Bistability and Colored Noise in Nonequilibrium Systems: Theory versus Precise Numerics

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(Received 27 October 1987)

We present novel calculations for the archetypal bistability dynamics driven by a correlated Gaussian random force (colored noise). Our focus is on the behavior at weak noise, which is difficult to solve theoretically as well as numerically. Precise numerical results for the smallest eigenvalue $\lambda_1(\tau)$ [or rate of escape $\Gamma(\tau)=\frac{1}{2}\lambda_1(\tau)$] at small to moderate to large noise correlation times τ are compared and interpreted versus a whole set of recent but *conflicting* theoretical predictions.

PACS numbers: 05.40.+j, 05.70.Ln

The study of dynamical systems perturbed by noise is of wide-ranging significance to the detailed understanding of onset and characterization of nonlinear phenomena. In recent years, there emerged the need for more realistic modelings of physical systems. In this context, the incorporation of a finite noise correlation time τ, or finite bandwidth (i.e., colored noise), that describes the separation of time scale(s) between the system dynamics and environmental and/or external perturbances is currently in the limelight from both the experimental and theoretical viewpoints.[1] In particular, for the large class of thermal equilibrium systems it has been demonstrated that memory friction, or—via the fluctuation-dissipation theorem—colored thermal noise, can modify substantially the classical, diffusive barrier transmission.[2]

The situation is even more difficult in stationary nonequilibrium systems which, generally, do not obey the condition of detailed balance. The archetypal situation is a Ginzburg-Landau-type bistable dynamics, being driven by exponentially correlated Gaussian noise, i.e., with $a>0$, $b>0$, we set

$$\dot{x}=ax-bx^3+\xi(t), \tag{1}$$

$$\text{with } \langle\xi(t)\xi(s)\rangle=(D/\tau)\exp(-|t-s|/\tau).$$

Here, τ is the correlation time and D denotes the noise intensity. In order to deal with dimensionless quantities we introduce instead of x, ξ, and t the dimensionless variables $x\rightarrow(b/a)^{1/2}x$, $\xi\rightarrow(b/a^3)^{1/2}\xi$, and $t\rightarrow at$, obeying the dynamics (1) where $a=b=1$ with a dimensionless noise intensity $D\rightarrow Db/a^2$ and a dimensionless noise correlation time $\tau\rightarrow a\tau$. In the following we shall stick to these dimensionless variables.

For $\tau=0$, everything is well known, i.e., (1) reduces to the (Smoluchowski) equilibrium dynamics. Finite noise correlation times (at weak noise), however, play a ubiquitous role in the statistical description of dye lasers, the ring-laser gyroscope,[3] or magnetic resonance.[4] Often, the physics is controlled by noise color τ of moderate to large strength,[3-5] i.e., realistic noise color means more than just a small correction to the white-noise limit ($\tau=0$). With $\tau>0$, the dynamics in (1) constitutes a *non-Markovian* process, which makes things much harder. Of course, we also could embed the dynamics in (1) into a two-dimensional Markovian dynamics.[6] This result, however, does not make our task any simpler: The enlarged dynamics contains even more information which is not controlled experimentally, such as an infinite number of different initial preparation schemes between "system," x, and "rest," ξ. The dynamics in (1) corresponds to one specific preparation where $\xi(0)$ is stationary and independent of $x(0)$. Moreover, our interest lies in the system dynamics $x(t)$ alone. The investigation of the nonlinear, noisy dynamics in (1) is difficult because of an entanglement of complications. In particular, (i) the exact stationary probability, $\bar{p}(x)$, for $x(t)$ is not known; (ii) the generator of the underlying Markov process is not a symmetric operator, i.e., complex eigenvalues are possible.

Over the last years we have witnessed a flood of papers, all attempting to describe the influence of such realistic noise color. The recent theoretical developments,[7-14] of which we cite only a representative sample, can be grouped into two parts: (1) construction of theories for small noise color τ, i.e., theories that provide small corrections to the white-noise limit[7,8]; (2) development of approximations covering noise color of small to moderate to large strength.[9,10]

In this work, our focus will be on the weak-noise behavior, since it is typical for realistic applications.[3,4] At weak noise, an interesting quantity is presented by the rate of escape, Γ, from the two metastable states $x_{1/2}=\pm1$. The rate Γ probes the colored-noise behavior within exponential sensitivity, i.e., it crucially depends on the precise form of $\bar{p}(x)$, and involves knowledge from regions with exponentially small probability around $x\cong0$.

