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Asymptotic Goodness-of-Fit Tests for Point Processes Based on Scaled Empirical K-Functions

Lothar Heinrich¹

Abstract

We study sequences of scaled edge-corrected empirical (generalized) K -functions (modifying Ripley's K -function) each of them constructed from a single observation of a d -dimensional fourth-order stationary point process in a sampling window W_n which grows together with some scaling rate unboundedly as $n \rightarrow \infty$. Under some natural assumptions it is shown that the normalized difference between scaled empirical and scaled theoretical K -function converges weakly to a mean zero Gaussian process with simple covariance function. This result suggests discrepancy measures between empirical and theoretical K -function with known limit distribution which allow to perform goodness-of-fit tests for checking a hypothesized point process based only on its intensity and (generalized) K -function. Similar test statistics are derived for testing the hypothesis that two independent point processes in W_n have the same distribution without explicit knowledge of their intensities and K -functions.

Keywords fourth-order Brillinger mixing point process · generalized K -function · edge-corrected empirical K -function · scaling rate · Skorohod-space $D[0, R]$ · functional central limit theorem · one- and two-sample tests for point processes.

AMS 2010 Subject Classifications Primary: 62 G 10 · 60 G 55; Secondary: 60 F 05 · 60 F 17

1 Introduction

In many fields of application statisticians are faced with irregular point patterns or point-like objects which are randomly distributed in large planar or spatial domains. Random point processes provide appropriate models to describe such phenomena. It is often assumed and in practice at least approximately justifiable that the distribution of the point pattern under consideration or at least its moment measures up to certain order are invariant under translations. In this paper all point processes and random variables are defined over a common probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ and \mathbf{E} , \mathbf{Var} , \mathbf{Cov} denote expectation, variance, covariance w.r.t. \mathbf{P} .

The aim of the paper is to establish asymptotic goodness-of-fit tests for checking point process hypotheses provided the hypothesized d -dimensional simple point process (short: PP) $N = \sum_{i \geq 1} \delta_{X_i}$ with distribution P (short: $N \sim P$) is fourth-order stationary with known intensity $\lambda := \mathbf{E}N([0, 1]^d) > 0$ and a known generalized K -function $0 \leq r \mapsto K_B(r) := \lambda^{-1} \alpha_{red}^{(2)}(rB)$, where $\alpha_{red}^{(2)}(\cdot)$ denotes the reduced second factorial moment measure which always exists when N

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is at least second-order (or weakly) stationary, see Daley and Vere-Jones (1988) for mathematical background of PP theory. Here, B is assumed to be an \mathbf{o} -symmetric, convex, compact set containing the origin $\mathbf{o} \in \mathbb{R}^d$ as inner point. Equivalently, B can be considered as closed unit ball w.r.t. a unique norm $\|\cdot\|_B$ defined by $\|x\|_B := \inf\{r > 0 : x \in rB\}$ for $x \in \mathbb{R}^d$. In case of the Euclidean norm $\|\cdot\|$ with unit ball B_e the subscript B is omitted and $K(r)$ coincides with Ripley's K -function as introduced in Ripley (1976) for motion-invariant PPs. Our approach is based on a single observation of N in a large sampling window W_n , where throughout this paper (W_n) forms a *convex averaging sequence* (short: CAS), i.e. $W_n \subseteq W_{n+1}$ and W_n is convex, compact in \mathbb{R}^d for all $n \geq 1$ such that $\varrho(W_n) := \sup\{r > 0 : x + rB_e \subseteq W_n \text{ for some } x \in W_n\} \xrightarrow[n \rightarrow \infty]{} \infty$.

Instead to impose one of the usual strong mixing conditions on $N \sim P$ we will assume that the the second, third and fourth reduced cumulant measure have bounded total variation and that the ratio $(N(W_n) - \lambda|W_n|)/\sqrt{|W_n|}$ is asymptotically normally distributed (as $n \rightarrow \infty$) with an *asymptotic variance* $\sigma^2 := \lim_{n \rightarrow \infty} \mathbf{Var}(N(W_n))/|W_n|$. Here and in what follows, $|\cdot|$ stands for the Lebesgue measure or volume in \mathbb{R}^d as well as for the absolute value of a real number.

A frequently used unbiased (edge-corrected) estimator of $\lambda^2 K_B(r)$ (so far for $B = B_e$) is the so-called *Ohser-Stoyan- or Horvitz-Thompson-type estimator* defined by

$$(\widehat{\lambda^2 K_B})_n(r) := \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{rB}(X_j - X_i)}{|(W_n - X_i) \cap (W_n - X_j)|}, \quad (1)$$

see Ohser and Stoyan (1981), Stein (1993), Diggle (2003), Chiu et al. (2013), Chapt. 4.7.4, or Illian et al. (2008), Chapt. 4.3.3, for further details and modifications. On the r.h.s of (1) the sum \sum^{\neq} runs over pairs of distinct indices and the indicator function $\mathbf{1}_{rB}(X_j - X_i)$ can be replaced by $\mathbf{1}_{[0,r]}(\|X_j - X_i\|_B)$. It is plausible that the use of several norms $\|\cdot\|_B$ provide more information on the PP to be examined. In this connection B plays a similar role like *structuring elements* in image analysis, see Chiu et al.(2013). It should be mentioned that there exist a number of further estimators based on second-order characteristics to obtain information on stationary point patterns (orientation, isotropy etc.), see e.g. Guan et al. (2006), Guan and Sherman (2007), Illian et al. (2008), Chapt. 4.5.3. Baddeley et al. (2000) introduced an inhomogeneous K -function and its empirical counterpart for intensity-reweighted stationary PPs, see also Gaetan and Guyon (2010), Chapt. 3.5.3, Adelfio and Schoenberg (2009) and Zhao and Wang (2010) for asymptotic properties.

In the most important case when $N \sim P$ is a stationary Poisson process the weak convergence of the empirical process $\sqrt{|W_n|}((\widehat{\lambda^2 K})_n(r) - \lambda^2|B_e|r^d)$ in the Skorohod space $D[0, R]$, see Billingsley (1968), to a Gaussian process has been proved in Heinrich (1991). The corresponding functional central limit (short: CLT) for the empirical process (1) can be show in the same way, see Heinrich (2013). The known limit distribution of $\sqrt{|W_n|} \max_{0 \leq r \leq R} |(\widehat{\lambda^2 K_B})_n(r) - \lambda^2|B|r^d|$ (in particular when λ^2 is replaced by an unbiased estimator) allows to perform Kolmogorov-Smirnov-type as well as Cramér-von Mises-type tests to check *complete spatial randomness (CSR)* of a given point pattern in W_n . Multiparameter versions of this and related asymptotic tests are derived in Heinrich (2015). Note that testing for CSR of a transformed PP can provide information on the original PP, see Schoenberg (1999). There exist further (non-asymptotic)

tests for CSR based on Ripley's K-function using different estimators supported by data analytic methods and simulation techniques, see e.g. Ho and Chiu (2010), Marcon et al. (2013), Wiegand et al. (2016) and further references therein.

Asymptotic normality of the family of random variables $\sqrt{|W_n|} ((\widehat{\lambda^2 K_B})_n(r) - \lambda^2 K_B(r))$ can be shown due to a more general CLT proved first by Jolivet (1980), see also Karr (1987) and Ki u and Mora (1999), if the underlying PP $N \sim P$ is Brillinger mixing, see Section 2. The main obstacle to construct a goodness-of-fit test as in the Poisson case is the rather complicated covariance function of the Gaussian limit process depending on λ and integrals w.r.t. $\gamma_{red}^{(k)}$, $k = 2, 3, 4$, see (7). A closed-term formula of this covariance function is given at the end of Appendix B, whereas a more explicit form of this function is known e.g. for α -determinantal PPs, see Heinrich (2016). An expression for the asymptotic variance of $\sqrt{|W_n|} (\widehat{\lambda^2 K_B})_n(r)$ for $r > 0$ is given in (45).

To realize the announced program of testing hypotheses on the distribution of non-Poisson PPs via their generalized K -functions $K_B(r)$ we allow that the range of r grows unboundedly with a scaling rate c_n^α , where $c_n := |W_n|^{1/d}$ and $0 < \alpha < 1$. More precisely, our main issue is to study the asymptotic behaviour of the sequence of the scaled (centered and normalized) empirical processes

$$\Delta_{B,n}^{(\alpha)}(r) := |W_n|^{1/2-\alpha} \left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 K_B(c_n^\alpha r) \right) \quad \text{for } 0 \leq r \leq R, \quad 0 \leq \alpha < 1 \quad (2)$$

as $n \rightarrow \infty$, where R is a freely chosen positive real number. Under mild (mixing) assumptions we shall show in Section 3 that the asymptotic variance and covariance of (2) does not depend on $\gamma_{red}^{(3)}$ and $\gamma_{red}^{(4)}$. Furthermore, (2) possesses a Gaussian limit for fixed $r > 0$ and in $D[0, R]$ for all $0 < \alpha < 1$, but only for $\alpha \leq 1/2$ this results are relevant for our tests. The latter restriction is meaningful because $\|\gamma_{red}^{(2)}\|_{var} < \infty$ implies for $\alpha > 1/2$ that

$$\sup_{0 \leq r \leq R} |W_n|^{1/2-\alpha} \left| \lambda^2 K_B(c_n^\alpha r) - \lambda^2 |c_n^\alpha r B| \right| = \lambda |W_n|^{1/2-\alpha} \sup_{0 \leq r \leq R} \left| \gamma_{red}^{(2)}(c_n^\alpha r B) \right| \xrightarrow[n \rightarrow \infty]{} 0. \quad (3)$$

This means that we can always take $K_B(r) = |r B| = |B| r^d$ in (2) which in turn prevents to distinguish between Poisson and non-Poisson PPs if $\alpha > 1/2$. On the other hand, such a separation is possible for $\alpha \in (0, 1/2]$, where the limit distribution of (2) depends on the intensity λ and asymptotic variance σ^2 of the PP $N \sim P$. In particular for $\alpha = 1/2$, under comparatively mild assumptions on $N \sim P$ we can show that

$$\frac{1}{\widehat{\lambda}_n \sqrt{(\widehat{\sigma^2})_n}} \max_{0 \leq r \leq \sqrt{c_n} R} \left| (\widehat{\lambda^2 K_B})_n(r) - \lambda^2 K_B(r) \right| \xrightarrow[n \rightarrow \infty]{d} 2 R^d |B| |\mathcal{N}(0, 1)|, \quad (4)$$

where $(\widehat{\sigma^2})_n$ and $\widehat{\lambda}_n$ are defined in (12) resp. (13), $\mathcal{N}(0, \tau^2)$ denotes a mean zero Gaussian random variable with variance τ^2 and $\xrightarrow[n \rightarrow \infty]{d}$ indicates *convergence in distribution* or *weak convergence*.

More general limit theorems are presented, proved and applied in the Sections 3 - 5 to construct the above-announced goodness-of-fit tests. This includes tests of the null hypothesis that the distributions P_a and P_b of two independent stationary PPs N_a and N_b coincide based on checking the coincidence of their K -functions $K_{a,B}$ and $K_{b,B}$ and their intensities λ_a and λ_b .

2 Definitions and Preliminary Results

A simple point process $N = \sum_{i \geq 1} \delta_{X_i}$ on \mathbb{R}^d is said to be *kth-order stationary* for some $k \geq 1$ if $\mathbf{E}N^k([0, 1]^d) < \infty$ and, for $j = 1, \dots, k$,

$$\alpha^{(j)}\left(\bigotimes_{i=1}^j (A_j + x)\right) = \alpha^{(j)}\left(\bigotimes_{i=1}^j A_i\right) \quad \text{for all } x \in \mathbb{R}^d \quad \text{and any bounded } A_1, \dots, A_j \in \mathcal{B}^d.$$

Here $\alpha^{(k)}(\cdot)$ denotes the *k-th factorial moment measure* which is defined on $[\mathbb{R}^{dk}, \mathcal{B}^{dk}]$ by

$$\alpha^{(k)}\left(\bigotimes_{i=1}^k A_i\right) = \mathbf{E} \sum_{i_1, \dots, i_k \geq 1}^{\neq} \mathbf{1}_{A_1}(X_{i_1}) \cdots \mathbf{1}_{A_k}(X_{i_k}),$$

where the sum \sum^{\neq} stretches over k -tuples of pairwise distinct indices.

The invariance of $\alpha^{(k)}(\cdot)$ against diagonal-shifts (if $k \geq 2$) allows the disintegration w.r.t. the Lebesgue measure $|\cdot|$ (multiplied additionally by λ) and defines the *reduced k-th factorial moment measure* $\alpha_{red}^{(k)}(\cdot)$ (on $[\mathbb{R}^{d(k-1)}, \mathcal{B}^{d(k-1)}]$) satisfying

$$\alpha^{(k)}\left(\bigotimes_{i=1}^k A_i\right) = \lambda \int_{A_k} \alpha_{red}^{(k)}\left(\bigotimes_{i=1}^{k-1} (A_i - x)\right) dx \quad \text{for any bounded } A_1, \dots, A_k \in \mathcal{B}^d,$$

where, for bounded $W, A_2, \dots, A_k \in \mathcal{B}^d$ with $|W| > 0$, the measure $\alpha_{red}^{(k)}(\cdot)$ can be expressed by

$$\alpha_{red}^{(k)}\left(\bigotimes_{i=2}^k A_i\right) = \frac{1}{\lambda |W|} \mathbf{E} \sum_{i_1, \dots, i_k \geq 1}^{\neq} \mathbf{1}_W(X_{i_1}) \mathbf{1}_{A_2}(X_{i_2} - X_{i_1}) \cdots \mathbf{1}_{A_k}(X_{i_k} - X_{i_1}).$$

Next we recall formal definition of *k-th factorial cumulant measure* $\gamma^{(k)}(\cdot)$ which is a signed measure on $[\mathbb{R}^{dk}, \mathcal{B}^{dk}]$ defined by

$$\gamma^{(k)}\left(\bigotimes_{i=1}^k A_i\right) = \sum_{j=1}^k (-1)^{j-1} (j-1)! \sum_{K_1 \cup \dots \cup K_j = \{1, \dots, k\}} \prod_{i=1}^j \alpha^{(\#K_i)}\left(\bigotimes_{k_i \in K_i} A_{k_i}\right), \quad (5)$$

where the inner sum is taken over all partitions of the set $\{1, \dots, k\}$ into j disjoint non-empty subsets K_1, \dots, K_j , see Daley and Vere-Jones (1988), Heinrich (2013).

A corresponding representation formula of $\alpha^{(k)}(\cdot)$ in terms of factorial cumulant measures can be derived from (5) by induction,

$$\alpha^{(k)}\left(\bigotimes_{i=1}^k A_i\right) = \sum_{j=1}^k \sum_{K_1 \cup \dots \cup K_j = \{1, \dots, k\}} \prod_{i=1}^j \gamma^{(\#K_i)}\left(\bigotimes_{k_i \in K_i} A_{k_i}\right). \quad (6)$$

Since k th-order stationarity implies diagonal-shift invariance of the locally finite signed measures $\gamma^{(j)}(\cdot)$ for $j = 1, \dots, k$, the above disintegration also applies for factorial cumulant measures. In this way a k th-order stationary PP is accompanied by reduced factorial cumulant measures $\gamma_{red}^{(j)}(\cdot)$ of order $j = 2, \dots, k$, defined by

$$\gamma^{(j)}\left(\times_{i=1}^j A_i\right) = \lambda \int_{A_j} \gamma_{red}^{(j)}\left(\times_{i=1}^{j-1} (A_i - x)\right) dx. \quad (7)$$

For $j = 1$ it makes sense to put $\gamma_{red}^{(1)}(\cdot) = \alpha_{red}^{(1)}(\cdot) \equiv 1$, whereas the signed measure $\gamma_{red}^{(2)}(\cdot) = \alpha_{red}^{(2)}(\cdot) - \lambda|\cdot|$ (called *reduced covariance measure*) reflects the dependence between two distinct atoms of N and $\gamma_{red}^{(j)}(\cdot)$ describes mutual dependences between j distinct atoms.

