# Algebraic notions of termination

**Jules Desharnais, Bernhard Möller, Georg Struth**

**Angaben zur Veröffentlichung / Publication details:**

Desharnais, Jules, Bernhard Möller, and Georg Struth. 2006. "Algebraic notions of termination." Augsburg: Institut für Informatik, Universität Augsburg.

UNIVERSITÄT AUGSBURG

# Algebraic Notions of Termination

J. Desharnais B. Möller G. Struth

Report 2006-23 October 2006

INSTITUT FÜR INFORMATIK
D-86135 AUGSBURG

# Algebraic Notions of Termination

Jules Desharnais[1], Bernhard Möller[2], Georg Struth[3]

[1]Département d'informatique et de génie logiciel, Université Laval, Québec QC G1K 7P4, Canada, email:Jules.Desharnais@ift.ulaval.ca

[2]Institut für Informatik, Universität Augsburg, Universitätsstr. 14, D-86135 Augsburg, Germany, email:moeller@informatik.uni-augsburg.de

[3]Department of Computer Science, The University of Sheffield, Sheffield S1 4DP, United Kingdom, email:g.struth@dcs.shef.ac.uk

**Abstract** Five algebraic notions of termination are formalised, analysed and compared: well-foundedness or Noetherity, Löb's formula, absence of infinite iteration, absence of divergence and normalisation. The study is based on modal semirings, which are additively idempotent semirings with forward and backward modal operators. To reason about infinite behaviour, semirings are extended to divergence semirings, divergence Kleene algebras and omega algebras. The resulting notions and techniques are used in calculational proofs of classical theorems of rewriting theory. These applications show that modal semirings are powerful tools for reasoning algebraically about the finite and infinite behaviour of programs and state transition systems.

**Keywords** Idempotent semirings, Kleene algebras, omega-algebras, divergence semirings, modal operators, well-foundedness, Noetherity, rewriting theory, program analysis, program termination.

## 1. Introduction

Idempotent semirings and Kleene algebras are fundamental structures in computer science with widespread applications. Roughly, idempotent semirings are rings without subtraction and with idempotent addition; Kleene algebras also provide an operation for finite iteration or reflexive transitive closure. Initially conceived as algebras of regular events [16], Kleene algebras have been extended by tests to model regular programs [17] and by infinite iteration to analyse reactive systems [6], program refinement [29] and rewriting systems [26, 27]. More recently, modal operators for idempotent semirings and Kleene algebras have been introduced [7, 9, 20] in order to model properties of programs and transition systems more conveniently and to link algebraic and relational formalisms with traditional approaches such as propositional dynamic logic and temporal logics.

Here, we propose modal semirings and modal Kleene algebras as tools for termination analysis of programs and transition systems: for formalising specifications and calculating proofs that involve termination, and for analysing and comparing different notions of termination. Benefits of this algebraic approach are simple abstract specifications, concise

We gratefully acknowledge partial support of this research by NSERC (Natural Sciences and Engineering Research Council of Canada) and within the Project InopSys (Interoperability of System Calculi) by DFG (Deutsche Forschungsgemeinschaft).

equational proofs, easy mechanisability and connections with automata-based decision procedures. Induction with respect to external measures, for instance, is avoided in favour of internal fixpoint reasoning. Particularly abstract, point-free, proofs can often be obtained in the algebra of modal operators.

The first contribution is a specification and comparison of five notions of termination in modal semirings and modal Kleene algebra.

(1) We translate the standard set-theoretic notions of Noetherity and well-foundedness and demonstrate their adequacy by several examples.
(2) We translate Löb's formula from modal logic (cf. [4]) and show its compatibility with the set-theoretic notions. We prove this modal correspondence result for a second-order frame property entirely by simple equational reasoning.
(3) We express termination as absence of infinite iteration in omega algebra [6]. This notion differs from the set-theoretic notion.
(4) We extend modal semirings to divergence semirings, thus describing the sources of possible non-termination in a state space. The associated notion of termination is proved compatible with the set-theoretic one.
(5) We express termination via normalisation. This is again compatible with the set-theoretic notion.