For this escape rate, there exists a series of conflicting predictions among different theories.[6-14] This state of affairs has ignited a passionate debate involving the merits of various approximation schemes. The conventional small-τ theories[7] have been critiqued in Ref. 6, particularly, for their inconsistent omission of non-Fokker-

Planck contributions and the failure to predict a colored-noise correction for the exponential leading part of the rate. In contrast, the theory in Ref. 9 predicts for moderate τ a dependence in the form $\Gamma(\tau)/\Gamma(\tau=0) \propto \exp(-\alpha\tau/D)$.

Recent digital[6] and analog simulations[9] have also added to the confusion: The digital simulations (error $\gtrsim 10\%$!) and the analog simulations are not very accurate, and more importantly, they do not involve the weak-noise regime, $1/4D \gg 1$. While analog simulations are quite powerful to predict the main features of stationary probabilities,[15] these encounter difficulties in simulating small noise intensities, and cannot resolve the asymptotic regime of small noise correlation times close to the white-noise limit—though for a different problem (periodic potential), there is one recent illuminating study for weak noise.[16] This study supports the findings of Ref. 9; however, the *quantitative* agreement found with theory[9,13b] is not quite decisive: The result in Ref. 16 involves a time scaling, $\tau \rightarrow \tau/4$, which lacks rigorous justification. In addition, the parameters values for the potential form have not been varied in order to check the agreement between decoupling theory[9] and numerics over a wider range of potential forms. Moreover, the small-τ asymptotics has not been addressed.

There is an urgent need for a detailed numerical weak-noise study of the problem in (1).[13a] In order to clarify and resolve the various conflicting predictions, and to test the numerous theories[6-14] we present here for the first time precise numerical results for the following: (a) The smallest nonvanishing eigenvalue $\lambda_1(\tau)$ of the bistable Fokker-Planck dynamics, which we find to be real for $\tau < 2$. At weak noise, $\lambda_1(\tau)$ is related to the escape rate by $\Gamma(\tau) = \frac{1}{2}\lambda_1(\tau)$.[17] (b) The asymptotic behavior of $\lambda_1(\tau)$ for $\tau \rightarrow 0$. (c) The accuracy of a steepest-descent approximation to $\Gamma(\tau)$ or $\lambda_1(\tau)$.

Following the matrix continued fraction method of Jung and Risken[18] (see also Risken[19]) we have numerically evaluated the eigenvalue $\lambda_1(\tau)$ (*error* $\leq 0.1\%$!) of the *enlarged two-dimensional Markovian dynamics,* see Fig. 1. In order to improve the rate of convergence we have refined the procedure in Ref. 18 by introducing a parametric form function $\rho_0(x) = \exp(-cx^2)$. The convergence of the matrix continued fraction is rather different for different values of the parameter of the form function. Comparing the digits of $\lambda_1(\tau)$ for different values of the parameter one readily tests the accuracy of $\lambda_1(\tau)$. This weak-noise numerics is nontrivial and rather cumbersome: It should be noted that in absence of knowledge of the stationary probability, efficient procedures such as the reactive flux method,[21] commonly used in equilibrium systems, cannot be invoked. A main difficulty arises because the exponential weight, i.e., the analog of the Boltzmann factor, is not known *a priori* for our archetypal nonequilibrium case. The D values used in Fig. 1 correspond to (white noise) Arrhenius factors of

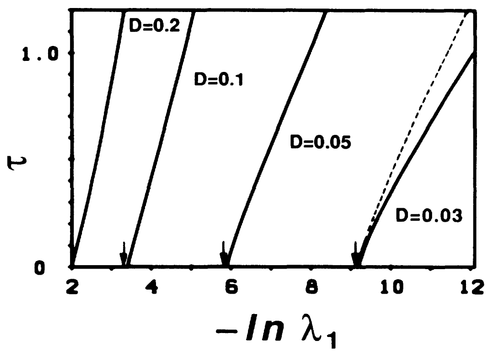

FIG. 1. First nonvanishing eigenvalue $\lambda_1(\tau)$ of the nonequilibrium bistable flow in (1) as a function of noise color τ. For $D < 0.1$ one finds the behavior in (2) with $\alpha \approx 0.10$ (see inset of Fig. 2). The arrows indicate the result (3) of the steepest-descent approximation at $\tau = 0$. The bridging result by Luciani and Verga (Ref. 20), i.e.,

$$\lambda_1(\tau) = (\sqrt{2}/\pi)(1+3\tau)^{-1/2} \times \exp[-(1/4D)(1+\tfrac{27}{16}\tau+\tfrac{1}{2}\tau^2)/(1+\tfrac{27}{16}\tau)]$$

is depicted by the dashed curve.