If both the positive and negative part $(\gamma_{red}^{(k)})^\pm(\cdot)$ of Jordan's decomposition of $\gamma_{red}^{(k)}(\cdot)$ are finite on $\mathbb{R}^{d(k-1)}$, then

$$\|\gamma_{red}^{(k)}\|_{var} := (\gamma_{red}^{(k)})^+(\mathbb{R}^{d(k-1)}) + (\gamma_{red}^{(k)})^-(\mathbb{R}^{d(k-1)})$$

is called the *total variation* of $\gamma_{red}^{(k)}(\cdot)$. A k th-order stationary PP $N = \sum_{i \geq 1} \delta_{X_i}$ on \mathbb{R}^d is said to be B_k -mixing if $\|\gamma_{red}^{(j)}\|_{var} < \infty$ for $j = 2, \dots, k$. B_k -mixing for all $k \geq 2$ is called *Brillinger-mixing*, see e.g. Jolivet (1980), Karr (1987), Heinrich (2016), Biscio and Lanvancier (2016),(2017) for numerous applications in PP statistics. It should be mentioned that B_k -mixing of $N \sim P$ even for all $k \geq 2$ does not imply the uniqueness of the stationary distribution P , see also Baddeley and Silverman (1984) for $k = 2$.

Next, we state some technical results which will be needed to prove the below Theorems 1-5.

Lemma 1 For any sequence r_n of positive numbers satisfying $r_n/\varrho(W_n) \xrightarrow{n \rightarrow \infty} 0$, we have

$$\frac{1}{|W_n|} \sup_{x \in r_n B} (|W_n| - |W_n \cap (W_n - x)|) = \mathcal{O}\left(\frac{r_n}{\varrho(W_n)}\right) \quad \text{as } n \rightarrow \infty.$$

Proof of Lemma 1 Since $B = \{x \in \mathbb{R}^d : \|x\|_B \leq 1\}$ is bounded there exist some $\kappa > 0$ (depending on B) such that $B \subseteq \kappa B_e$. By definition of Minkowski-subtraction \ominus , see Chiu et al., p. 6, it follows that

$$W_n \ominus (r \kappa B_e) \subseteq W_n \ominus (r B) = \bigcap_{y \in r B} (W_n - y) \subseteq W_n \cap (W_n - x) \quad \text{for any } x \in r B.$$

Applying the first inequality (50) of Lemma 5 to $K = W_n$ we see that

$$\sup_{x \in r B} \left(1 - \frac{|W_n \cap (W_n - x)|}{|W_n|}\right) \leq 1 - \frac{|W_n \ominus (r \kappa B_e)|}{|W_n|} \leq \frac{d \kappa r}{\varrho(W_n)} \quad \text{for any } r \in [0, \varrho(W_n)/\kappa].$$

Setting $r = r_n$ and letting $n \rightarrow \infty$ complete the proof of Lemma 1. \square

The following Lemma 2 turns out crucial to prove the Theorems 1 and 2 in Section 3.

Lemma 2 Let $N = \sum_{i \geq 1} \delta_{X_i}$ be a B_4 -mixing PP on \mathbb{R}^d . Further, assume that $c_n^\alpha / \varrho(W_n) \xrightarrow[n \rightarrow \infty]{} 0$ for some $0 < \alpha < 1$. Then

$$\lim_{n \rightarrow \infty} \mathbf{E} \left(\Delta_{B,n}^{(\alpha)}(r) - 2 \frac{\alpha_{red}^{(2)}(c_n^\alpha r B)}{|W_n|^\alpha} \frac{N(W_n) - \lambda |W_n|}{\sqrt{|W_n|}} \right)^2 = 0 \quad \text{for all } r \geq 0.$$

The next result becomes relevant in proving the tightness of the sequence $(\Delta_{B,n}^{(\alpha)})_n$.

Lemma 3 Let $N = \sum_{i \geq 1} \delta_{X_i}$ be a B_4 -mixing PP on \mathbb{R}^d . Further, let there exist a uniformly bounded Lebesgue density of $\alpha_{red}^{(2)}(\cdot)$, i.e. there exist a constant $a_0 > 0$ such that

$$\alpha_{red}^{(2)}(A) \leq a_0 |A| \quad \text{for all bounded } A \in \mathcal{B}^d. \quad (8)$$

Then, for $0 \leq \alpha < 1$, $0 \leq s < t \leq R$ and all $n \geq n_0(R)$,

$$\mathbf{E} \left(\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s) \right)^2 \leq a_1 |(tB) \setminus (sB)|^2 + a_2 |(tB) \setminus (sB)| |W_n|^{-\alpha} + a_3 |W_n|^{-2\alpha}, \quad (9)$$

where the constants a_1, a_2, a_3 depend on $a_0, \lambda, \|\gamma_{red}^{(2)}\|_{var}, \|\gamma_{red}^{(3)}\|_{var}, \|\gamma_{red}^{(4)}\|_{var}$.

Remark 1 If, in addition to (8), the Lebesgue densities $\kappa^{(3)}$ and $\kappa^{(4)}$ of $\gamma_{red}^{(3)}(\cdot)$ resp. $\gamma_{red}^{(4)}(\cdot)$ exist such that, for all $x, y \in \mathbb{R}^d$,

$$|\kappa^{(3)}(x, y)| \leq a_{01} \quad \text{and} \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\kappa^{(4)}(x, y + z, z)| dz \right)^2 dy dx \leq a_{02}, \quad (10)$$

then, for $0 \leq \alpha < 1$, $0 \leq s < t \leq R$ and all $n \geq n_0(R)$,

$$\mathbf{E} \left(\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s) \right)^2 \leq a_1 |(tB) \setminus (sB)|^2 + a_2 |(tB) \setminus (sB)| |W_n|^{-\alpha}, \quad (11)$$

where the constants a_0, a_1, a_2 may differ from those in (8) and (9), respectively.

The proofs of Lemma 2 and Lemma 3 as well as of (11) are postponed to Section 5.

Finally, we introduce an estimator of the asymptotic variance $\sigma^2 = \lim_{n \rightarrow \infty} \mathbf{Var}(N(W_n))/|W_n| = \lambda(1 + \gamma_{red}^{(2)}(\mathbb{R}^d))$ which generalizes a corresponding estimator for cubic windows $W_n = [-n, n]^d$ defined and studied in Heinrich (1994), Heinrich and Prokešová (2010).

Let $w : \mathbb{R}^d \mapsto [0, \infty)$ be a Borel-measurable, bounded, symmetric function with bounded support satisfying $\lim_{x \rightarrow \mathbf{o}} w(x) = w(\mathbf{o}) = 1$, e.g. $w(x) = \mathbf{1}_B(x)$ or $w(x) = (1 - \|x\|) \mathbf{1}_B(x)$.

$$(\widehat{\sigma^2})_n := \widehat{\lambda}_n + \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j)}{|(W_n - X_i) \cap (W_n - X_j)|} w\left(\frac{X_j - X_i}{b_n c_n}\right) - (\widehat{\lambda^2})_n (b_n c_n)^d \int_{\mathbb{R}^d} w(x) dx, \quad (12)$$

where (b_n) is a positive null sequence of bandwidths such that $c_n b_n \xrightarrow[n \rightarrow \infty]{} \infty$, and

$$\widehat{\lambda}_n := \frac{N(W_n)}{|W_n|} \quad \text{and} \quad (\widehat{\lambda^2})_n := \frac{(N(W_n) - 1)N(W_n)}{|W_n|^2}. \quad (13)$$

The asymptotic properties of $(\widehat{\sigma^2})_n$ summarized in

Lemma 4 Let (W_n) be a CAS in \mathbb{R}^d with inball radius $\varrho(W_n)$ and $N = \sum_{i \geq 1} \delta_{X_i}$ be a B_2 -mixing PP on \mathbb{R}^d . Then $(\widehat{\sigma^2})_n$ is asymptotically unbiased for σ^2 , i.e.

$$\mathbf{E}(\widehat{\sigma^2})_n \xrightarrow{n \rightarrow \infty} \sigma^2.$$

If, additionally, the PP N is B_4 -mixing and $c_n b_n^2 \xrightarrow{n \rightarrow \infty} 0$ such that $c_n b_n / \varrho(W_n) \xrightarrow{n \rightarrow \infty} 0$, then $(\widehat{\sigma^2})_n$ is mean square consistent, i.e.

$$\mathbf{E}((\widehat{\sigma^2})_n - \sigma^2)^2 \xrightarrow{n \rightarrow \infty} 0.$$

Proof of Lemma 4 In view of the definitions of factorial moment and cumulant measures and their reduced versions it is easily seen after some rearrangements that

$$\mathbf{E}(\widehat{\sigma^2})_n = \lambda + \lambda \int_{\mathbb{R}^d} w\left(\frac{y}{b_n c_n}\right) \gamma_{red}^{(2)}(dy) - \frac{b_n^d}{|W_n|} \gamma^{(2)}(W_n \times W_n) \int_{\mathbb{R}^d} w(x) dx.$$

Since $\|\gamma_{red}^{(2)}\|_{var} < \infty$, $b_n \xrightarrow{n \rightarrow \infty} 0$ and $b_n c_n \xrightarrow{n \rightarrow \infty} \infty$, the properties of the function $w(\cdot)$ imply the limit $\mathbf{E}(\widehat{\sigma^2})_n \xrightarrow{n \rightarrow \infty} \lambda(1 + \gamma_{red}^{(2)}(\mathbb{R}^d))$. This proves the first assertion of Lemma 4.

The proof of the second assertion is somewhat more complicated because integrals w.r.t. third- and fourth-order factorial moment and cumulant measures have to be estimated. However, apart from obvious changes, this proof equals almost verbatim that of Theorem 1 in [18] (which states Lemma 4 for $W_n = [-n, n]^d$). Therefore we omit the details and refer the reader to Heinrich and Prokešová (2010). \square

3 Limit Theorems for Scaled Empirical K-Functions

Theorem 1 Let $N = \sum_{i \geq 1} \delta_{X_i}$ be a B_4 -mixing PP on \mathbb{R}^d satisfying

$$\sqrt{|W_n|}(\widehat{\lambda}_n - \lambda) = \frac{N(W_n) - \lambda|W_n|}{\sqrt{|W_n|}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \sigma^2) \quad \text{with} \quad \sigma^2 = \lambda(1 + \gamma_{red}^{(2)}(\mathbb{R}^d)). \quad (14)$$

If $c_n^\alpha / \varrho(W_n) \xrightarrow{n \rightarrow \infty} 0$ for some $\alpha \in (0, 1)$, then

$$\Delta_{B,n}^{(\alpha)}(r) \xrightarrow[n \rightarrow \infty]{d} G_B(r) := 2\lambda|B|r^d \mathcal{N}(0, \sigma^2) \quad \text{for any fixed } r \geq 0 \quad (15)$$

and

$$(\Delta_{B,n}^{(\alpha)}(r_i))_{i=1}^k \xrightarrow[n \rightarrow \infty]{d} (G_B(r_i))_{i=1}^k \quad \text{for any fixed } k\text{-tuple } r_1, \dots, r_k \geq 0, k \geq 1. \quad (16)$$

Proof of Theorem 1 Lemma 2 and Chebyshev's inequality imply that

$$X_n(r) := \Delta_{B,n}^{(\alpha)}(r) - 2 \frac{\alpha_{red}^{(2)}(c_n^\alpha r B)}{|W_n|^\alpha} \frac{N(W_n) - \lambda |W_n|}{\sqrt{|W_n|}} \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0 \quad \text{for all } r \geq 0. \quad (17)$$

In other words, $X_n(r)$ converges in probability to zero, that is, $\mathbf{P}(|X_n(r)| \geq \varepsilon) \xrightarrow[n \rightarrow \infty]{} 0$ for all $\varepsilon > 0$. Now, we rearrange the terms in (17) so that

$$\Delta_{B,n}^{(\alpha)}(r) = X_n(r) + Y_n(r) + 2\lambda |r B| \frac{N(W_n) - \lambda |W_n|}{\sqrt{|W_n|}}, \quad (18)$$

where

$$Y_n(r) := 2\gamma_{red}^{(2)}(c_n^\alpha r B) \frac{N(W_n) - \lambda |W_n|}{|W_n|^{\alpha+1/2}}.$$

Since $\alpha > 0$, it follows from $\|\gamma_{red}^{(2)}\|_{var} < \infty$ and (14) that $Y_n(r)$ also converges in probability to zero. Finally, condition (14) and Slutsky's lemma, see Billingsley (1968), Chapt. 1, provide the assertion (15) of Theorem 1. To prove (16) we use the Cramér-Wold device, see Billingsley (1968), Chapt. 1, and show that, for any $\xi_1, \dots, \xi_k \in \mathbb{R}^1$ satisfying $\xi_1^2 + \dots + \xi_k^2 \neq 0$, the linear combination

$$\sum_{i=1}^k \xi_i \Delta_{B,n}^{(\alpha)}(r_i) = \sum_{i=1}^k \xi_i X_n(r_i) + \sum_{i=1}^k \xi_i Y_n(r_i) + 2\lambda |B| \frac{N(W_n) - \lambda |W_n|}{\sqrt{|W_n|}} \sum_{i=1}^k \xi_i r_i^d$$

converges in distribution to the Gaussian random variable $2\lambda |B| \sum_{i=1}^k \xi_i r_i^d \mathcal{N}(0, \sigma^2)$. As before in the proof of (15), this follows immediately by applying (14), $X_n(r_i) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0$ and $Y_n(r_i) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0$ for $i = 1, \dots, k$ together with Slutsky's lemma. \square

Corollary 1 The multivariate CLT (16) and (14) give rise to define a χ^2 -type test statistic

$$T_{B,n}^{(\alpha)}(r_1, \dots, r_k) := \frac{1}{\lambda^2 \sigma^2} \sum_{i=1}^k \left(\frac{\Delta_{B,n}^{(\alpha)}(r_i) - \Delta_{B,n}^{(\alpha)}(r_{i-1})}{r_i^d - r_{i-1}^d} \right)^2 + \frac{|W_n|}{\sigma^2} (\hat{\lambda}_n - \lambda)^2.$$

For any $0 := r_0 < r_1 < \dots < r_k < \infty$, $k \geq 1$ and all $\alpha \in (0, 1)$ it holds

$$T_{B,n}^{(\alpha)}(r_1, \dots, r_k) \xrightarrow[n \rightarrow \infty]{d} (4k|B|^2 + 1) \mathcal{N}(0, 1)^2,$$

where the limit remains the same when λ^2 and σ^2 are replaced by their mean-square consistent estimators (13) and (12), respectively, provided the assumptions of Lemma 4 are satisfied.

Proof of Corollary 1 Using (18) for $r \in \{r_1, \dots, r_k\}$ gives

$$\frac{\Delta_{B,n}^{(\alpha)}(r_i) - \Delta_{B,n}^{(\alpha)}(r_{i-1})}{r_i^d - r_{i-1}^d} = 2\lambda |B| \sqrt{|W_n|} (\hat{\lambda}_n - \lambda) + \frac{X_n(r_i) + Y_n(r_i) - X_n(r_{i-1}) - Y_n(r_{i-1})}{r_i^d - r_{i-1}^d}$$

for $i = 1, \dots, k$. By squaring the previous equality and summing up all squares we get together with (14) and Slutsky's lemma the assertion of Corollary 1. \square

Remark 2 The CLT (14) for the number $N(W_n)$ can be verified for various classes of stationary PPs such as Poisson cluster processes, see Heinrich (1988), or α -determinantal PPs, see e.g. Heinrich (2016) and references therein. Note that for any Brillinger-mixing PP condition (14) is satisfied, see e.g. Jolivet (1980), Karr (1987) or Biscio and Lavancier (2017).