This analysis shows that modal semirings and modal Kleene algebras are powerful tools for analysing and integrating notions of termination. Their rich model classes, as investigated in [9], and the flexibility to switch between relation-style and modal reasoning makes the present approach more general than previous relation-based [10, 22], non-modal [6, 29] and mono-modal ones [12] which inspired this work.

The second contribution is an application of our termination techniques in rewriting theory, continuing previous research [11, 26, 27] on abstract reduction systems. Here, we prove the well-founded union theorem of Bachmair and Dershowitz [2] and a variant of Newman's lemma for non-symmetric rewriting [25] in modal Kleene algebra and divergence Kleene algebra. While the calculational proof of the commutative union theorem is novel, that of Newman's lemma requires less machinery than previous ones [10, 22]. Together with the results from [27], this exercise shows that large parts of abstract reduction can conveniently be modelled in variants of modal Kleene algebra.

The remainder of this text is organised as follows. Section 2 defines idempotent semirings, tests and modal operators together with their basic properties, symmetries and dualities. Section 3 adds unbounded finite iteration to yield (modal) Kleene algebras. Section 4 translates the set-theoretic notion of Noetherity to modal semirings and presents some basic properties. Sections 5 to 9 introduce and compare notions of termination based on modal logic, omega algebra, divergence semirings and normalisation. In particular, the novel concepts of divergence semiring and divergence Kleene algebra are introduced in Section 7 and a basic calculus for these structures is outlined in Section 8. Section 10 and Section 11 present calculational proofs of the well-founded union theorem and of Newman's lemma. Section 12 uses normalisation to relate confluence properties with normal forms. Section 13 contains a conclusion and an outlook.

## 2. Modal Semirings

We start with the definition of the algebraic structure that underlies the other algebras introduced in this paper, that of idempotent semirings.

**Definition 2.1.** Let $S = (S, +, \cdot, 0, 1)$ be an algebra.

(1) $S$ is a *semiring* if
   - $(S, +, 0)$ is a commutative monoid,
   - $(S, \cdot, 1)$ is a monoid,
   - multiplication distributes over addition from the left and right and
   - 0 is a left and right zero of multiplication.

(2) $S$ is an *idempotent semiring* if $S$ is a semiring and addition is idempotent.

We will usually omit the multiplication symbol. Two properties of semirings are particularly interesting for our purposes.

- Every semiring induces an *opposite semiring* in which the order of multiplication is swapped. For every statement that holds in a semiring there is a dual one that holds in its opposite.
- Every idempotent semiring $S$ admits a *natural order* $\leq$ defined by $a \leq b$ iff $a+b = b$ for all $a, b \in S$. This turns $(S, +)$ into a semilattice. One can show that $\leq$ is the only order for which addition is isotone in both arguments and for which 0 is the least element.

Idempotent semirings provide an algebraic model of sequential composition and angelic non-deterministic choice of actions. A prominent semiring is formed by set-theoretic relations under union and composition. Relations serve as a standard semantics for programs and transition systems and as Kripke frames for modal logics. Another important semiring is formed by the set of formal languages over an alphabet under union and concatenation.

General semiring elements model sets of transitions. Assertions or sets of states are modelled by special semiring elements called tests [17]; they should form a Boolean subalgebra of the overall semiring. From the semiring of relations one knows that tests can conveniently be represented as partial identity relations, i.e., as elements below the multiplicative unit 1. Moreover, join and meet of these elements coincide with their sum and product, respectively. This motivates the following definitions of complements and tests.

**Definition 2.2.** A *test* in a semiring $S$ is an element $p \leq 1$ that has a *complement* relative to 1, i.e., there is a $q \in S$ with $p + q = 1$ and $pq = 0 = qp$. The set of all tests of $S$ is denoted by $\mathsf{test}(S)$.