$\phi_0 = 1/4D = 2.5$ for $D = 0.1$, $\phi_0 = 5$ for $D = 0.05$, and $\phi_0 = \frac{25}{3}$ for $D = 0.03$. We note from the numerics that $\ln(\lambda_1(\tau))$ varies *inversely proportional to the noise intensity D,* i.e.,

$$\lambda_1(\tau) \propto \exp(-\alpha\tau/D), \quad 1.5 > \tau \gtrsim 0.2. \tag{2}$$

Thus, the Arrhenius factor becomes modified by the noise color in a form that is in *qualitative agreement* with the decoupling approximation of Ref. 9, i.e., the Arrhenius factor exhibits a dependence proportional to $\exp(-\alpha\tau/D)$. The factor α is plotted in the inset of Fig. 2. For weak noise, it approaches a value of $\alpha \cong 0.10$. The prediction of the lowest-order decoupling theory[9] exceeds this value by about a factor of 5. This difference in α is the result of higher-order, non-Fokker-Planck contributions which have been neglected in the lowest-order decoupling scheme, but do affect the precise value of the constant α in (2). α is not precisely independent of the noise color τ[20] (e.g., note the slight upturn in Fig. 1 for $D \geq 0.1$), but exhibits a weak dependence on τ. The study of $\lambda_1(\tau)$ as $\tau \rightarrow \infty$ reveals that α, at weak noise, converges very slowly to a limiting value $\alpha = \frac{2}{27} \approx 0.074$.[20,22]

Some comments are in order for the noise intensity $D = 0.1$. This D value corresponds to an Arrhenius factor of $\phi_0 = 2.5$, i.e., it is not truly in the weak-noise regime. This noise intensity, however, has been used previously[6,11,12] in comparisons of theories with digital[6] and analog simulations.[9] In this case, the numerical result of $\lambda_1(\tau) = \lambda_1(0)\exp(-1.27\tau)$, $\tau \leq 1$, closely corresponds, by change, to the value obtained via an *ad hoc* exponentiation[12] of the prefactor in Ref. 6; i.e., $\lambda_1(\tau)$

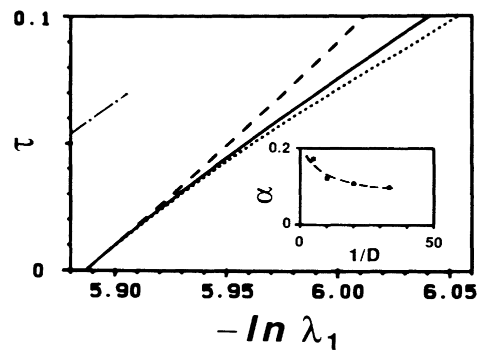

FIG. 2. The asymptotic behavior of $\lambda_1(\tau)$ for vanishing noise color at $D=0.05$. Solid line, numerical result; dashed line, $\lambda_1^{\mathrm{EFP}}(\tau)$ from Ref. 8; dotted line, $\lambda_1^{\mathrm{EFP}}(\tau)$ from Ref. 6; dash-dotted line, steepest-descent prediction $\lambda_1(\tau)=(\sqrt{2}/\pi)(1-1.5\tau)\exp(-1/4D)$.

$=\lambda_1(0)\exp(-1.5\tau)$.

In Fig. 1 we also depict (arrows) the steepest-descent approximation for the white-noise result, i.e.,

$$\lambda_1(\tau=0)=(\sqrt{2}/\pi)\exp(-1/4D). \tag{3}$$

This steepest-descent approximation exceeds the numerical result: For $D=0.05$ ($\phi_0=5$) the difference amounts to 9.4%, whereas for $D=0.03$ the error is reduced to 4.6%. The noise intensity D at which the error is below 1% has been evaluated to be $D\leq 0.016$ (i.e., $\phi_0 \gtrsim 15$).