It is noteworthy that the squared intensity λ^2 occurring in $\Delta_{B,n}^{(\alpha)}(r)$ cannot be replaced by the asymptotically unbiased estimator (13) without changing the limit as reveals

Remark 3 Under the conditions of Lemma 2 we have

$$\lim_{n \rightarrow \infty} |W_n|^{1-2\alpha} \mathbf{E} \left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - (\widehat{\lambda^2})_n K_B(c_n^\alpha r) \right)^2 = 0 \quad \text{for all } r \geq 0. \quad (19)$$

The proof of (19) is based on the same arguments and techniques as used to prove Lemma 2. For the interested reader a detailed proof is given Appendix B.

On the other hand, (14) implies that $(\widehat{\lambda^2})_n$ is asymptotically normally distributed which follows easily by applying Slutsky's lemma:

$$\sqrt{|W_n|} \left((\widehat{\lambda^2})_n - \lambda^2 \right) = \sqrt{|W_n|} (\widehat{\lambda}_n - \lambda) (\widehat{\lambda}_n + \lambda) - \frac{\widehat{\lambda}_n}{\sqrt{|W_n|}} \xrightarrow[n \rightarrow \infty]{d} 2\lambda \mathcal{N}(0, \sigma^2).$$

Theorem 2 In addition to the assumptions of Theorem 1, suppose that (8) is satisfied. Then, for $1/4 < \alpha < 1$, the weak convergence

$$\{\Delta_{B,n}^{(\alpha)}(r) : r \in [0, R]\} \xrightarrow[n \rightarrow \infty]{d} \{G_B(r) := 2\lambda |B| r^d \mathcal{N}(0, \sigma^2) : r \in [0, R]\} \quad (20)$$

holds in the Skorohod-space $D[0, R]$. If, in addition to (8), the Lebesgue densities of $\gamma_{red}^{(3)}$ and $\gamma_{red}^{(4)}$ exist and satisfy (10), then the weak convergence (20) in $D[0, R]$ holds for $0 < \alpha < 1$.

Proof of Theorem 2 The multivariate CLT (16) states that all finite-dimensional distributions of $(\Delta_{B,n}^{(\alpha)}(r))_{r \geq 0}$ converge to the corresponding distributions of the continuous mean zero Gaussian process $(G_B(r))_{r \geq 0}$ with covariance function $\mathbf{E}G_B(s)G_B(t) = 4\lambda^2\sigma^2|B|^2s^d t^d$. To verify the tightness of the sequence of random functions $\Delta_{B,n}^{(\alpha)} := (\Delta_{B,n}^{(\alpha)}(r))_{r \in [0, R]}$ in the Skorohod-space $D[0, R]$, see Billingsley (1968), Chapt. 3, we shall show that for each $\varepsilon > 0$ and $\eta > 0$ there exists a $\delta \in (0, 1)$, such that

$$\mathbf{P}(w(\Delta_{B,n}^{(\alpha)}, \delta) \geq \varepsilon) \leq \eta \quad \text{for all sufficiently large } n, \quad (21)$$

where $w(f, \delta) := \sup\{|f(t) - f(s)| : |t - s| \leq \delta, 0 \leq s, t \leq R\}$ denotes the modulus of continuity of a so-called càdlàg-functions $f : [0, R] \mapsto \mathbb{R}^1$ being right-continuous on $[0, R]$ and having limits from the left on $(0, R]$, i.e. $f \in D[0, R]$.

Since rB is compact for all $r \geq 0$ and $sB \subseteq tB$ for $s \leq t$, the function $r \mapsto (\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$ is piecewise constant, non-decreasing, right-continuous on $[0, R]$ (\mathbf{P} -a.s.) with upward-jumps

at $c_n^{-\alpha} \|X_i - X_j\|_B$ for $i \neq j$. Furthermore, $\lambda K_B(c_n^\alpha r) = \alpha_{red}^{(2)}(c_n^\alpha r B)$ is also non-decreasing and right-continuous (even absolutely continuous due (8)) for $r \geq 0$. Hence, by (1) we have $\Delta_{B,n}^{(\alpha)}(0) = 0$ and $\mathbf{P}(\Delta_{B,n}^{(\alpha)} \in D[0, R]) = 1$ for all $n \geq 1$.

Since $|(tB) \setminus (sB)| = (t^d - s^d) |B| \leq d R^{d-1} |B| (t - s)$ for $0 \leq s \leq t \leq R$, Lemma 3 yields

$$\mathbf{E}(\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s))^2 \leq a_1 (d R^{d-1} |B|)^2 (t - s)^2 + d R^{d-1} |B| \frac{a_2 (t - s)}{|W_n|^\alpha} + \frac{a_3}{|W_n|^{2\alpha}}.$$

For given $\varepsilon \in (0, 1)$, for some $\beta \in (0, 1)$ and sufficiently large $n_0 = n_0(\varepsilon)$, we choose $s, t \in [0, R]$ such that $\varepsilon/|W_n|^{2\alpha} \leq |t - s|^{1+\beta} \leq 1$ for all $n \geq n_0$. Such a choice of $s, t \in [0, R]$ implies $|t - s|/|W_n|^\alpha \leq |t - s|^{1+\beta}/\varepsilon$ and $(t - s)^2 \leq |t - s|^{1+\beta}/\varepsilon$ so that

$$\mathbf{E}(\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s))^2 \leq a |t - s|^{1+\beta}/\varepsilon, \quad (22)$$

where $a = a_1 (d R^{d-1} |B|)^2 + a_2 d R^{d-1} + a_3$ depends on d, R, B, λ and $\|\gamma_{red}^{(k)}\|_{var}$, $k = 2, 3, 4$.

Since $(\lambda^2 K_B)_n(c_n^\alpha r)$ as well as $\lambda^2 K_B(c_n^\alpha r)$ is non-decreasing in $r \geq 0$, we find after a short calculation that, for any $s \leq t \leq u$,

$$|\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)| \leq |\Delta_{B,n}^{(\alpha)}(u) - \Delta_{B,n}^{(\alpha)}(s)| + |W_n|^{1/2-\alpha} \lambda^2 (K_B(c_n^\alpha u) - K_B(c_n^\alpha s)), \quad (23)$$

where the second term on the r.h.s. is bounded by $a_0 \lambda |W_n|^{1/2} |(uB) \setminus (sB)|$ due to (8).

In the next step we determine an upper bound of $\sup_{s \leq t \leq s+mh} |\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)|$, where m is a positive integer and the number h is chosen such that $\varepsilon/|W_n|^{2\alpha} \leq h^{1+\beta}$ and $h \leq \min\{1, R\}$. We consider the random variables $\zeta_k := \Delta_{B,n}^{(\alpha)}(s + kh) - \Delta_{B,n}^{(\alpha)}(s + (k-1)h)$ for $k = 1, \dots, m$ (for $s \leq R$ and with $s + kh$ replaced by R if it exceeds R). From (22) it follows

$$\mathbf{P}(|\Delta_{B,n}^{(\alpha)}(s + jh) - \Delta_{B,n}^{(\alpha)}(s + ih)| \geq \tau) = \mathbf{P}(|\zeta_{i+1} + \dots + \zeta_j| \geq \tau) \leq \frac{a}{\varepsilon \tau^2} ((j - i)h)^{1+\beta}$$

for any $\tau > 0$ and $1 \leq i \leq j \leq m$, where m does not exceed $\lfloor R/h \rfloor + 1$. Now, we are in a position to apply Theorem 12.2 in Billingsley (1968), p. 94, which admits the estimate

$$\mathbf{P}(\max_{1 \leq k \leq m} |\Delta_{B,n}^{(\alpha)}(s + kh) - \Delta_{B,n}^{(\alpha)}(s)| \geq \tau) = \mathbf{P}(\max_{1 \leq k \leq m} |\zeta_1 + \dots + \zeta_k| \geq \tau) \leq \frac{a_4 a}{\varepsilon \tau^2} (mh)^{1+\beta},$$

where a_4 (occurring in Theorem 12.2) only depends on β . Using (23) for $u = t + h$ yields

$$|\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)| \leq |\Delta_{B,n}^{(\alpha)}(s + h) - \Delta_{B,n}^{(\alpha)}(s)| + a_5 |W_n|^{1/2} h \quad \text{with } a_5 = a_0 \lambda |B| d R^{d-1},$$

whence we conclude that

$$\sup_{s \leq t \leq s+mh} |\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)| \leq 3 \max_{1 \leq k \leq m} |\Delta_{B,n}^{(\alpha)}(s+kh) - \Delta_{B,n}^{(\alpha)}(s)| + a_5 |W_n|^{1/2} h.$$

To obtain the latter estimate we simply use the fact the sup is taken in at least one of the intervals $[s + (k-1)h, s+kh]$ for $k = 1, \dots, m$. We specify now the choice of h, m and $\delta \in (0, \min\{1, R\})$ for given (small enough) $\varepsilon > 0, \eta > 0$ as follows: Since $\alpha > 1/4$ we may put $\beta = 4\alpha - 1 > 0$ for $\alpha < 1/2$ (and $\beta = \alpha$ for $\alpha \in [1/2, 1)$) and hence h can be chosen to fulfill the inequality $\varepsilon^{\frac{1}{1+\beta}}/|W_n|^{\frac{2\alpha}{1+\beta}} \leq h < \varepsilon/a_5 |W_n|^{1/2}$ for $n \geq n_0(\varepsilon)$. This choice of h combined with the previous estimates for $\tau = \varepsilon$ gives

$$\mathbf{P}\left(\sup_{s \leq t \leq s+mh} |\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)| \geq 4\varepsilon\right) \leq \mathbf{P}\left(\max_{1 \leq k \leq m} |\zeta_1 + \dots + \zeta_k| \geq \varepsilon\right) \leq \frac{a_4 a}{\varepsilon^3} (mh)^{1+\beta}.$$

Setting $m = \lfloor (\varepsilon^3 \eta)^{1/\beta} / h \rfloor$ and $\delta = mh$ (such that $0 < \delta \leq (\varepsilon^3 \eta)^{1/\beta} < \min\{1, R\}$) we get that

$$\mathbf{P}\left(\sup_{s \leq t \leq s+\delta} |\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)| \geq 4\varepsilon\right) \leq \delta \frac{a_4 a}{\varepsilon^3} (mh)^\beta \leq a_4 a \delta \eta \quad \text{for } n \geq n_0(\varepsilon), s \in [0, R-\delta].$$

Finally, applying the Corollary to Theorem 8.3 in Billingsley (1968) we obtain that for any $\varepsilon, \eta > 0$ there exists a $\delta = \delta(\varepsilon, \eta) > 0$ such that

$$\mathbf{P}(w(\Delta_{B,n}^{(\alpha)}, \delta) \geq 12\varepsilon) \leq (1 + \lfloor R/\delta \rfloor) a_4 a \delta \eta \leq a_4 a (1 + R) \eta \quad \text{for all sufficiently large } n,$$

which coincides with (21) up to the factors 12 and $a_4 a (1 + R)$, respectively. Therefore, the sequence $(\Delta_{B,n}^{(\alpha)})_{n \geq 1}$ is tight in $D(0, R)$ and together with the convergence of its the finite-dimensional distributions the proof of (20) for $1/4 < \alpha < 1$ is complete. The validity of (20) can be extended to $0 < \alpha \leq 1/4$ under the additional assumption (10). To see this, let $\alpha \in (0, 1)$. For given $\varepsilon \in (0, 1)$ we choose $s, t \in [0, R]$ such that $\varepsilon/|W_n|^\alpha \leq |t-s|^\alpha \leq 1$ for all $n \geq n_0$. In view of (11) such a choice of $s, t \in [0, R]$ enables us to replace (22) by

$$\mathbf{E}\left(\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s)\right)^2 \leq b |t-s|^{1+\alpha}/\varepsilon \quad \text{with } b = a_1 (dR^{d-1} |B|)^2 + a_2 dR^{d-1}, \quad (24)$$

where the constants a_1, a_2 are from (11). Now, the increment h can be chosen to fulfill the inequality $\varepsilon^{\frac{1}{\alpha}}/|W_n| \leq h < \varepsilon/a_5 |W_n|^{1/2}$ for $n \geq n_0(\varepsilon)$. The remaining steps are the same as before with $\beta = \alpha$. This terminates the proof of Theorem 2. \square

Corollary 2 Let $V[[0, R] \mapsto [0, \infty)$ be non-decreasing, right-continuous function such that $V(R) < \infty$. Under the assumptions of Theorem 2 (additionally with (10)) we have for $1/4 < \alpha \leq 1/2$ ($0 < \alpha \leq 1/2$)

$$\int_0^R (\Delta_{B,n}^{(\alpha)}(r))^2 dV(r) \xrightarrow[n \rightarrow \infty]{d} 4 \lambda^2 |B|^2 \int_0^R r^{2d} dV(r) \mathcal{N}(0, \sigma^2)^2. \quad (25)$$

Corollary 3 Let v be a càdlàg-function on $[0, R]$, i.e. $v \in D[0, R]$. Under the assumptions of Theorem 2 (additionally with (10)) we have for $1/4 < \alpha \leq 1/2$ ($0 < \alpha \leq 1/2$)

$$\sup_{0 \leq r \leq R} |\Delta_{B,n}^{(\alpha)}(r) v(r)| \xrightarrow[n \rightarrow \infty]{d} 2 \lambda |B| \sup_{0 \leq r \leq R} |r^d v(r)| |\mathcal{N}(0, \sigma^2)|. \quad (26)$$

The proofs of Corollary 2 and 3 rely on the fact that both functionals $F_1(f) = \int_0^R f^2(r) dV(r)$ and $F_2(f) = \sup_{0 \leq r \leq R} |f(r) v(r)|$ are continuous on $D[0, R]$ in the following sense: Let (f_n) be a sequence in $D[0, R]$ having a continuous limit f w.r.t. to the Skorohod metric in $D[0, R]$, see Chapt. 3 in Billingsley (1968) for a precise definition. By virtue of this definition it can be shown that $F_i(f_n) \xrightarrow[n \rightarrow \infty]{} F_i(f)$ for $i = 1, 2$. Since the limit process $\{G_B(r) : r \in [0, R]\}$ in Theorem 2 is (\mathbf{P} -a.s.) continuous, a direct application of the *continuous mapping theorem*, see Theorem 5.1 in Billingsley (1968), yields the limit theorems (25) and (26). \square

Remark 4 The trivial weight functions $V(r) = r$ in (25) and $v(r) = 1$ in (26) provide analogues to the classical Cramér-von Mises and Kolmogorov-Smirnov test statistic. By means of the functions V and v one can place higher weights on $|\Delta_{B,n}^{(\alpha)}(r)|$ in particular for small r -values, e.g. $v(r) = e^{-ar}$ for $a \geq d/R$ or $V'(r) = e^{-br^{2d+1}}$ for $b > 0$.

4 Goodness-of-fit-tests for checking point process hypotheses

Aim: Testing the null hypothesis $H_0 : P = P_0$ vs. $H_1 : P \neq P_0$ by checking the goodness-of-fit of the (known) K -function $K_{0,B}(\cdot)$ and intensity λ_0 of $N \sim P_0$.

For this purpose we change the notation (2) and put

$$\Delta_{0,B,n}^{(\alpha)}(r) := |W_n|^{1/2-\alpha} \left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda_0^2 K_{0,B}(c_n^\alpha r) \right) \quad \text{for } 0 \leq r \leq R, \quad 0 \leq \alpha < 1.$$

As consequence of (3) in order to distinguish between PPs with different K_B -function, the below Theorems 3 - 5 are formulated only for $0 < \alpha \leq 1/2$. In the particular case $\alpha = 1/2$ the normalizing sequence $|W_n|^{1/2-\alpha}$ is constant equal to 1.

