|v_\lambda(q)|^2=\mathcal{G}_a(q)$ is independent of λ, and $\omega_{q\lambda}$ is independent of q. In particular [see (2.6)]

$$\begin{aligned}\mathcal{E}(q,\omega_n) &= -2\mathcal{G}_a(q)A(\omega_n) \ , \\ A(\omega_n) &= \int \frac{d\omega'}{2\pi} \frac{2\omega'}{\omega'^2+\omega_n^2} \frac{1}{2e} I_n(\omega') \ .\end{aligned} \tag{C4}$$

However, the expression for $A(\omega_n)$, as it stands, is meaningless since $I_n(\omega)\propto\omega$ for $|\omega|\to\infty$.

This feature excludes a straightforward perturbation expansion. For instance, we have worked out the self-energy in first and second order; and each order required additional interpretations if one insisted on obtaining finite results.

On the other hand, it appears that the mobility of the object can be calculated without difficulties. Let us introduce a perturbation by a vector potential $A(\tau)$ such that $\dot{A}$ represents a time-dependent but spatially homogeneous force. Then, the mobility (in Matsubara frequencies) can be written as

$$\mu(\omega_n) = \frac{1}{\omega_n \mathcal{M}_0} \left[1 + \frac{1}{\mathcal{M}_0 n_0} \chi(\omega_n) \right] , \tag{C5}$$

where

$$\chi(\omega_n) = \int_0^{1/T} d\tau \, e^{i\omega_n \tau} \langle\langle T_\tau \hat{p}(\tau)\hat{p}(0) \rangle\rangle \ , \tag{C6}$$

and where

$$n_0 = \int \frac{dk}{2\pi} e^{-(\epsilon_k-\zeta)/T} \tag{C7}$$

is the density of the objects.

In zeroth order, we have

$$\chi^{(0)}(\omega_n) = -T \sum_{\omega_{n'}} \int \frac{dk}{2\pi} k \mathcal{D}_0(k,\omega_{n'}) \mathcal{D}_0(k,\omega_{n'}-\omega_n) k = 0 \ . \tag{C8}$$

In first order, there are three diagrams (see Fig. 3). Obviously, the two self-energy diagrams diverge as well as the vertex-correction diagram. However, after having done the frequency summations, we obtain for the total sum

$$\chi^{(1)}(\omega_n) = -\int \frac{dk\,dq\,d\omega'}{(2\pi)^3} \frac{2kq\,\mathcal{G}_a(q) 2\omega' I_n(\omega')/2e}{\omega'(\epsilon_{k-q}-\epsilon_k+\omega')[(\epsilon_{k-q}-\epsilon_k+\omega')^2+\omega_n^2]} \times \frac{1}{4}\left[\coth\frac{\omega'}{2T}+\coth\frac{\epsilon_{k-q}-\zeta}{2}\right]\left[\coth\frac{\epsilon_{k-q}-\zeta+\omega'}{2T}-\coth\frac{\epsilon_k-\zeta}{2T}\right] \quad \text{(C9)}$$

which is finite. Let us now take the limit $\zeta \to -\infty$, $T \to 0$, where it follows that

$$\chi^{(1)}(\omega_n) = -4n_0 \int \frac{dq\,d\omega'}{(2\pi)^2} \frac{|I_n(\omega')/2e|\,\mathcal{G}_a(q) q^2}{(|\omega'|+\epsilon_q)[(|\omega'|+\epsilon_q)^2+\omega_n^2]} . \quad \text{(C10)}$$

The analytical continuation is easily done by letting $\omega_n \to -i\Omega+0$.

Consider now the case where there is a gap ω_g in the spectrum of the environmental oscillators $I_n(\omega)=0$ for $|\omega|<\omega_g$, and where $\mathcal{G}_a(q)$ differs from zero only for small-q values such that effectively $\epsilon_q \ll \omega_g$. Then, we obtain

$$\chi^{(1)}(0) = -4n_0 \int \frac{d\omega'}{2\pi}\frac{I_n(\omega')/2e}{\omega'^3}\int \frac{dq}{2\pi}\mathcal{G}_a(q) q^2 = -4n_0(M_{qp})\left[\frac{\pi^2}{2}\times 1.10\right] , \quad \text{(C11)}$$

where the factors in the parenthesis correspond to the two integrals. Inserting the result (C11) in the expression (C5), we obtain in the limit $\Omega \to 0$

$$\mu(-i\Omega+0) = \frac{1}{(-i\Omega+0)\mathcal{M}_0}(1-\overline{\mathcal{M}}/\mathcal{M}_0) = \frac{1}{(-i\Omega+0)(\mathcal{M}_0+\overline{\mathcal{M}})} \quad \text{(C12)}$$

which corresponds to free acceleration of the object. Note that $\overline{\mathcal{M}} = 2\pi^2 \times 1.10 M_{qp}$ agrees with the quasiparticle-induced mass of a vortex as defined by Eq. (5.4), except for minor changes due to differences in the model.