Straightforward calculations show that $\mathsf{test}(S)$ is closed under $+$ and $\cdot$ and has 0 and 1 as its least and greatest elements. Moreover, the complement of a test $p$ is uniquely determined by this definition; we denote it by $\neg p$. Hence $\mathsf{test}(S)$ indeed forms a Boolean algebra, i.e., a complemented distributive lattice. We will consistently write $a, b, c \ldots$ for arbitrary semiring elements and $p, q, r, \ldots$ for Boolean elements. Also, we will freely use the standard Boolean operations on $\mathsf{test}(S)$, for instance implication $p \to q = \neg p + q$ and relative complementation $p - q = p \cdot \neg q$, with their usual laws. It should be noted that $\neg$ as a unary operator binds more tightly than $+$ or $\cdot$ .

The above definition of tests deviates slightly from that in [17] in that it does not allow an arbitrary Boolean algebra of subidentities as $\mathsf{test}(S)$ but only the maximal complemented one. The reason is that the axiomatisation of the modal operators to be presented next forces this maximality anyway (see [9]).

To obtain further properties we briefly recall the following notion (see e.g. [18]). A *Galois connection* is a pair of mappings $f^\flat : B \to A$ and $f^\sharp : A \to B$ between partial orders

$(A, \leq_A)$ and $(B, \leq_B)$ such that, for all $a \in A$ and $b \in B$,

$$f^\flat(b) \leq_A a \Leftrightarrow b \leq_B f^\sharp(a).$$

Then $f^\flat$ and $f^\sharp$ are called the *lower* and *upper adjoints* of the Galois connection, resp.

In the remainder we omit the indices of the partial order relations involved. Moreover, we will freely use the standard pointwise lifting of partial orders to functions. Lower and upper adjoints enjoy many generic properties. The following ones are particularly useful for our purposes.

(1) $f^\flat(x) = \mathsf{inf}\,\{y : x \leq f^\sharp(y)\}$ and $f^\sharp(y) = \mathsf{sup}\,\{x : f^\flat(x) \leq y\}$, whence lower and upper adjoints uniquely determine each other.
(2) $f^\flat$ and $f^\sharp$ satisfy the *cancellation properties* $f^\flat \circ f^\sharp \leq id$ and, by order duality, $id \leq f^\sharp \circ f^\flat$.
(3) Lower adjoints are *completely additive*, i.e., preserve all existing suprema. Dually, upper adjoints are *completely multiplicative*, i.e., preserve existing infima.

Since the function $(p\,\cdot) = \lambda x\,.\,p \cdot x$ on tests is the lower adjoint in the Galois connection $p \cdot q \leq r \Leftrightarrow q \leq p \to r$ and the function $(p+) = \lambda x\,.\,p + x$ on tests is the upper adjoint in the Galois connection $q - p \leq r \Leftrightarrow q \leq p + r$, we obtain

$$(p\,\cdot) \text{ is completely additive,} \qquad (p+) \text{ is completely multiplicative.} \tag{2.1}$$

We now turn to modalities. Forward and backward diamond operators can be introduced as abstract preimage and image operators.

**Definition 2.3.** A semiring is called *modal* if for every element $a \in S$ there are operators $|a\rangle, \langle a| : \mathsf{test}(S) \to \mathsf{test}(S)$ obeying the following axioms:

$$|a\rangle p \leq q \Leftrightarrow \neg q a p \leq 0, \qquad \langle a|p \leq q \Leftrightarrow p a \neg q \leq 0, \tag{dia1}$$

$$|a\rangle(|b\rangle p) = |ab\rangle p, \qquad \langle a|(\langle b|p) = \langle ba|p. \tag{dia2}$$

This axiomatisation is equivalent to the purely equational, domain-based one in [9], since we can define the domain and codomain of an element $a$ as

$$\mathsf{dom}\, a = |a\rangle 1, \qquad \mathsf{cod}\, a = \langle a|1.$$

Conversely,

$$|a\rangle p = \mathsf{dom}(ap), \qquad \langle a|p = \mathsf{cod}(pa).$$

In particular, the diamonds are uniquely characterised by the axioms. The algebra of modal operators over an idempotent semiring has been studied in detail in [20]. Here, we only present a brief synopsis of the relevant properties.