First we establish a χ^2 -type test as an immediate outcome of Corollary 1. According to this under the null hypothesis H_0 and provided that $N \sim P_0$ satisfies the conditions of Theorem 1 and Lemma 4, the test statistic

$$\widehat{T_{0,B,n}^{(\alpha)}}(r_1, \dots, r_k) := \frac{1}{(\widehat{\lambda^2})_n (\widehat{\sigma^2})_n} \sum_{i=1}^k \left(\frac{\Delta_{0,B,n}^{(\alpha)}(r_i) - \Delta_{0,B,n}^{(\alpha)}(r_{i-1})}{r_i^d - r_{i-1}^d} \right)^2 + \frac{|W_n| (\widehat{\lambda}_n - \lambda_0)^2}{(\widehat{\sigma^2})_n}$$

possesses the limit distribution function $F_{k,B}(x) := \mathbf{P}((4k|B|^2 + 1)\mathcal{N}(0,1)^2 \leq x)$ (as $n \rightarrow \infty$) for any $0 := r_0 < r_1 < \dots < r_k < \infty$, $k \geq 1$ and all $\alpha \in (0, 1/2]$. Given a *significance level* $\gamma > 0$ (= probability of the *type I error*), H_0 is accepted (for sufficiently large n) if $\widehat{T_{0,B,n}^{(\alpha)}}(r_1, \dots, r_k) \leq (4k|B|^2 + 1)z_{1-\gamma/2}^2$, where z_p denotes the p -quantile of the standard normal distribution $\Phi(x) = \mathbf{P}(\mathcal{N}(0,1) \leq x)$. This follows from $F_{k,B}((4k|B|^2 + 1)z_{1-\gamma/2}^2) = 1 - \mathbf{P}(|\mathcal{N}(0,1)| > z_{1-\gamma/2}) = 1 - \gamma$.

Theorem 3 Let $N_0 \sim P_0$ satisfy the conditions of Theorem 2 (additionally with (10)) and Lemma 4. Then, under the null hypothesis H_0 , we have for $1/4 < \alpha \leq 1/2$ ($0 < \alpha \leq 1/2$)

$$\frac{\int_0^R (\Delta_{0,B,n}^{(\alpha)}(r))^2 dV(r)}{(\widehat{\lambda^2})_n (\widehat{\sigma^2})_n} + \frac{|W_n| (\widehat{\lambda}_n - \lambda_0)^2}{(\widehat{\sigma^2})_n} \xrightarrow[n \rightarrow \infty]{d} \left(4|B|^2 \int_0^R r^{2d} dV(r) + 1 \right) \mathcal{N}(0,1)^2 \quad (27)$$

and

$$\frac{\sup_{0 \leq r \leq R} |\Delta_{0,B,n}^{(\alpha)}(r) v(r)|}{\widehat{\lambda}_n \sqrt{(\widehat{\sigma^2})_n}} + \frac{\sqrt{|W_n|} |\widehat{\lambda}_n - \lambda_0|}{\sqrt{(\widehat{\sigma^2})_n}} \xrightarrow[n \rightarrow \infty]{d} \left(2|B| \sup_{0 \leq r \leq R} |r^d v(r)| + 1 \right) |\mathcal{N}(0,1)|, \quad (28)$$

where the weight functions V and v are the same as in Corollary 2 and 3, respectively.

Proof of Theorem 3 In view of Slutsky's lemma it suffices to prove (27) when $(\widehat{\lambda^2})_n$ and $(\widehat{\sigma^2})_n$ are replaced by λ_0^2 and σ_0^2 (= asymptotic variance of $N(W_n)/\sqrt{|W_n|}$ under H_0).

Using the relations (17) and (18) under $H_0 : N \sim P_0$ we obtain after some rearrangements the following identity

$$\begin{aligned} \frac{1}{\lambda_0^2} \int_0^R (\Delta_{0,B,n}^{(\alpha)}(r))^2 dV(r) + |W_n| (\widehat{\lambda}_n - \lambda_0)^2 &= \left(\frac{N(W_n) - \lambda_0 |W_n|}{\sqrt{|W_n|}} \right)^2 \left(\int_0^R 4|r|B|^2 dV(r) + 1 \right) \\ &+ \int_0^R \left(\frac{X_n(r) + Y_n(r)}{\lambda_0} \right) \left(\frac{X_n(r) + Y_n(r)}{\lambda_0} + 4|r|B| \frac{N(W_n) - \lambda_0 |W_n|}{\sqrt{|W_n|}} \right) dV(r). \end{aligned} \quad (29)$$

Now, we apply the tightness of the sequence $\{\Delta_{0,B,n}^{(\alpha)}(r) : r \in [0, R]\}$ which implies that $\sup_{0 \leq r \leq R} |X_n(r)| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0$ and $\sup_{0 \leq r \leq R} |Y_n(r)| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0$. Thus, the integral in (29) tends to zero in probability. Finally, the assumption (14) combined once more with Slutsky's lemma shows the normal convergence

$$\frac{1}{\lambda_0^2} \int_0^R (\Delta_{0,B,n}^{(\alpha)}(r))^2 dV(r) + |W_n| (\widehat{\lambda}_n - \lambda_0)^2 \xrightarrow[n \rightarrow \infty]{d} \left(\int_0^R 4|r|B|^2 dV(r) + 1 \right) \mathcal{N}(0, \sigma_0^2)^2$$

which is equivalent to (27). The proof of the second assertion (28) relies on the same arguments as before. The relations (17) and (18) (under $H_0 : N \sim P_0$) allows to derive the (ω -wise) estimate

$$\left| \frac{\sup_{0 \leq r \leq R} |\Delta_{0,B,n}^{(\alpha)}(r) v(r)|}{\lambda_0} + \sqrt{|W_n|} |\widehat{\lambda}_n - \lambda_0| - (2|B| \sup_{0 \leq r \leq R} |r^d v(r)| + 1) \frac{|N(W_n) - \lambda_0 |W_n||}{\sqrt{|W_n|}} \right|$$

$$\leq \frac{1}{\lambda_0} \sup_{0 \leq r \leq R} |(X_n(r) + Y_n(r)) v(r)| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0.$$

Together with (14) and Slutsky's lemma it follows the distributional convergence

$$\frac{\sup_{0 \leq r \leq R} |\Delta_{0,B,n}^{(\alpha)}(r) v(r)|}{\lambda_0} + \sqrt{|W_n|} |\widehat{\lambda}_n - \lambda_0| \xrightarrow[n \rightarrow \infty]{d} (2|B| \sup_{0 \leq r \leq R} |r^d v(r)| + 1) |\mathcal{N}(0, \sigma_0^2)|,$$

which turns out to be equivalent to (28)). Thus, the proof of Theorem 3 is complete. \square

A further application of the Theorem 2 consists in testing the hypothesis that two independent PPs $N_a \sim P_a$ and $N_b \sim P_b$ possess the same distribution, that is, testing $H_0^{a,b} : P_a = P_b$ vs. $H_1^{a,b} : P_a \neq P_b$. Note that the parameters λ_a, σ_a^2 and the K -function $K_{a,B}$ and their estimators are related to P_a and an observation of N_a in W_n which also applies for quantities with subscript b . It should be mentioned that first attempts to tackle this two-sample test problem for stationary PPs were made in Doss (1989).

Theorem 4 Let $N_a \sim P_a$ and $N_b \sim P_b$ be two independent B_4 -mixing PPs on \mathbb{R}^d which can be observed in W_n and satisfy the conditions of Theorem 2 (additionally with (10)) and Lemma 4. Then, under the null hypothesis $H_0^{a,b}$, we have for $1/4 < \alpha \leq 1/2$ ($0 < \alpha \leq 1/2$)

$$\frac{|W_n| \int_0^R ((\widehat{\lambda}_a^2 K_a)_n(c_n^\alpha r) - (\widehat{\lambda}_b^2 K_b)_n(c_n^\alpha r))^2 dV(r)}{|W_n|^{2\alpha} ((\widehat{\lambda}_a^2)_n (\widehat{\sigma}_a^2)_n + (\widehat{\lambda}_b^2)_n (\widehat{\sigma}_b^2)_n)} + \frac{|W_n| ((\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n)^2}{(\widehat{\sigma}_a^2)_n + (\widehat{\sigma}_b^2)_n} \quad (30)$$

$$\xrightarrow[n \rightarrow \infty]{d} \left(4|B|^2 \int_0^R r^{2d} dV(r) + 1 \right) \mathcal{N}(0, 1)^2.$$

Proof of Theorem 4 In case of the PP $N_a \sim P_a$ we use the abbreviation

$$\Delta_{a,B,n}^{(\alpha)}(r) := |W_n|^{1/2-\alpha} ((\widehat{\lambda}_a^2 K_{a,B})_n(c_n^\alpha r) - \lambda_a^2 K_{a,B}(c_n^\alpha r)) \quad \text{for } 0 \leq r \leq R$$

and correspondingly $\Delta_{b,B,n}^{(\alpha)}(r)$ for $N_b \sim P_b$. Further, define $Q_{a,n} := (N_a(W_n) - \lambda_a |W_n|)/\sqrt{|W_n|}$ and likewise $Q_{b,n}$. Using the relations (17) and (18) for both PPs N_a and N_b (with obvious changes of the notation) we arrive at

$$\Delta_{a,B,n}^{(\alpha)}(r) - \Delta_{b,B,n}^{(\alpha)}(r) = Z_n^{a,b}(r) + 2|rB|(\lambda_a Q_{a,n} - \lambda_b Q_{b,n}), \quad (31)$$

where $Z_n^{a,b}(r) := X_{a,n}(r) - X_{b,n}(r) + Y_{a,n}(r) - Y_{b,n}(r)$. Under $H_0^{a,b}$ (which implies $\lambda_a = \lambda_b$ and $K_{a,B} = K_{b,B}$) we may use the (ω -wise) equality $\sqrt{|W_n|}((\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n) = Q_{a,n} - Q_{b,n}$. After

squaring, integrating w.r.t. dV and dividing both sides of (31) by $\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2$ and provided $H_0^{a,b} : P_a = P_b$ is true, we obtain the equality

$$\begin{aligned} & \frac{|W_n| ((\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n)^2}{\sigma_a^2 + \sigma_b^2} + \int_0^R \frac{(\Delta_{a,B,n}^{(\alpha)}(r) - \Delta_{b,B,n}^{(\alpha)}(r))^2}{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2} dV(r) = \frac{(Q_{a,n} - Q_{b,n})^2}{\sigma_a^2 + \sigma_b^2} \\ & + 4 \int_0^R |rB|^2 dV(r) \frac{(Q_{a,n} - Q_{b,n})^2}{\sigma_a^2 + \sigma_b^2} + \int_0^R \frac{Z_n^{a,b}(r) (Z_n^{a,b}(r) + 4|rB|(\lambda_a Q_{a,n} - \lambda_b Q_{b,n}))}{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2} dV(r), \end{aligned}$$

where $\Delta_{a,B,n}^{(\alpha)}(r) - \Delta_{b,B,n}^{(\alpha)}(r)$ can be replaced by $|W_n|^{1-2\alpha} ((\widehat{\lambda}_a^2 K_a)_n(c_n^\alpha r) - (\widehat{\lambda}_b^2 K_b)_n(c_n^\alpha r))$ for all $0 \leq r \leq R$. The assumptions of Theorem 2 put on both PPs $N_a \sim P_a$ and $N_b \sim P_b$ and their independence imply

$$\sup_{0 \leq r \leq R} |Z_n^{a,b}(r)| \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0 \quad \text{and} \quad \frac{Q_{a,n} - Q_{b,n}}{\sqrt{\sigma_a^2 + \sigma_b^2}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

whence in summary we conclude that

$$\begin{aligned} & \frac{|W_n|^{1-2\alpha}}{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2} \int_0^R ((\widehat{\lambda}_a^2 K_a)_n(c_n^\alpha r) - (\widehat{\lambda}_b^2 K_b)_n(c_n^\alpha r))^2 dV(r) + \frac{|W_n| ((\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n)^2}{\sigma_a^2 + \sigma_b^2} \\ & \xrightarrow[n \rightarrow \infty]{d} \left(4|B|^2 \int_0^R r^{2d} dV(r) + 1 \right) \mathcal{N}(0, 1)^2. \end{aligned}$$

After replacing the unknown parameters λ_a^2, λ_b^2 and σ_a^2, σ_b^2 by the mean-square consistent estimators given in (13) and (12), respectively, and applying of Slutsky's lemma, the proof of Theorem 4 is complete. \square

Theorem 5 Let $N_a \sim P_a$ and $N_b \sim P_b$ be two independent B_4 -mixing PPs on \mathbb{R}^d which can be observed in W_n and satisfy the conditions of Theorem 2 (additionally with (10)) and Lemma 4. Then, under the null hypothesis $H_0^{a,b}$, we have for $1/4 < \alpha \leq 1/2$ ($0 < \alpha \leq 1/2$)

$$\begin{aligned} & \frac{\sqrt{|W_n|} \sup_{0 \leq r \leq R} |((\widehat{\lambda}_a^2 K_a)_n(c_n^\alpha r) - (\widehat{\lambda}_b^2 K_b)_n(c_n^\alpha r)) v(r)|}{|W_n|^\alpha \sqrt{(\widehat{\lambda}_a^2)_n (\widehat{\sigma}_a^2)_n + (\widehat{\lambda}_b^2)_n (\widehat{\sigma}_b^2)_n}} + \frac{\sqrt{|W_n|} |(\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n|}{\sqrt{(\widehat{\sigma}_a^2)_n + (\widehat{\sigma}_b^2)_n}} \\ & \xrightarrow[n \rightarrow \infty]{d} \left(2|B| \sup_{0 \leq r \leq R} |r^d v(r)| + 1 \right) |\mathcal{N}(0, 1)|. \end{aligned} \tag{32}$$

Proof of Theorem 5 With the notation introduced in the proof of Theorem 4 and (31) it is easily seen that

$$\left| \frac{\sup_{0 \leq r \leq R} |(\Delta_{a,B,n}^{(\alpha)}(r) - \Delta_{b,B,n}^{(\alpha)}(r)) v(r)|}{\sqrt{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2}} - 2|B| \sup_{0 \leq r \leq R} |r^d v(r)| \frac{|\lambda_a Q_{a,n} - \lambda_b Q_{b,n}|}{\sqrt{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2}} \right|$$

is bounded by the sequence $\sup_{0 \leq r \leq R} |Z_n^{a,b}(r) v(r)| / \sqrt{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2}$ tending to zero in probability.

Obviously, just as in the foregoing proof, if $H_0^{a,b}$ is true and if (14) is satisfied for both of the independent PPs N_a and N_b we have

$$\frac{\sqrt{|W_n|} |(\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n|}{\sqrt{\sigma_a^2 + \sigma_b^2}} = \frac{|\lambda_a Q_{a,n} - \lambda_b Q_{b,n}|}{\sqrt{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2}} = \frac{|Q_{a,n} - Q_{b,n}|}{\sqrt{\sigma_a^2 + \sigma_b^2}} \xrightarrow[n \rightarrow \infty]{d} |\mathcal{N}(0, 1)|.$$

Hence, if $H_0^{a,b}$ is true and all the assumptions of Theorem 4 are fulfilled,

$$\begin{aligned} & \frac{\sqrt{|W_n|} \sup_{0 \leq r \leq R} |((\widehat{\lambda}_a^2 K_a)_n(c_n^\alpha r) - (\widehat{\lambda}_b^2 K_b)_n(c_n^\alpha r)) v(r)|}{|W_n|^\alpha \sqrt{\lambda_a^2 \sigma_a^2 + \lambda_b^2 \sigma_b^2}} + \frac{\sqrt{|W_n|} |(\widehat{\lambda}_a)_n - (\widehat{\lambda}_b)_n|}{\sqrt{\sigma_a^2 + \sigma_b^2}} \\ & \xrightarrow[n \rightarrow \infty]{d} \left(2|B| \sup_{0 \leq r \leq R} |r^d v(r)| + 1 \right) |\mathcal{N}(0, 1)|. \end{aligned}$$

The arguments at the of the proof of Theorem 4 confirm the assertion stated in Theorem 5. \square

Remark 5 The observation windows of N_a and N_b need not to be the same. Theorems 4 and 5 remain valid when the independent PPs N_a and N_b are given in (W_n^a) and (W_n^b) (each forming a CAS) such that $|W_n^a| = |W_n^b|$ for sufficiently large n .