Next we investigate the behavior for vanishing noise color. In Fig. 2 we depict for $D=0.05$ the behavior of $\lambda_1(\tau)$ as $\tau\to 0$, with τ/D being small. Note that the limiting asymptotics of $\tau\to 0$, but with $\tau/D\gg 1$, is clearly beyond our numerical possibilities. For comparison, we have also evaluated numerically the eigenvalue $\lambda_1^{\mathrm{EFP}}(\tau)$ of the corresponding effective Fokker-Planck (EFP) approximation of the conventional small-τ theory[6,7] (dotted line), and for the small-τ theory due to Fox[8] (dashed line). Furthermore, we have calculated numerically the integral expression for the corresponding mean first-passage time (MFPT), $T^{\mathrm{EFP}}(\tau)$, [see Eq. (4.2) in Ref. 6] and checked the relation $\lambda_1^{\mathrm{EFP}}(\tau)\cong 1/T^{\mathrm{EFP}}(\tau)$, being valid at weak noise.[17] For $D=0.05$ we find that $T^{\mathrm{EFP}}(\tau=0)$ exceeds $[\lambda_1^{\mathrm{EFP}}(\tau=0)]^{-1}$ by only 0.4% and the agreement becomes even better with increasing noise color τ. At $D=0.03$ the equality holds virtually without error. As $\tau/D\to 0$, there exist in the literature several predictions for the behavior of $\lambda_1(\tau)$ or $T(\tau)$, respectively. Setting

$$\lambda_1(\tau)=\lambda_1(0)(1-\beta\tau) \quad \text{for } \tau\to 0, \quad \tau/D\ll 1, \tag{4}$$

we find from Fig. 2 an initial slope of $\beta=1.33\pm 0.01$. For $D=0.03$ we obtain similar results with $\beta=1.44\pm 0.02$. This β value can be compared with published (and as yet unpublished) results valid for vanishing noise color: All these small-τ theories[6,8,11-14] yield in the limit $\tau\to 0$, $\tau/D\ll 1$ the unique steepest-descent prediction $\beta=1.5$.

The authors of Ref. 13d derive an expression for the MFPT to reach the line ($x=0$, $\xi<0$) starting at $x=1$. Their MFPT exhibits a behavior $\beta=1.165\tau^{-0.5}+1.5$ as $\tau\to 0$. This result, however, does not compare directly with the eigenvalue $\lambda_1(\tau)$ being related to the MFPT to reach the *separatrix.*[17]

In conclusion, we have presented a numerical study for the generic colored-noise bistability in (1). Our emphasis has been both on the small-τ and the moderate to large-τ behavior. Many of the various confusing theoretical predictions have been tested against our precise study at weak noise. In the limit of vanishing noise color $\tau\to 0$, $\tau/D\ll 1$, there now exists agreement among all advocates of small-τ theories.[6,8,11-14] With increasing noise color, however, there occurs a crossover regime to a behavior $\lambda_1(\tau)\propto\exp(-\alpha\tau/D)$, with $\alpha\approx 0.1$. This behavior is followed by yet another very slow crossover at very large τ to a limiting law with $\alpha=\frac{2}{27}=0.074$. Clearly, it is difficult to derive an approximation for moderate to large noise color which accurately reproduces the precise numerical results for the non-Markovian dynamics in (1). In the latter regime, the decoupling approximation in Ref. 9, the unified approximation in Ref. 10, and the large-τ studies in Refs. 20 and 22 serve as useful guides to explore this difficult, but physically most relevant, regime[3-5] of noise color of moderate strength.

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$$\lambda_1(\tau)=(\sqrt{2}/\pi)(1+3\tau)^{-1/2}\times\exp[-(4D)^{-1}(1+\tfrac{27}{16}\tau+\tfrac{1}{2}\tau^2)/(1+\tfrac{27}{16}\tau)],$$

i.e.,

$$\lambda_1(\tau)=(\sqrt{2}/\pi)(3\tau)^{-1/2}\times\exp[-(4D)^{-1}(0.8244+\tfrac{8}{27}\tau+0.104/\tau)]$$

as $\tau\to\infty$. The value of the asymptotic slope $d[\ln\lambda_1(\tau)]/d\tau=2/27D=0.074/D$ has also been determined by G. P. Tsironis and P. Grigolini (to be published).

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$$\lambda_1(\tau)=[\tfrac{27}{2}\pi D(\tau+\tfrac{1}{2})]^{-1/2}\exp[-(4D)^{-1}(\tfrac{4}{9}+\tfrac{8}{27}\tau)]$$

[P. Hänggi, P. Jung, and F. Marchesoni, "Escape Driven by Strongly Correlated Noise" (to be published)].