First, we define forward and backward box operators as de Morgan duals of the diamonds:

$$|a]p = \neg|a\rangle\neg p, \qquad [a|p = \neg\langle a|\neg p.$$

Clearly, forward and backward operators of the same kind are dual with respect to opposition. Moreover, the modal operators form the following Galois connections with diamonds as lower and boxes as upper adjoints:

$$|a\rangle p \leq q \Leftrightarrow p \leq [a|q, \qquad \langle a|p \leq q \Leftrightarrow p \leq |a]q.$$

As a consequence, diamonds are (completely) additive and strict and boxes are (completely) multiplicative and co-strict:

$$\begin{aligned} |a\rangle(p+q) &= |a\rangle p + |a\rangle q, & \langle a|(p+q) &= \langle a|p + \langle a|q,\\ |a](pq) &= |a]p \cdot |a]q, & [a|(pq) &= [a|p \cdot [a|q,\\ |a\rangle 0 &= 0, & \langle a|0 &= 0,\\ |a]1 &= 1, & [a|1 &= 1. \end{aligned}$$

Every additive endofunction $f$ and every multiplicative endofunction $g$ on a Boolean algebra satisfy, for all elements $p$ and $q$, the useful identities

$$f(p) - f(q) \le f(p-q), \qquad g(p \to q) \le g(p) \to g(q). \tag{2.2}$$

As last properties we list the behaviour of box and diamond for tests. For $p, q \in \mathsf{test}(S)$,

$$|q\rangle p = qp = \langle q|p, \qquad |q]p = q \to p = [q|p. \tag{2.3}$$

Many properties of modal semirings can be expressed more succinctly in the endofunction space $\mathsf{test}(S) \to \mathsf{test}(S)$. The semiring operations are lifted as

$$(f+g)(p) = f(p) + g(p), \qquad (f \sqcap g)(p) = f(p) \cdot g(p), \qquad (f \cdot g)(p) = f(g(p))$$

and likewise for the other Boolean operations. In particular $\mathsf{1} = |1\rangle = \langle 1|$ and $\mathsf{0} = |0\rangle = \langle 0|$.

This yields closure conditions like $|a+b\rangle = |a\rangle + |b\rangle$ for addition, and covariant and contravariant laws $|ab\rangle = |a\rangle|b\rangle$ and $\langle ab| = \langle b|\langle a|$ for multiplication.

The lifting yields some further interesting operator-level laws and their duals. The Galois connections extend to endofunctions $f$ and $g$ on $\mathsf{test}(S)$:

$$|a\rangle f \le g \Leftrightarrow f \le [a|g, \qquad \langle a|f \le g \Leftrightarrow f \le |a]g. \tag{2.4}$$

This implies the following cancellation properties:

$$|a\rangle[a| \le \mathsf{1} \le [a||a\rangle, \qquad \langle a||a] \le \mathsf{1} \le |a]\langle a|. \tag{2.5}$$

Consequently, we have the co-Galois connections

$$\begin{aligned} f|a] \le g \Leftrightarrow f \le g\langle a| &\qquad \text{if } f \text{ and } g \text{ are isotone,}\\ f|a\rangle \le g \Leftrightarrow f \le g[a| &\qquad \text{if } f \text{ and } g \text{ are antitone.} \end{aligned}$$

Moreover, diamonds are isotone and boxes are antitone, i.e.,

$$a \le b \Rightarrow |a\rangle \le |b\rangle, \quad \text{and} \quad a \le b \Rightarrow |b] \le |a]. \tag{2.6}$$

Diamonds and boxes satisfy variants of (2.2), i.e.,

$$|a\rangle f - |a\rangle g \le |a\rangle(f-g), \qquad |a](f \to g) \le |a]f \to |a]g. \tag{2.7}$$

Finally, the above laws entail the following lifting property.