Remark 6 Let there be given a significance level $\gamma > 0$ (= probability of the type I error). Then, H_0 resp. $H_0^{a,b}$ is accepted (for large n) if the test statistic on the l.h.s. of (27) resp. (30) does not exceed $(4|B|^2 \int_0^R r^{2d} dV + 1) z_{1-\gamma/2}^2$ (or if the test statistic on the l.h.s. of (28) resp. (32) does not exceed $(2|B| \sup_{r \in [0, R]} |r^d v(r)| + 1) z_{1-\gamma/2}$), where z_p is defined by $\Phi(z_p) = p$.

5 Proofs of Lemma 2 and Lemma 3

Proof of Lemma 2 We start with the following decomposition

$$\begin{aligned} & \mathbf{E} \left[|W_n| (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - N(W_n) \alpha_{red}^{(2)}(c_n^\alpha r B) - \alpha_{red}^{(2)}(c_n^\alpha r B) (N(W_n) - \lambda |W_n|) \right]^2 \\ & = (\alpha_{red}^{(2)}(c_n^\alpha r B))^2 \mathbf{Var}(N(W_n)) + \mathbf{E} \left[|W_n| (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - N(W_n) \alpha_{red}^{(2)}(c_n^\alpha r B) \right]^2 \\ & - 2 \alpha_{red}^{(2)}(c_n^\alpha r B) \mathbf{E} \left[(N(W_n) - \lambda |W_n|) \left[|W_n| (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - N(W_n) \alpha_{red}^{(2)}(c_n^\alpha r B) \right] \right] \\ & = T_n^{(1)}(r B) + T_n^{(2)}(r B) - T_n^{(3)}(r B). \end{aligned}$$

The assertion of Lemma 2 follows from the three limits

$$\lim_{n \rightarrow \infty} |W_n|^{-(1+2\alpha)} T_n^{(1)}(rB) = \lim_{n \rightarrow \infty} |W_n|^{-(1+2\alpha)} T_n^{(2)}(rB) = \lambda^3 |rB|^2 (1 + \gamma_{red}^{(2)}(\mathbb{R}^d)) \quad (33)$$

and

$$\lim_{n \rightarrow \infty} |W_n|^{-(1+2\alpha)} T_n^{(3)}(rB) = 2\lambda^3 |rB|^2 (1 + \gamma_{red}^{(2)}(\mathbb{R}^d)). \quad (34)$$

Using the second-order stationarity of $N = \sum_{i \geq 1} \delta_{X_i}$ and the definitions of $\gamma^{(2)}$ and $\gamma_{red}^{(2)}$ we get

$$\mathbf{Var}(N(W_n)) = \mathbf{E}(N(W_n) - \lambda|W_n|)^2 = \lambda|W_n| + \gamma^{(2)}(W_n \times W_n) = \lambda|W_n| +$$

$$\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y+x) \gamma_{red}^{(2)}(dy) dx = \lambda|W_n| + \lambda \int_{\mathbb{R}^d} |W_n \cap (W_n - y)| \gamma_{red}^{(2)}(dy),$$

whence in view of $\|\gamma_{red}^{(2)}\|_{var} < \infty$, Lemma 1 and Lebesgue's convergence theorem it follows that

$$\frac{\alpha_{red}^{(2)}(c_n^\alpha rB)}{|W_n|^\alpha} = \lambda|rB| + \frac{\gamma_{red}^{(2)}(c_n^\alpha rB)}{|W_n|^\alpha} \xrightarrow{n \rightarrow \infty} \lambda|rB| \quad \text{for } \alpha > 0 \quad (35)$$

and

$$\frac{\gamma^{(2)}(W_n \times W_n)}{|W_n|} = \frac{\lambda}{|W_n|} \int_{\mathbb{R}^d} |W_n \cap (W_n - y)| \gamma_{red}^{(2)}(dy) \xrightarrow{n \rightarrow \infty} \lambda \gamma_{red}^{(2)}(\mathbb{R}^d).$$

The latter two relations prove the first limit of (33).

Taking into account the unbiasedness of $(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$, i.e. $\mathbf{E}(\widehat{\lambda^2 K_B})_n(r) = \lambda^2 K_B(r) = \lambda \alpha_{red}^{(2)}(rB)$, and $\mathbf{E}N(W_n) = \lambda|W_n|$ we can write $T_n^{(3)}(rB)$ as follows:

$$\begin{aligned} T_n^{(3)}(rB) &= 2\alpha_{red}^{(2)}(c_n^\alpha rB) [|W_n| \mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \mathbf{E}N^2(W_n) \alpha_{red}^{(2)}(c_n^\alpha rB)] \\ &= 2\alpha_{red}^{(2)}(c_n^\alpha rB) |W_n| [\mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 |W_n| \alpha_{red}^{(2)}(c_n^\alpha rB)] \\ &\quad - 2(\alpha_{red}^{(2)}(c_n^\alpha rB))^2 \mathbf{Var}(N(W_n)). \end{aligned}$$

Combining (35) and $\mathbf{Var}(N(W_n))/|W_n| \xrightarrow{n \rightarrow \infty} \lambda(1 + \gamma_{red}^{(2)}(\mathbb{R}^d))$ reveals that (34) is equivalent to

$$\lim_{n \rightarrow \infty} |W_n|^{-\alpha} [\mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 |W_n| \alpha_{red}^{(2)}(c_n^\alpha rB)] = 2\lambda^2 |rB| (1 + \gamma_{red}^{(2)}(\mathbb{R}^d)). \quad (36)$$

For this purpose we rewrite $\mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$ as follows

$$\begin{aligned}
\mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) &= \mathbf{E} \sum_{\substack{i,j,k \geq 1 \\ i \neq j}} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k)}{|(W_n - X_i) \cap (W_n - X_j)|} \mathbf{1}_{c_n^\alpha r B}(X_j - X_i) \\
&= \mathbf{E} \sum_{i,j,k \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k)}{|(W_n - X_i) \cap (W_n - X_j)|} \mathbf{1}_{c_n^\alpha r B}(X_j - X_i) \\
&+ \mathbf{E} \sum_{i,k \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_k) \mathbf{1}_{c_n^\alpha r B}(X_k - X_i)}{|(W_n - X_i) \cap (W_n - X_k)|} + \mathbf{E} \sum_{j,k \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k) \mathbf{1}_{c_n^\alpha r B}(X_k - X_j)}{|(W_n - X_k) \cap (W_n - X_j)|} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(z)}{|W_n \cap (W_n - y + x)|} \mathbf{1}_{c_n^\alpha r B}(y - x) \alpha^{(3)}(d(x, y, z)) \\
&+ 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y)}{|W_n \cap (W_n - y + x)|} \mathbf{1}_{c_n^\alpha r B}(y - x) \alpha^{(2)}(d(x, y)) \\
&= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y+x) \mathbf{1}_{W_n}(z+x)}{|W_n \cap (W_n - y)|} \mathbf{1}_{c_n^\alpha r B}(y) \alpha_{red}^{(3)}(d(y, z)) dx \\
&+ 2\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y+x)}{|W_n \cap (W_n - y)|} \mathbf{1}_{c_n^\alpha r B}(y) \alpha_{red}^{(2)}(dy) dx \\
&= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|W_n \cap (W_n - y) \cap (W_n - z)| \mathbf{1}_{c_n^\alpha r B}(y)}{|W_n \cap (W_n - y)|} \alpha_{red}^{(3)}(d(y, z)) + 2\lambda \alpha_{red}^{(2)}(c_n^\alpha r B). \quad (37)
\end{aligned}$$

Disintegration of formula (6) for $k = 3$ on both sides w.r.t. the third component leads to

$$\begin{aligned}
\alpha_{red}^{(3)}(A_1 \times A_2) &= \gamma_{red}^{(3)}(A_1 \times A_2) + \lambda |A_1| \gamma_{red}^{(2)}(A_2) + \gamma^{(2)}(A_1 \times A_2) + \lambda \gamma_{red}^{(2)}(A_1) |A_2| + \lambda^2 |A_1| |A_2| \\
&= \gamma_{red}^{(3)}(A_1 \times A_2) + \lambda |A_1| \gamma_{red}^{(2)}(A_2) + \gamma^{(2)}(A_1 \times A_2) + \lambda \alpha_{red}^{(2)}(A_1) |A_2| \quad (38)
\end{aligned}$$

for any bounded $A_1, A_2 \in \mathcal{B}^d$. Thus, according to (38) we rewrite (37) so that $\mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$ takes the form

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|W_n \cap (W_n - y) \cap (W_n - z)| \mathbf{1}_{c_n^\alpha r B}(y)}{|W_n \cap (W_n - y)|} [\gamma_{red}^{(3)}(d(y, z)) + \lambda \gamma_{red}^{(2)}(dz) dy + \gamma^{(2)}(d(y, z))] \\
&+ \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|W_n \cap (W_n - y) \cap (W_n - z)| \mathbf{1}_{c_n^\alpha r B}(y)}{|W_n \cap (W_n - y)|} \alpha_{red}^{(2)}(dy) dz + 2\lambda \alpha_{red}^{(2)}(c_n^\alpha r B).
\end{aligned}$$

Since $\int_{\mathbb{R}^d} |W_n \cap (W_n - y) \cap (W_n - z)| dz = |W_n \cap (W_n - y)| |W_n|$ multiple application of Fubini's theorem yields the estimate

$$\left| \mathbf{E}N(W_n) (\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 |W_n| \alpha_{red}^{(2)}(c_n^\alpha r B) - 2\lambda \alpha_{red}^{(2)}(c_n^\alpha r B) - 2\lambda^2 |c_n^\alpha r B| \gamma_{red}^{(2)}(\mathbb{R}^d) \right|$$

$$\leq \lambda \|\gamma_{red}^{(3)}\|_{var} + \lambda \int_{\mathbb{R}^d} \int_{c_n^\alpha r B} \left(1 - \frac{|W_n \cap (W_n - y) \cap (W_n - z)|}{|W_n \cap (W_n - y)|}\right) dy \gamma_{red}^{(2)}(dz).$$

From Lemma 1 we see that, for any $y \in c_n^\alpha r B$ and $z \in \mathbb{R}^d$,

$$1 - \frac{|W_n \cap (W_n - y) \cap (W_n - z)|}{|W_n|} \leq 1 - \frac{|W_n \cap (W_n - y)|}{|W_n|} + 1 - \frac{|W_n \cap (W_n - z)|}{|W_n|} \xrightarrow{n \rightarrow \infty} 0,$$

whence together with (35) and Lebesgue's convergence theorem it follows (36) so that (34) is proved.

In order to verify the second limit of (33) we use again the relations $\mathbf{E}[\widehat{(\lambda^2 K_B)}_n(c_n^\alpha r)] = \lambda \alpha_{red}^{(2)}(c_n^\alpha r B)$ and $\mathbf{E}N(W_n) = \lambda |W_n|$ which allows to write $T_n^{(2)}(r B)$ in the following form:

$$\begin{aligned} T_n^{(2)}(r B) &= \mathbf{E} \left[|W_n| (\widehat{(\lambda^2 K_B)}_n(c_n^\alpha r)) - N(W_n) \alpha_{red}^{(2)}(c_n^\alpha r B) \right]^2 \\ &= |W_n|^2 \mathbf{E} [(\widehat{(\lambda^2 K_B)}_n(c_n^\alpha r))^2] - (\alpha_{red}^{(2)}(c_n^\alpha r B))^2 \mathbf{E} N^2(W_n) - T_n^{(3)}(r B) \\ &= |W_n|^2 \mathbf{Var} [(\widehat{(\lambda^2 K_B)}_n(c_n^\alpha r))] - \mathbf{Var}(N(W_n)) (\alpha_{red}^{(2)}(c_n^\alpha r B))^2 - T_n^{(3)}(r B). \end{aligned}$$

Having in mind the just proved limit (34) and the limits $\mathbf{Var}(N(W_n))/|W_n| \xrightarrow{n \rightarrow \infty} \lambda(1 + \gamma_{red}^{(2)}(\mathbb{R}^d))$ and (35), we need to show that

$$\lim_{n \rightarrow \infty} |W_n|^{1-2\alpha} \mathbf{Var} [(\widehat{(\lambda^2 K_B)}_n(c_n^\alpha r))] = 4 \lambda^3 |r B|^2 (1 + \gamma_{red}^{(2)}(\mathbb{R}^d)). \quad (39)$$

According to the definition of the estimator $(\widehat{(\lambda^2 K_B)}_n(r))$ we first rewrite its second moment as expectations of multiple sums over pairwise distinct atoms of N which can be expressed in terms of integrals w.r.t. the factorial moment measures $\alpha^{(k)}(\cdot)$ for $k = 2, 3, 4$. In this way we get

$$\begin{aligned} \mathbf{E} [(\widehat{(\lambda^2 K_B)}_n(r))^2] &= \mathbf{E} \sum_{\substack{i,j,k,\ell \geq 1 \\ i \neq j, k \neq \ell}} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k) \mathbf{1}_{W_n}(X_\ell) \mathbf{1}_{rB}(X_j - X_i) \mathbf{1}_{rB}(X_\ell - X_k)}{|(W_n - X_i) \cap (W_n - X_j)| |(W_n - X_k) \cap (W_n - X_\ell)|} \\ &= \mathbf{E} \sum_{i,j,k,\ell \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k) \mathbf{1}_{W_n}(X_\ell) \mathbf{1}_{rB}(X_j - X_i) \mathbf{1}_{rB}(X_\ell - X_k)}{|(W_n - X_i) \cap (W_n - X_j)| |(W_n - X_k) \cap (W_n - X_\ell)|} \\ &\quad + 4 \mathbf{E} \sum_{i,j,k \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{W_n}(X_k) \mathbf{1}_{rB}(X_j - X_i) \mathbf{1}_{rB}(X_k - X_i)}{|(W_n - X_i) \cap (W_n - X_j)| |(W_n - X_i) \cap (W_n - X_k)|} \\ &\quad + 2 \mathbf{E} \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{rB}(X_j - X_i)}{|(W_n - X_i) \cap (W_n - X_j)|^2}, \end{aligned}$$

$$[\mathbf{E}(\widehat{(\lambda^2 K_B)}_n(r))]^2 = \int_{W_n} \int_{W_n} \int_{W_n} \int_{W_n} \frac{\mathbf{1}_{rB}(y-x) \mathbf{1}_{rB}(v-u) \alpha^{(2)}(d(x,y)) \alpha^{(2)}(d(u,v))}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|}$$

and

$$\begin{aligned}
\mathbf{Var}[(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)] &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(u) \mathbf{1}_{W_n}(v) \mathbf{1}_{c_n^\alpha r B}(y-x)}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|} \\
&\quad \times \mathbf{1}_{c_n^\alpha r B}(v-u) [\alpha^{(4)}(d(x, y, u, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(u, v))] \\
&\quad + 4 \int_{W_n} \int_{W_n} \int_{W_n} \frac{\mathbf{1}_{c_n^\alpha r B}(y-x) \mathbf{1}_{c_n^\alpha r B}(z-x) \alpha^{(3)}(d(x, y, z))}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - z + x)|} \\
&\quad + 2 \int_{W_n} \int_{W_n} \frac{\mathbf{1}_{c_n^\alpha r B}(y-x) \alpha^{(2)}(d(x, y))}{|W_n \cap (W_n - y + x)|^2} \\
&= T_n^{(4)}(r B) + 4 T_n^{(5)}(r B) + 2 T_n^{(6)}(r B). \tag{40}
\end{aligned}$$