As pointed out in the Introduction, for small junctions $\mathcal{M}_0/\overline{\mathcal{M}} = M_0/M_{qp} = C_0/C_{qp} \ll 1$, and first order perturbation theory is insufficient. It seems that the adiabatic approximation in the Lagrangian formulation allows a comprehensive approach even in the case of strong coupling. However, it is not clear whether the condition $|\Omega| \ll \omega_g$ is sufficient and whether it is also necessary to satisfy $\epsilon_q = q^2/2\mathcal{M}_0 \ll \omega_g$, perhaps in the very much milder form $q^2/2[\mathcal{M}_0+\overline{\mathcal{M}}] \ll \omega_g$. Note that the milder condition is marginally satisfied in the vortex problem.

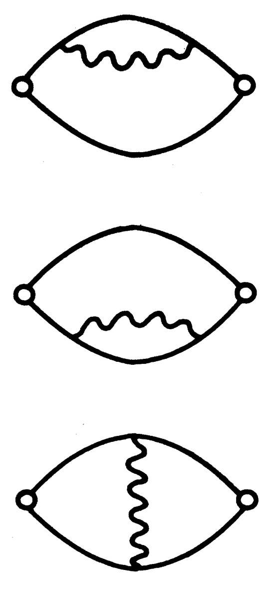

FIG. 3. Diagrams contributing in first order to $\chi(\omega_n)$; compare (C9).

For a comparison, let us also consider the "large" polaron. The Hamiltonian in the form (C2) has been set forward by Fröhlich,[30] and later Feynman[31] established the effective action of the type (C1). As before, we may present the "environmental" Green's function in the form (C4) where we have to set, however,

$$\mathcal{G}_a(\mathbf{q}) = \frac{2\pi\alpha}{(2\mathcal{M}_0)^{1/2}}\frac{\omega_0^{3/2}}{\mathbf{q}^2} , \qquad A(\omega_n) = \int \frac{d\omega'}{2\pi}\frac{2\omega'}{\omega'^2+\omega_n^2}\pi[\delta(\omega'-\omega_0)-\delta(\omega'+\omega_0)] . \quad \text{(C13)}$$

Above, ω_0 is the frequency of the optical phonons and α the dimensionless coupling constant. In the present case $A(\omega_n)$ exists, and we may calculate the self-energy from which we obtain an effective mass $\mathcal{M}^*$ which is in lowest order given by $\mathcal{M}^*/\mathcal{M}_0 = 1+\alpha/6$.

We may also calculate the correction to the mobility. Inserting the expression (C13) in relation (C10), we obtain

$$\frac{1}{n_0}\chi^{(1)}(0) = -\frac{2\pi\alpha\omega_0^{3/2}}{(2\mathcal{M}_0)^{1/2}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{[\omega_0+\epsilon_q]^3} . \quad \text{(C14)}$$

We recognize that presently, it is not possible to neglect ϵ_q in the denominator; this is an important difference in comparison with the vortex problem. Evaluating the integral, and inserting the result in Eq. (C5), we obtain

$$\mu(-i\Omega+0) = \frac{1}{(-i\Omega+0)\mathcal{M}_0}(1-\alpha/2) = \frac{1}{(-i\Omega+0)\mathcal{M}_0(1+\alpha/2)} . \quad \text{(C15)}$$

Again, we find the mobility of a free object; however, the transport mass $\mathcal{M}_{tr}=\mathcal{M}_0(1+\alpha/2)$ differs from the effective mass $\mathcal{M}^*$.

Of course, free acceleration is specific to the case where there is a gap in the environmental excitation spectrum. In this case, the zero-temperature limit is meaningful. On the other hand, a freely accelerated object may easily reach higher velocities where the response is nonlinear. However, it seems that the nonlinear response of a strongly coupled polaron poses an almost intractable problem.[32]

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