**Proposition 2.4.** *The (forward) diamonds on a modal semiring form an idempotent semiring.*

The point-free style and the properties of the operator algebra will yield more concise specifications and proofs in the following sections.

## 3. Modal Kleene Algebras

Kleene algebras are idempotent semirings with an additional operation of finite iteration. This allows modelling loops. Algebras that describe infinite iteration will be defined in Section 6.

**Definition 3.1** ([16])**.** A *left-inductive Kleene algebra* is a structure $(S, {}^*)$ such that $S$ is an idempotent semiring and the star operation ${}^* : S \to S$ satisfies, for all $a, b, c \in S$, the *left unfold* and *left induction* axioms

$$1 + aa^* \leq a^*, \qquad b + ac \leq c \Rightarrow a^*b \leq c.$$

*Right-inductive Kleene algebras* are their duals with respect to opposition, i.e., they satisfy the *right unfold* and *right induction* axioms $1 + a^*a \leq a^*$ and $b + ca \leq c \Rightarrow ba^* \leq c$.

By these axioms, $a^*b = \mu x.b + ax$ and $ba^* = \mu x.b + xa$, resp., where $\mu$ is the least fixpoint operator. Hence the star operation is isotone with respect to the natural order.

Two prominent left-inductive Kleene algebras are formed by enriching the relation semiring by the reflexive transitive closure operation and the language semiring by finite repetition, i.e., Kleene's original star operation. Proposition 3.5 shows that modal operators form left-inductive Kleene algebras as well. Various further models are discussed in [9].

It can be shown [19] that in a left-inductive Kleene algebra the star satisfies $aa^* = a^*a$; consequently, also the right unfold law $1 + a^*a \leq a^*$ holds.

**Definition 3.2.** In a left-inductive Kleene algebra, the *transitive closure* of $a$ is

$$a^+ = aa^* (= a^*a).$$

We will freely use the well-known standard properties of $a^+$.

**Definition 3.3** ([16])**.** A *Kleene algebra* is a left-inductive and right-inductive Kleene algebra.

**Definition 3.4.** A Kleene algebra $S$ is called *modal* if $S$ is a modal semiring.

It turns out that no extra axiom for the interaction between star and the modal operators is necessary, since the following properties can be shown [9]:

$$p + |a\rangle|a^*\rangle p = |a^*\rangle p, \qquad p + |a^*\rangle|a\rangle p = |a^*\rangle p, \qquad q + |a\rangle p \leq p \Rightarrow |a^*\rangle q \leq p. \tag{3.1}$$

These are used to prove the following statement [20].

**Proposition 3.5.** *The (forward) diamonds on a left-inductive modal Kleene algebra form a left-inductive Kleene algebra.*

In fact, $1 + |a\rangle|a^*\rangle = |a^*\rangle$, $1 + |a^*\rangle|a\rangle = |a^*\rangle$ and $f + |a\rangle g \leq g \Rightarrow |a^*\rangle f \leq g$ hold for arbitrary endofunctions $f$ and $g$ on a test algebra. This justifies setting $|a\rangle^* = |a^*\rangle$. Variants for the other modalities follow by duality.

The operator-level left star induction law is equivalent to the induction axiom of propositional dynamic logic

$$|a\rangle^* - 1 \leq |a\rangle^*(|a\rangle - 1). \tag{3.2}$$

## 4. Termination via Noetherity

In this section we abstract the notions of well-foundedness and Noetherity from the relation semiring to modal semirings. By definition, a relation $a$ on a set $p$ is well-founded iff every non-empty subset of $p$ has an $a$-minimal element. It is a standard exercise to show that this is equivalent to the absence of infinitely descending $a$-chains.

An element of set $p$ is $a$-minimal in $p$ if it has no $a$-predecessor in $p$, or, equivalently, if it is not in the image $\langle a|p$ is of $p$ under $a$. This motivates the following definition.