We first treat the asymptotic behaviour of $T_n^{(5)}(r B)$ and $T_n^{(6)}(r B)$. After some obvious rearrangements combined with Lemma 1 and (35) we obtain

$$\begin{aligned}
\frac{|W_n| T_n^{(6)}(r B)}{|W_n|^{2\alpha}} &= \frac{\lambda |W_n|}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y+x) \mathbf{1}_{c_n^\alpha r B}(y)}{|W_n \cap (W_n - y)|^2} \alpha_{red}^{(2)}(dy) dx \\
&= \frac{\lambda |W_n|}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{c_n^\alpha r B}(y) \alpha_{red}^{(2)}(dy)}{|W_n \cap (W_n - y)|} \quad (\xrightarrow{n \rightarrow \infty} \lambda \alpha_{red}^{(2)}(r B) \text{ for } \alpha = 0) \\
&\leq \lambda \sup_{y \in c_n^\alpha r B} \frac{|W_n|}{|W_n \cap (W_n - y)|} \frac{\alpha_{red}^{(2)}(c_n^\alpha r B)}{|W_n|^{2\alpha}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \alpha > 0. \tag{41}
\end{aligned}$$

Similarly, using the differential reduction formula $\alpha^{(3)}(d(x, y, z)) = \lambda \alpha_{red}^{(3)}(dy - x, dz - x) dx$ and Lemma 1 we find that

$$\begin{aligned}
\frac{|W_n| T_n^{(5)}(r B)}{|W_n|^{2\alpha}} &= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(z) |W_n| |W_n \cap (W_n - y) \cap (W_n - z)|}{|W_n|^{2\alpha} |W_n \cap (W_n - y)| |W_n \cap (W_n - z)|} \alpha_{red}^{(3)}(d(y, z)) \\
&= \frac{\lambda \alpha_{red}^{(3)}(c_n^\alpha r B \times c_n^\alpha r B)}{|W_n|^{2\alpha}} (1 + \theta \varepsilon_n(r)), \tag{42}
\end{aligned}$$

where $\theta \in [-1, 1]$ is suitably chosen and

$$\varepsilon_n(r) := \sup_{y, z \in c_n^\alpha r B} \left| \frac{|W_n| |W_n \cap (W_n - y) \cap (W_n - z)|}{|W_n \cap (W_n - y)| |W_n \cap (W_n - z)|} - 1 \right| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for any fixed } r > 0,$$

which shows that $|W_n| T_n^{(5)}(r B) \xrightarrow{n \rightarrow \infty} \lambda \alpha_{red}^{(3)}(r B \times r B)$ for $\alpha = 0$. On the other hand, by (38),

$$\begin{aligned}
\lambda \alpha_{red}^{(3)}(c_n^\alpha r B \times c_n^\alpha r B) &= \lambda^2 \alpha_{red}^{(2)}(c_n^\alpha r B) |c_n^\alpha r B| + \lambda^2 \gamma_{red}^{(2)}(c_n^\alpha r B) |c_n^\alpha r B| \\
&\quad + \lambda \gamma_{red}^{(3)}(c_n^\alpha r B \times c_n^\alpha r B) + \lambda^2 \int_{\mathbb{R}^d} |c_n^\alpha r B \cap (c_n^\alpha r B - y)| \gamma_{red}^{(2)}(dy).
\end{aligned}$$

Hence, together with (35) it is easily seen that $|W_n|^{1-2\alpha} T_n^{(5)}(rB) \xrightarrow{n \rightarrow \infty} \lambda^3 |rB|^2$.

To prove (39) it remains to verify the limit

$$\lim_{n \rightarrow \infty} \frac{|W_n| T_n^{(4)}(rB)}{|W_n|^{2\alpha}} = 4\lambda^3 |rB|^2 \gamma_{red}^{(2)}(\mathbb{R}^d) \quad \text{for } \alpha > 0. \quad (43)$$

For proving this we make use of the decomposition

$$\begin{aligned} & \alpha^{(4)}(d(x, y, u, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(u, v)) = \gamma^{(4)}(d(x, y, u, v)) \\ & + \lambda [dx \gamma^{(3)}(d(y, u, v)) + dy \gamma^{(3)}(d(x, u, v)) + du \gamma^{(3)}(d(x, y, v)) + dv \gamma^{(3)}(d(x, y, u))] \\ & + \gamma^{(2)}(d(x, u)) \gamma^{(2)}(d(y, v)) + \gamma^{(2)}(d(x, v)) \gamma^{(2)}(d(y, u)) \\ & + \lambda^2 [dx du \gamma^{(2)}(d(y, v)) + dx dv \gamma^{(2)}(d(y, u)) + dy du \gamma^{(2)}(d(x, v)) + dy dv \gamma^{(2)}(d(x, u))], \end{aligned} \quad (44)$$

which is obtained by applying the formula (6) for $k = 2$ and $k = 4$ with $A_1 = dx$, $A_2 = dy$, $A_3 = du$ and $A_4 = dv$. After rewriting the integrals in terms of reduced cumulant measures and in view of the assumptions $\|\gamma_{red}^{(k)}\|_{var} < \infty$ for $k = 2, 3, 4$ it turns out that, for $\alpha > 0$, only the below four integrals contribute with a non-zero limit to the r.h.s. of (43):

$$\begin{aligned} & \frac{\lambda^2 |W_n|}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(u) \mathbf{1}_{W_n}(v) \mathbf{1}_{c_n^\alpha rB}(y-x) \mathbf{1}_{c_n^\alpha rB}(v-u)}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|} \\ & \times [\gamma^{(2)}(d(y, v)) dx du + dx dv \gamma^{(2)}(d(y, u)) + dy du \gamma^{(2)}(d(x, v)) + dy dv \gamma^{(2)}(d(x, u))] \\ & = \frac{4\lambda^3 |W_n|}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x+y) \mathbf{1}_{W_n}(u+v+y) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(v+y)}{|W_n \cap (W_n + x)| |W_n \cap (W_n + u)|} \\ & \quad \times \mathbf{1}_{c_n^\alpha rB}(x) \mathbf{1}_{c_n^\alpha rB}(u) \gamma_{red}^{(2)}(dv) dx du \\ & = 4\lambda^3 \int_{\mathbb{R}^d} \int_{rB} \int_{rB} \frac{|W_n| |W_n \cap (W_n - c_n^\alpha x) \cap (W_n - v) \cap (W_n - c_n^\alpha u - v)|}{|W_n \cap (W_n - c_n^\alpha x)| |W_n \cap (W_n - c_n^\alpha u)|} dx du \gamma_{red}^{(2)}(dv) \\ & \xrightarrow{n \rightarrow \infty} 4\lambda^3 |rB|^2 \gamma_{red}^{(2)}(\mathbb{R}^d) \quad \text{for } \alpha \geq 0, \end{aligned}$$

where the latter limit is justified by Lemma 1 and Lebesgue's convergence theorem. Thus, (43) is proved which completes the proof of Lemma 2. \square

For $\alpha = 0$ the remaining seven integrals on the r.h.s. of (44) possesses non-zero limits in general:

$$\begin{aligned}
& |W_n| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(u) \mathbf{1}_{W_n}(v) \mathbf{1}_{rB}(y-x) \mathbf{1}_{rB}(v-u)}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|} [\gamma^{(4)}(d(x, y, u, v)) \\
& \quad + 2\lambda \gamma^{(3)}(d(y, u, v)) dx + 2\lambda \gamma^{(3)}(d(x, y, v)) du + 2\gamma^{(2)}(d(x, u)) \gamma^{(2)}(d(y, v))] \\
& \xrightarrow{n \rightarrow \infty} \lambda \int_{\mathbb{R}^{3d}} \mathbf{1}_{rB}(v-u) \mathbf{1}_{rB}(y) \gamma_{red}^{(4)}(d(y, u, v)) + 2\lambda^2 |rB| \int_{\mathbb{R}^{2d}} \mathbf{1}_{rB}(v-u) \gamma_{red}^{(3)}(d(u, v)) \\
& \quad + 2\lambda^2 |rB| \gamma_{red}^{(3)}(rB \times \mathbb{R}^d) + 2\lambda^2 \int_{\mathbb{R}^{2d}} |(rB-u) \cap (rB-v)| \gamma_{red}^{(2)}(du) \gamma_{red}^{(2)}(dv).
\end{aligned}$$

All above limits obtained for $\alpha = 0$ and inserted in (40) are summarized in

Remark 7 For any B_4 -mixing PP $N = \sum_{i \geq 1} \delta_{X_i}$ on \mathbb{R}^d the asymptotic variance

$$\tau_B^2(r) := \lim_{n \rightarrow \infty} |W_n| \mathbf{Var}(\widehat{(\lambda^2 K_B)_n}(r))$$

exists and takes the form

$$\begin{aligned}
\tau_B^2(r) &= \lambda \int_{\mathbb{R}^{3d}} \mathbf{1}_{rB}(y-x) \mathbf{1}_{rB}(z) \gamma_{red}^{(4)}(d(x, y, z)) + 4\lambda^2 |rB| \gamma_{red}^{(3)}(rB \times \mathbb{R}^d) \\
& \quad + 2\lambda^3 \int_{\mathbb{R}^{2d}} |(rB-x) \cap (rB-y)| \gamma_{red}^{(2)}(dx) \gamma_{red}^{(2)}(dy) + 4\lambda^3 |rB|^2 \gamma_{red}^{(2)}(\mathbb{R}^d) \\
& \quad + 4\lambda \alpha_{red}^{(3)}(rB \times rB) + 2\lambda \alpha_{red}^{(2)}(rB). \tag{45}
\end{aligned}$$

Proof of Lemma 3 For notational ease we put

$$\widehat{(\lambda \alpha_{red}^{(2)})_n}(A) := \sum_{i, j \geq 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{c_n^\alpha A}(X_j - X_i)}{|(W_n - X_i) \cap (W_n - X_j)|} \quad \text{for bounded } A \in \mathcal{B}^d.$$

From (2) and the definition of $\alpha_{red}^{(2)}(\cdot)$ we get, for $0 \leq s \leq t$,

$$\Delta_{B,n}^{(\alpha)}(t) - \Delta_{B,n}^{(\alpha)}(s) = |W_n|^{1/2-\alpha} \widehat{(\lambda \alpha_{red}^{(2)})_n}((tB) \setminus (sB)) \quad \text{and} \quad \mathbf{E} \widehat{(\lambda \alpha_{red}^{(2)})_n}(A) = \lambda \alpha_{red}^{(2)}(c_n^\alpha A).$$

Therefore, it suffices to show that

$$|W_n|^{1-2\alpha} \mathbf{Var}(\widehat{(\lambda \alpha_{red}^{(2)})_n}(A)) \leq a_1 |A|^2 + a_2 |A| |W_n|^{-\alpha} + a_3 |W_n|^{-2\alpha} \tag{46}$$

for any bounded, \mathbf{o} -symmetric (i.e. $A = -A$) $A \in \mathcal{B}^d$.

For this purpose we use the decomposition (40) and replace the ball rB by a bounded $A \in \mathcal{B}^d$. Due to Lemma 1 we find some n_0 (depending on $\sup\{\|x\| : x \in A\}$) such that $\sup_{y \in c_n^\alpha A} |W_n|/|W_n \cap (W_n - y)| \leq 2$ for all $n \geq n_0$. From (41) with A instead of rB together with (8) we arrive at

$$|W_n|^{1-2\alpha} T_n^{(6)}(A) \leq 2\lambda \alpha_{red}^{(2)}(c_n^\alpha A) |W_n|^{-2\alpha} \leq 2a_0 \lambda |A| |W_n|^{-\alpha} \quad \text{for } n \geq n_0. \tag{47}$$

In the same way we modify (42), where $1 + \theta \varepsilon_n(r)$ can be replaced by 2 for $n \geq n_0$ (due to the argument used before), and obtain

$$|W_n|^{1-2\alpha} T_n^{(5)}(A) \leq 2 \lambda \alpha_{red}^{(3)}(c_n^\alpha A \times c_n^\alpha A) |W_n|^{-2\alpha} \quad \text{for } n \geq n_0.$$

Condition (8) applied to (38) leads to

$$\alpha_{red}^{(3)}(A_1 \times A_2) \leq \|\gamma_{red}^{(3)}\|_{var} + 3 \lambda a_0 |A_1| |A_2| - 2 \lambda^2 |A_1| |A_2|,$$

whence it follows that

$$|W_n|^{1-2\alpha} T_n^{(5)}(A) \leq 2 \lambda \|\gamma_{red}^{(3)}\|_{var} |W_n|^{-2\alpha} + 6 \lambda^2 a_0 |A|^2 \quad \text{for } n \geq n_0. \quad (48)$$

On the other hand, (38) and the first condition of (10) yield $\alpha_{red}^{(3)}(A_1 \times A_2) \leq \max\{a_{01} + 2 \lambda^2, 3 \lambda a_0\} |A_1| |A_2|$ so that $|W_n|^{1-2\alpha} T_n^{(5)}(A) \leq a_1 |A|^2$ for $n \geq n_0$ for some constant $a_1 > 0$.

To obtain suitable bounds of $T_n^{(4)}(A)$ (defined in (40) for $A = rB$) for any bounded, \mathbf{o} -symmetric set $A \in \mathcal{B}^d$ we have to estimate all integrals w.r.t. the measures on the r.h.s of (44). As before we choose n_0 such that $|W_n| \leq 2 \inf_{x \in c_n^\alpha A} |W_n \cap (W_n - x)|$ for $n \geq n_0$. By the \mathbf{o} -symmetry of A it is easily checked that

$$T_n^{(4)}(A) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(u) \mathbf{1}_{W_n}(v) \mathbf{1}_{c_n^\alpha A}(y-x) \mathbf{1}_{c_n^\alpha A}(v-u)}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|} \times$$

$$[\gamma^{(4)}(d(x, y, u, v)) + 4 \lambda \gamma^{(3)}(d(y, u, v)) dx + 2 \gamma^{(2)}(d(x, u)) \gamma^{(2)}(d(y, v)) + 4 \lambda^2 \gamma^{(2)}(d(y, v)) d(x, u)].$$

After passing to the reduced factorial cumulant measures $\gamma_{red}^{(k)}(\cdot)$ and using that $\|\gamma_{red}^{(k)}\|_{var} < \infty$ for $k = 2, 3, 4$ we obtain after some straightforward, but lengthy calculations the estimate

$$|W_n| \frac{T_n^{(4)}(A)}{|W_n|^{2\alpha}} \leq 2 \lambda \frac{\|\gamma_{red}^{(4)}\|_{var}}{|W_n|^{2\alpha}} + 8 \lambda^2 |A| \frac{\|\gamma_{red}^{(3)}\|_{var}}{|W_n|^\alpha} + 8 \lambda^2 |A| \frac{\|\gamma_{red}^{(2)}\|_{var}^2}{|W_n|^\alpha} + 8 \lambda^3 |A|^2 \|\gamma_{red}^{(2)}\|_{var} \quad (49)$$

for all $n \geq n_0$. Clearly, (8) implies that the reduced covariance measure $\gamma_{red}^{(2)}(\cdot) = \alpha_{red}^{(2)}(\cdot) - \lambda |\cdot|$ admits the estimate $|\gamma_{red}^{(2)}(A)| \leq \max\{a_0, \lambda\} |A|$ for all bounded $A \in \mathcal{B}^d$. After inserting the latter on the r.h.s. of (49) we obtain together with (47), (48) the estimate (46). Setting $A = (tB) \setminus (sB)$ in (46) terminates the proof of Lemma 3. \square

To prove (11) we use $\gamma^{(4)}(d(x, y, u, v)) = \lambda \kappa^{(4)}(y - x, u - x, v - x) d(v, u, y, x)$ and (10) to find an appropriate bound of

$$\begin{aligned}
& \frac{|W_n|}{|W_n|^{2\alpha}} \left| \int_{\mathbb{R}^{4d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{W_n}(u) \mathbf{1}_{W_n}(v) \mathbf{1}_{c_n^\alpha A}(y-x) \mathbf{1}_{c_n^\alpha A}(v-u)}{|W_n \cap (W_n - y + x)| |W_n \cap (W_n - v + u)|} \gamma^{(4)}(d(x, y, u, v)) \right| \\
& \leq \frac{|W_n|}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{|W_n \cap (W_n - y) \cap (W_n - u) \cap (W_n - v + u)| \mathbf{1}_{c_n^\alpha A}(y) \mathbf{1}_{c_n^\alpha A}(v)}{|W_n \cap (W_n - y)| |W_n \cap (W_n - v)|} \\
& \quad \times |\kappa^{(4)}(y, u, v + u)| d(v, u, y) \\
& \leq \frac{2}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \mathbf{1}_{c_n^\alpha A}(y) \mathbf{1}_{c_n^\alpha A}(v) \int_{\mathbb{R}^d} |\kappa^{(4)}(y, u, v + u)| du d(v, y).
\end{aligned}$$

An application of the Cauchy-Schwarz inequality combined with the second condition of (10) yields the following bound of the latter line:

$$\frac{2|c_n^\alpha A|}{|W_n|^{2\alpha}} \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} |\kappa^{(4)}(y, u, v + u)| du \right)^2 d(v, y) \right)^{1/2} \leq 2\sqrt{a_{02}} \frac{|A|}{|W_n|^\alpha}.$$

Together with the foregoing estimates of the other terms of $T_n^{(4)}(A)$ we obtain estimate (46) without $a_3 |W_n|^{-2\alpha}$ but changed constants a_1 and a_2 . For $A = (tB) \setminus (sB)$ this coincides with the desired relation (11). \square

6 Appendix A

In this appendix we state and prove a result from convex geometry which has relevance for the statistical analysis of spatial PPs and random closed sets observed on increasing compact convex sampling windows. Unfortunately, we could not find an appropriate reference to the below estimates (50) in the huge literature on convex geometry.

Let $K \neq \emptyset$ be a *convex body* (compact convex set) in \mathbb{R}^d with inball radius $\varrho(K) > 0$. In this case the corresponding *quermassintegrals* $W_k(K)$ are also positive for $k = 0, 1, \dots, d$, where $W_0(\cdot) = |\cdot|$ and $dW_1(\cdot) = \mathcal{H}^{d-1}(\partial(\cdot))$, stands for the d -dimensional volume and the $(d-1)$ -dimensional surface content, respectively. Let denote $K \oplus (rB_e) := \{x \in \mathbb{R}^d : (rB_e) + x \cap K \neq \emptyset\}$ resp. $K \ominus (rB_e) := \{x \in K : (rB_e) + x \subseteq K\}$ the dilation resp. erosion of K by a closed Euclidean ball rB_e with radius $r \geq 0$ centered at the origin \mathbf{o} . The following lemma gives bounds of the deviation of the volumes $|K \oplus (rB_e)|$ and $|K \ominus (rB_e)|$ from $|K|$ in terms of the ratio $r/\varrho(K)$.

Lemma 5 With the above notation the following two inequalities hold for $0 \leq r \leq \varrho(K)$:

$$0 \leq 1 - \frac{|K \ominus (rB_e)|}{|K|} \leq \frac{dr}{\varrho(K)} \quad \text{and} \quad \frac{r}{\varrho(K)} \leq \frac{|K \oplus (rB_e)|}{|K|} - 1 \leq \frac{(2^d - 1)r}{\varrho(K)}. \quad (50)$$

Proof of Lemma 5 We start by recalling the fact that the function $r \mapsto |K \ominus rB_e|$ has a continuous derivative $\mathcal{H}^{d-1}(\partial(K \ominus (rB_e)))$ for $0 \leq r < \varrho(K)$, see Hadwiger (1957), p. 207. This and the monotonicity of the quermassintegral $W_1(\cdot)$ yields

$$0 \leq |K| - |K \ominus (rB_e)| = \int_0^r \mathcal{H}^{d-1}(\partial(K \ominus (\rho B_e))) d\rho \leq r \mathcal{H}^{d-1}(\partial K) \quad \text{for } 0 \leq r \leq \varrho(K). \quad (51)$$

Since $|K \ominus (\varrho(K)B_e)| = 0$ we get immediately that $|K| \leq \varrho(K) \mathcal{H}^{d-1}(\partial K)$. On the other hand, it was proved in Wills (1970) that $d|K| \geq \varrho(K) \mathcal{H}^{d-1}(\partial K)$ which implies the inclusion

$$\frac{1}{\varrho(K)} \leq \frac{\mathcal{H}^{d-1}(\partial K)}{|K|} \leq \frac{d}{\varrho(K)}. \quad (52)$$

Combining (51) and (52) yields the first inequality of the Lemma. To verify the second assertion we make use of the famous Steiner formula, see Hadwiger (1957), which reads as follows:

$$r \mathcal{H}^{d-1}(\partial K) = r d W_1(K) \leq |K \oplus (rB_e)| - |K| = \sum_{k=1}^d \binom{d}{k} W_k(K) r^k \quad (53)$$

Now, we apply of the well-known inequality $W_{k-1}(K) W_{k+1}(K) \leq (W_k(K))^2$ for $k = 1, \dots, d-1$, see Hadwiger (1957), p. 282, which implies together with the right-hand inequality of (52) that

$$W_k(K) \leq W_0(K) \left(\frac{W_1(K)}{W_0(K)} \right)^k = |K| \left(\frac{\mathcal{H}^{d-1}(\partial K)}{d|K|} \right)^k \leq \frac{|K|}{(\varrho(K))^k} \quad \text{for } k = 1, \dots, d.$$

Inserting this estimate on the r.h.s. of (53) we arrive at

$$|K \oplus (rB_e)| - |K| \leq |K| \left(\left(1 + \frac{r}{\varrho(K)} \right)^d - 1 \right) = \frac{r|K|}{\varrho(K)} \sum_{k=0}^{d-1} \left(1 + \frac{r}{\varrho(K)} \right)^k.$$

Since $0 \leq r/\varrho(K) \leq 1$, the latter sum is bounded by $2^d - 1$. This and the previous estimate combined with the lower bounds in (52) and (53) yield the second estimate of (50) which completes the proof the Lemma 5. \square

Corollary 4 It holds that $|\partial K \oplus (rB_e)| \leq (2^d - 1 + d) r \mathcal{H}^{d-1}(\partial K)$ for $r \geq 0$.

Proof of Corollary 4 Combining both inequalities of Lemma 5 leads to

$$|\partial K \oplus (rB_e)| = |K \oplus (rB_e)| - |K| + |K| - |K \ominus (rB_e)| \leq (2^d - 1 + d) r \frac{|K|}{\varrho(K)}$$

and the l.h.s. of (52) confirms the assertion of Corollary 4. \square

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7 Appendix B

In this second appendix we study the asymptotic behaviour of two further estimators of $\lambda^2 K_B(r)$ which are slightly different from (1). We consider their scaled and as unscaled version separately. To start with we define a (so-called naive) estimator $(\widetilde{\lambda^2 K_B})_n(r)$ which also turns out unbiased and is easier to calculate,

$$(\widetilde{\lambda^2 K_B})_n(r) := \frac{1}{|W_n|} \sum_{i \geq 1} \mathbf{1}_{W_n}(X_i) (N - \delta_{X_i})(r B + X_i) = \frac{1}{|W_n|} \sum_{i, j \geq 1}^{\neq} \mathbf{1}_{W_n}(X_i) \mathbf{1}_{rB}(X_j - X_i).$$

However, in order to calculate $(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)$ for $0 \leq r \leq R$ and $0 \leq \alpha < 1$ the PP $N = \sum_{i \geq 1} \delta_{X_i}$ must be observable in the larger window $W_n \oplus (c_n^\alpha R B)$ or the original window W_n must be replaced by the eroded window $W_n \ominus (c_n^\alpha R B)$ (which means ‘minus-sampling’). To avoid minus-sampling and as an further alternative to $(\widetilde{\lambda^2 K_B})_n(r)$ or $(\widehat{\lambda^2 K_B})_n(r)$ the estimator

$$(\overline{\lambda^2 K_B})_n(r) := \frac{1}{|W_n|} \sum_{i, j \geq 1}^{\neq} \mathbf{1}_{W_n}(X_i) \mathbf{1}_{W_n}(X_j) \mathbf{1}_{rB}(X_j - X_i)$$

is sometimes used although it is no longer unbiased but at least still asymptotically unbiased in the following sense: Provided that $c_n^\alpha / \varrho(W_n) \xrightarrow{n \rightarrow \infty} 0$ for some $0 \leq \alpha < 1$ it follows by means of Lemma 1 that, as $n \rightarrow \infty$,

$$\mathbf{E}(\overline{\lambda^2 K_B})_n(c_n^\alpha r) = \lambda \int_{c_n^\alpha r B} \frac{|W_n \cap (W_n - x)|}{|W_n|} \alpha_{red}^{(2)}(dx) = \lambda^2 K_B(c_n^\alpha r) \left(1 + \mathcal{O}\left(\frac{c_n^\alpha}{\varrho(W_n)}\right) \right).$$

Lemma 1B Let $N = \sum_{i \geq 1} \delta_{X_i}$ be a B_4 -mixing PP on \mathbb{R}^d . Further, let (W_n) be a CAS in \mathbb{R}^d with inball radius $\varrho(W_n)$ and $0 \leq \alpha < 1$. If $c_n^\alpha / \varrho(W_n) \xrightarrow{n \rightarrow \infty} 0$, then

$$\lim_{n \rightarrow \infty} |W_n|^{1-2\alpha} \mathbf{E} \left[(\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r) \right]^2 = 0$$

and

$$\lim_{n \rightarrow \infty} |W_n|^{1-2\alpha} \mathbf{Var} \left[(\overline{\lambda^2 K_B})_n(c_n^\alpha r) - (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r) \right]^2 = 0.$$

Proof of Lemma 1B We first express the variances of $(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)$ and $(\overline{\lambda^2 K_B})_n(c_n^\alpha r)$ in terms of factorial moment and cumulant measures of order $k = 2, 3, 4$. For this we apply general formula for the covariance $C(g, h) := \mathbf{Cov}\left(\sum_{i, j \geq 1}^{\neq} g(X_i, X_j), \sum_{k, \ell \geq 1}^{\neq} h(X_k, X_\ell)\right)$ which has been proved in Heinrich (1988), p. 97. To avoid ambiguities we present this formula once more for fourth-order stationary PP’s and Borel-measurable functions g, h on $\mathbb{R}^d \times \mathbb{R}^d$ such that the integrals on the r.h.s. exist:

$$\begin{aligned}
C(g, h) &= \int_{\mathbb{R}^{4d}} g(x, y) h(u, v) [\alpha^{(4)}(d(x, y, u, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(u, v))] \quad (54) \\
&+ \int_{\mathbb{R}^{3d}} g(x, y) [h(x, u) + h(y, u) + h(u, x) + h(u, y)] \alpha^{(3)}(d(x, y, u)) \\
&+ \int_{\mathbb{R}^{2d}} g(x, y) [h(x, y) + h(y, x)] \alpha^{(2)}(d(x, y)),
\end{aligned}$$

where the signed measure $\alpha^{(4)} - \alpha^{(2)} \times \alpha^{(2)}$ in (54) allows the decomposition (44).

We use (54) together with (44) for $g(x, y) = h(x, y) = |W_n|^{-1} \mathbf{1}_{W_n}(x) \mathbf{1}_{c_n^\alpha r B}(y - x)$ and get

$$\begin{aligned}
&|W_n|^{1-2\alpha} \mathbf{Var}[(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)] \\
&= \frac{1}{|W_n|^{1+2\alpha}} \left[\int_{\mathbb{R}^{4d}} \mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(u) \mathbf{1}_{c_n^\alpha r B}(y - x) \mathbf{1}_{c_n^\alpha r B}(v - u) (\alpha^{(4)} - \alpha^{(2)} \times \alpha^{(2)})(d(x, y, u, v)) \right. \\
&+ \int_{\mathbb{R}^{3d}} \left(\mathbf{1}_{W_n}(x)(1 + \mathbf{1}_{W_n}(u) + \mathbf{1}_{W_n}(y)) + \mathbf{1}_{W_n}(u)\mathbf{1}_{W_n}(y) \right) \mathbf{1}_{c_n^\alpha r B}(y - x) \mathbf{1}_{c_n^\alpha r B}(u - x) \alpha^{(3)}(d(x, y, u)) \\
&+ \left. \int_{\mathbb{R}^{2d}} \mathbf{1}_{W_n}(x) [1 + \mathbf{1}_{W_n}(y)] \mathbf{1}_{c_n^\alpha r B}(y - x) \alpha^{(2)}(d(x, y)) \right] \\
&= \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{|W_n \cap (W_n - u)|}{|W_n|} \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(v - u) (\alpha_{red}^{(4)}(d(y, u, v)) - \alpha_{red}^{(2)}(dy) \alpha^{(2)}(d(u, v))) \\
&+ \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \left(1 + \frac{|W_n \cap (W_n - y)|}{|W_n|} + \frac{|W_n \cap (W_n - u)|}{|W_n|} + \frac{|(W_n - y) \cap (W_n - u)|}{|W_n|} \right) \\
&\times \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(u) \alpha_{red}^{(3)}(d(y, u)) + \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \left(1 + \frac{|W_n \cap (W_n - y)|}{|W_n|} \right) \mathbf{1}_{c_n^\alpha r B}(x) \alpha_{red}^{(2)}(dx) \\
&\xrightarrow{n \rightarrow \infty} \begin{cases} \tau_B^2(r) & \text{for } \alpha = 0, \\ 4\lambda^3 (1 + \gamma_{red}^{(2)}(\mathbb{R}^d)) |r B|^2 = 4\lambda^2 \sigma^2 |r B|^2 & \text{for } \alpha > 0. \end{cases} \quad (55)
\end{aligned}$$

Both limits in (55) follow from (38) and (44) combined with a multiple use of Lemma 1 (for $\alpha = 0$ and $\alpha > 0$ separately). Comparing (55) with (39) and (45) reveals that

$$\lim_{n \rightarrow \infty} \frac{|W_n|}{|W_n|^{2\alpha}} \mathbf{Var}[(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)] = \lim_{n \rightarrow \infty} \frac{|W_n|}{|W_n|^{2\alpha}} \mathbf{Var}[(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)] \quad \text{for } 0 \leq \alpha < 1. \quad (56)$$

From (54) and (44) for $g(x, y) = h(x, y) = |W_n|^{-1} \mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y - x)$ we get in the

same way as before

$$\begin{aligned}
& |W_n|^{1-2\alpha} \mathbf{Var}[(\overline{\lambda^2 K_B})_n(c_n^\alpha r)] \\
&= \frac{1}{|W_n|^{1+2\alpha}} \left[\int_{W_n^4} \mathbf{1}_{c_n^\alpha r B}(y-x) \mathbf{1}_{c_n^\alpha r B}(v-u) (\alpha^{(4)} - \alpha^{(2)} \times \alpha^{(2)})(d(x, y, u, v)) \right. \\
&\quad \left. + 4 \int_{W_n^3} \mathbf{1}_{c_n^\alpha r B}(y-x) \mathbf{1}_{c_n^\alpha r B}(u-x) \alpha^{(3)}(d(x, y, u)) + 2 \int_{W_n^2} \mathbf{1}_{c_n^\alpha r B}(y-x) \alpha^{(2)}(d(x, y)) \right] \\
&= \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{|W_n \cap (W_n - y) \cap (W_n - u) \cap (W_n - v)|}{|W_n|} \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(v-u) \\
&\quad \times (\alpha_{red}^{(4)}(d(y, u, v)) - \alpha_{red}^{(2)}(dy) \alpha^{(2)}(d(u, v))) \\
&\quad + \frac{4\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \frac{|W_n \cap (W_n - y) \cap (W_n - u)|}{|W_n|} \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(u) \alpha_{red}^{(3)}(d(y, u)) \\
&\quad + \frac{2\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^d} \frac{|W_n \cap (W_n - x)|}{|W_n|} \mathbf{1}_{c_n^\alpha r B}(x) \alpha_{red}^{(2)}(dx) \xrightarrow{n \rightarrow \infty} \begin{cases} \tau_B^2(r) & \text{for } \alpha = 0, \\ 4\lambda^2 \sigma^2 |r B|^2 & \text{for } \alpha > 0. \end{cases}
\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{|W_n|}{|W_n|^{2\alpha}} \mathbf{Var}[(\overline{\lambda^2 K_B})_n(c_n^\alpha r)] = \lim_{n \rightarrow \infty} \frac{|W_n|}{|W_n|^{2\alpha}} \mathbf{Var}[(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)] \text{ for } 0 \leq \alpha < 1. \quad (57)$$

Next we regard the asymptotic behaviour of the covariance of $(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$ and $(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)$. For doing this we use (54) with $g(x, y) = \mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y-x) |W_n \cap (W_n - y + x)|^{-1}$ and $h(u, v) = |W_n|^{-1} \mathbf{1}_{W_n}(u) \mathbf{1}_{c_n^\alpha r B}(v-u)$ which yields

$$\begin{aligned}
& |W_n|^{1-2\alpha} \mathbf{Cov}((\widehat{\lambda^2 K_B})_n(c_n^\alpha r), (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)) \\
&= \frac{1}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{4d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y-x)}{|W_n \cap (W_n - y + x)|} \mathbf{1}_{W_n}(u) \mathbf{1}_{c_n^\alpha r B}(v-u) (\alpha^{(4)} - \alpha^{(2)} \times \alpha^{(2)})(d(x, y, u, v)) \\
&\quad + \frac{2}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y-x)}{|W_n \cap (W_n - y + x)|} (1 + \mathbf{1}_{W_n}(u)) \mathbf{1}_{c_n^\alpha r B}(u-x) \alpha^{(3)}(d(x, y, u)) \\
&\quad + \frac{2}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y-x)}{|W_n \cap (W_n - y + x)|} \alpha^{(2)}(d(x, y))
\end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{|W_n \cap (W_n - y) \cap (W_n - u)|}{|W_n \cap (W_n - y)|} \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(v - u) \\
&\times [\alpha_{red}^{(4)}(d(y, u, v)) - \alpha_{red}^{(2)}(dy) \alpha^{(2)}(d(u, v))] \\
&+ \frac{2\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \left(1 + \frac{|W_n \cap (W_n - y) \cap (W_n - u)|}{|W_n \cap (W_n - y)|}\right) \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(u) \alpha_{red}^{(3)}(d(y, u)) \\
&+ \frac{2\lambda \alpha_{red}^{(2)}(c_n^\alpha r B)}{|W_n|^{2\alpha}} \xrightarrow{n \rightarrow \infty} \begin{cases} \tau_B^2(r) & \text{for } \alpha = 0, \\ 4\lambda^2 \sigma^2 |r B|^2 & \text{for } \alpha > 0. \end{cases} \tag{58}
\end{aligned}$$

To obtain the limit (58) we have used quite the same arguments as in the proof of (55). Once again we employ (54) with $g(x, y) = |W_n|^{-1} \mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y - x)$ and $h(u, v) = |W_n|^{-1} \mathbf{1}_{W_n}(u) \mathbf{1}_{c_n^\alpha r B}(v - u)$. After performing almost the same calculations as before we arrive at

$$\begin{aligned}
&|W_n|^{1-2\alpha} \mathbf{Cov}((\overline{\lambda^2 K_B})_n(c_n^\alpha r), (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)) \\
&= \frac{1}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{4d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y - x)}{|W_n|} \mathbf{1}_{W_n}(u) \mathbf{1}_{c_n^\alpha r B}(v - u) (\alpha^{(4)} - \alpha^{(2)} \times \alpha^{(2)})(d(x, y, u, v)) \\
&+ \frac{2}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y - x)}{|W_n|} (1 + \mathbf{1}_{W_n}(u)) \mathbf{1}_{c_n^\alpha r B}(u - x) \alpha^{(3)}(d(x, y, u)) \\
&+ \frac{2}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \frac{\mathbf{1}_{W_n}(x) \mathbf{1}_{W_n}(y) \mathbf{1}_{c_n^\alpha r B}(y - x)}{|W_n|} \alpha^{(2)}(d(x, y)) \\
&= \frac{\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{3d}} \frac{|W_n \cap (W_n - y) \cap (W_n - u)|}{|W_n|} \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(v - u) \\
&\times [\alpha_{red}^{(4)}(d(y, u, v)) - \alpha_{red}^{(2)}(dy) \alpha^{(2)}(d(u, v))] \\
&+ \frac{2\lambda}{|W_n|^{2\alpha}} \int_{\mathbb{R}^{2d}} \left(1 + \frac{|W_n \cap (W_n - y) \cap (W_n - u)|}{|W_n|}\right) \mathbf{1}_{c_n^\alpha r B}(y) \mathbf{1}_{c_n^\alpha r B}(u) \alpha_{red}^{(3)}(d(y, u)) \\
&+ \frac{2\lambda}{|W_n|^{2\alpha}} \int_{c_n^\alpha r B} \frac{|W_n \cap (W_n - y)|}{|W_n|} \alpha^{(2)}(d(x, y)) \xrightarrow{n \rightarrow \infty} \begin{cases} \tau_B^2(r) & \text{for } \alpha = 0, \\ 4\lambda^2 \sigma^2 |r B|^2 & \text{for } \alpha > 0. \end{cases} \tag{59}
\end{aligned}$$

Since the limits (58) and (59) are identical and coincide with the asymptotic variances (45) and (56) for $0 \leq \alpha < 1$ the first assertion of Lemma 2B is proved due to the well-known relation

$$\begin{aligned} \mathbf{E}[(\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)]^2 &= \mathbf{Var}[(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)] + \mathbf{Var}[(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)] \\ &\quad - 2 \mathbf{Cov}((\widehat{\lambda^2 K_B})_n(c_n^\alpha r), (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)). \end{aligned}$$

In just the same way the identity

$$\begin{aligned} \mathbf{Var}[(\overline{\lambda^2 K_B})_n(c_n^\alpha r) - (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)]^2 &= \mathbf{Var}[(\overline{\lambda^2 K_B})_n(c_n^\alpha r)] + \mathbf{Var}[(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)] \\ &\quad - 2 \mathbf{Cov}((\overline{\lambda^2 K_B})_n(c_n^\alpha r), (\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)) \end{aligned}$$

implies the second assertion of Lemma 1B which completes the proof of Lemma 1B. \square

Lemma 2B Under the conditions of Lemma 2 we have

$$\lim_{n \rightarrow \infty} |W_n|^{1-2\alpha} \mathbf{E}((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - (\widehat{\lambda^2})_n K_B(c_n^\alpha r))^2 = 0 \quad \text{for all } r \geq 0. \quad (60)$$

Furthermore, $(\widehat{\lambda^2 K_B})_n(c_n^\alpha r)$ can be replaced by $(\widetilde{\lambda^2 K_B})_n(c_n^\alpha r)$.

Proof of Lemma 2B By $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$ and $(\widehat{\lambda^2})_n = (\widehat{\lambda}_n)^2 - \widehat{\lambda}_n/|W_n|$,

$$\begin{aligned} &|W_n|^{1-2\alpha} \mathbf{E}((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - (\widehat{\lambda^2})_n K_B(c_n^\alpha r))^2 \\ &= |W_n|^{1-2\alpha} \mathbf{E}\left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 K_B(c_n^\alpha r) - ((\widehat{\lambda^2})_n - \lambda^2) K_B(c_n^\alpha r)\right)^2 \\ &= |W_n|^{1-2\alpha} \mathbf{E}\left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 K_B(c_n^\alpha r) - 2\lambda K_B(c_n^\alpha r)(\widehat{\lambda}_n - \lambda) \right. \\ &\quad \left. - K_B(c_n^\alpha r)((\widehat{\lambda^2})_n - \lambda^2 - 2\lambda(\widehat{\lambda}_n - \lambda))\right)^2 \\ &\leq 3|W_n|^{1-2\alpha} \mathbf{E}\left((\widehat{\lambda^2 K_B})_n(c_n^\alpha r) - \lambda^2 K_B(c_n^\alpha r) - 2\lambda K_B(c_n^\alpha r)(\widehat{\lambda}_n - \lambda)\right)^2 \\ &\quad + |W_n|^{1-2\alpha} \mathbf{E}(K_B(c_n^\alpha r)(\widehat{\lambda}_n - \lambda)^2)^2 + 3|W_n|^{-(1+2\alpha)} \mathbf{E}(K_B(c_n^\alpha r)\widehat{\lambda}_n)^2. \quad (61) \end{aligned}$$

Clearly, $|W_n| \mathbf{E}(\widehat{\lambda}_n - \lambda)^4 = |W_n|^{-3} \mathbf{E}(N(W_n) - \lambda|W_n|)^4$ and it is easily checked that $\mathbf{E}(N(W_n) - \lambda|W_n|)^4 = \mathbf{Cum}_4(N(W_n)) + 3(\mathbf{Var}(N(W_n)))^2$, where $\mathbf{Cum}_4(X) = \frac{d^4}{dt^4} \log \mathbf{E}e^{itX}|_{t=0}$ denotes the fourth cumulant of the random variable X . The fourth cumulant of $N(W_n)$ can be expressed as linear combination of the factorial cumulant measures $\gamma^{(k)}$ applied to k -fold Cartesian product W_n^k and $\mathbf{E}N(W_n)$. A somewhat lengthy calculation based on (5) shows that

$$\mathbf{Cum}_4(N(W_n)) = \gamma^{(4)}(W_n \times \cdots \times W_n) + 6\gamma^{(3)}(W_n \times W_n \times W_n) + 7\gamma^{(2)}(W_n \times W_n) + \mathbf{E}N(W_n).$$

The B_4 -mixing property of the PP $N = \sum_{i \geq 1} \delta_{X_i}$ implies immediately the estimates

$$|\mathbf{Cum}_4(N(W_n))| \leq \lambda |W_n| (\gamma_{red}^{(4)}((\mathbb{R}^d)^3) + 6\gamma_{red}^{(3)}((\mathbb{R}^d)^2) + 7\gamma_{red}^{(2)}(\mathbb{R}^d) + 1)$$

and $\mathbf{Var}(N(W_n)) = \gamma_2(W_n \times W_n) + \mathbf{E}N(W_n) \leq \lambda |W_n| (\gamma_{red}^{(2)}(\mathbb{R}^d) + 1)$. These two estimates combined with (35) yield

$$|W_n|^{1-2\alpha} \mathbf{E} (K_B(c_n^\alpha r) (\widehat{\lambda}_n - \lambda)^2)^2 = \left(\frac{K_B(c_n^\alpha r)}{|W_n|^\alpha} \right)^2 |W_n| \mathbf{E} (\widehat{\lambda}_n - \lambda)^4 = \mathcal{O}(|W_n|^{-1})$$

and

$$|W_n|^{-(1+2\alpha)} \mathbf{E} (K_B(c_n^\alpha r) \widehat{\lambda}_n)^2 = \left(\frac{K_B(c_n^\alpha r)}{|W_n|^\alpha} \right)^2 \frac{\mathbf{E}(N(W_n))^2}{|W_n|^3} = \mathcal{O}(|W_n|^{-1})$$

as $n \rightarrow \infty$. Lemma 2 and the latter two relations show that the r.h.s. of (61) converges to zero which terminates the proof (60). Finally, by the norm inequality in L^2 it follows

$$\begin{aligned} \left(\mathbf{E} \left((\widehat{(\lambda^2 K_B)_n}(c_n^\alpha r) - (\widehat{\lambda^2})_n K_B(c_n^\alpha r)) \right)^2 \right)^{1/2} &\leq \left(\mathbf{E} \left((\widehat{(\lambda^2 K_B)_n}(c_n^\alpha r) - (\widehat{\lambda^2 K_B})_n(c_n^\alpha r)) \right)^2 \right)^{1/2} \\ &+ \left(\mathbf{E} \left((\widehat{(\lambda^2 K_B)_n}(c_n^\alpha r) - (\widehat{\lambda^2})_n K_B(c_n^\alpha r)) \right)^2 \right)^{1/2}. \end{aligned}$$

After multiplying both sides by $|W_n|^{1/2+\alpha}$ the first limit of Lemma 1B and (60) complete the proof of Lemma 2B. \square

At the end of this Appendix B we give a structural formula for the covariance function of the Gaussian limit process of $\sqrt{|W_n|} ((\widehat{\lambda^2 K_B})_n(r) - \lambda^2 K_B(r))$ for $r \geq 0$, which can be obtained simply by repeating the proof of (45) taking into account some obvious changes.

Lemma 3B For any B_4 -mixing PP $N = \sum_{i \geq 1} \delta_{X_i}$ on \mathbb{R}^d the asymptotic covariance

$$\tau_B(s, t) := \lim_{n \rightarrow \infty} |W_n| \mathbf{Cov} \left((\widehat{(\lambda^2 K_B)_n}(s), (\widehat{(\lambda^2 K_B)_n}(t)) \right)$$

exists for any $s, t \geq 0$ and takes the form

$$\begin{aligned} \tau_B(s, t) &= \lambda \int_{\mathbb{R}^{3d}} \mathbf{1}_{sB}(y-x) \mathbf{1}_{tB}(z) \gamma_{red}^{(4)}(d(x, y, z)) + 2 \lambda^2 |sB| \gamma_{red}^{(3)}(tB \times \mathbb{R}^d) \\ &+ 2 \lambda^2 |tB| \gamma_{red}^{(3)}(sB \times \mathbb{R}^d) + 2 \lambda^3 \int_{\mathbb{R}^{2d}} |(sB-x) \cap (tB-y)| \gamma_{red}^{(2)}(dx) \gamma_{red}^{(2)}(dy) \\ &+ 4 \lambda^3 |sB| |tB| \gamma_{red}^{(2)}(\mathbb{R}^d) + 4 \lambda \alpha_{red}^{(3)}(sB \times tB) + 2 \lambda \alpha_{red}^{(2)}((s \wedge t)B). \end{aligned} \quad (62)$$

We notice that $\int_{\mathbb{R}^{3d}} \mathbf{1}_{sB}(y-x) \mathbf{1}_{tB}(z) \gamma_{red}^{(4)}(d(x, y, z)) = \int_{\mathbb{R}^{3d}} \mathbf{1}_{tB}(y-x) \mathbf{1}_{sB}(z) \gamma_{red}^{(4)}(d(x, y, z))$ for all $s, t \geq 0$ due to the symmetry properties of $\gamma_{red}^{(4)}(\cdot)$.

In case of stationary Poisson cluster PPs and (α) -determinantal PPs the reduced cumulant measures $\gamma_{red}^{(2)}, \gamma_{red}^{(3)}, \gamma_{red}^{(4)}$ resp. their Lebesgue densities have a comparatively simple shape leading to more compact representations of $\tau_B(s, t)$, see [15], [21] or [5, 6]. In the special case of a stationary Poisson PP with intensity $\lambda > 0$ (implying $\gamma_{red}^{(k)}(\cdot) \equiv 0$ for $k = 2, 3, 4$) we get

$$\tau_B(s, t) = 2 \lambda^2 (s \wedge t)^d |B| (1 + 2 \lambda (s \vee t)^d |B|) \quad \text{for } s, t \geq 0, \text{ see [16], [19].}$$