**Definition 4.1.** For a modal semiring $S$ and $a \in S, p \in \mathsf{test}(S)$, the *a-minimal part* of $p$ is $\mathsf{min}_a\, p = p - \langle a|p$. In point-free style this means $\mathsf{min}_a = 1 - \langle a|$. Dually, the *a-maximal part* is determined by $\mathsf{max}_a = 1 - |a\rangle$.

On the one hand, therefore, $a$ is well-founded iff $\mathsf{min}_a\, p$ is non-empty whenever $p$ is. On the other hand, an infinitely descending $a$-chain corresponds to a $p \neq 0$ for which $\mathsf{min}_a\, p = 0$. Thus absence of infinitely descending $a$-chains means that 0 is the only $p$ that satisfies $\mathsf{min}_a\, p \leq 0$.

Since well-foundedness and Noetherity are dual with respect to opposition and since we are mainly interested in termination, i.e., absence of strictly ascending sequences of actions, we will restrict our attention to Noetherity.

**Definition 4.2.** An element $a$ of a modal semiring $S$ is *Noetherian* if, for all $p \in \mathsf{test}(S)$,

$$\mathsf{max}_a\, p \leq 0 \Rightarrow p \leq 0.$$

Dually, $a$ is *well-founded* if $\mathsf{min}_a\, p \leq 0$ implies $p \leq 0$.

Similar definitions in related structures have been given in [1, 10, 12, 22].

To connect these notions to fixpoint-theoretic ones we briefly recall the following. Let $f$ be an endofunction on a partial order $(A, \leq)$. Then $a \in A$ is a *pre-fixpoint* of $f$ if $f(a) \leq a$. The notion of *post-fixpoint* is order-dual, and $a$ is a *fixpoint* of $f$ if it is both a pre- and a post-fixpoint. Now the following result is immediate from the definitions.

**Corollary 4.3.** *Assume a modal semiring $S$ and $a \in S, p \in \mathsf{test}(S)$.*

(1) $\mathsf{max}_a\, p \leq 0$ *iff $p$ is a post-fixpoint of the endofunction $|a\rangle$ on $\mathsf{test}(S)$.*
(2) *$a$ is Noetherian iff 0 is the unique post-fixpoint of $|a\rangle$, in other words, iff for all $p \in \mathsf{test}(S)$,*

$$p \leq |a\rangle p \Rightarrow p \leq 0.$$

We now relate Noetherity and finite iteration.

**Lemma 4.4.** *Assume a modal Kleene algebra $S$ and $a \in S, p \in \mathsf{test}(S)$. Define the endofunction $h_p : \mathsf{test}(S) \to \mathsf{test}(S)$ by $h_p(x) = p + |a\rangle x$.*

(1) $\mu h_p = |a^*\rangle p$.
(2) *If the greatest fixpoint $\nu|a\rangle$ of $|a\rangle$ exists then $h_p$ has the greatest fixpoint $\nu h_p = \mu h_p + \nu|a\rangle$.*
(3) *If $a$ is Noetherian then $h_p$ has the unique fixpoint $\mu h_p$.*
(4) *If for all $p$ the function $h_p$ has a unique fixpoint then $a$ is Noetherian.*

*Proof.*

(1) This follows from (3.1).
(2) The proof uses the principle of *greatest fixpoint fusion* (see e.g. [3] for the dual principle of least fixpoint fusion). Consider partial orders $(A, \leq_A)$ and $(B, \leq_B)$ and let $f : A \to B$,

$g : A \to A$ and $h : B \to B$ be isotone mappings. Assume that the greatest fixpoints $\nu g$ and $\nu h$ of $g$ and $h$ exist and that $f$ is completely multiplicative. Then

$$f \circ g = h \circ f \Rightarrow f(\nu g) = \nu h.$$

Now let $q = |a\rangle^* p$. Since $f = (q+)$ is completely multiplicative by (2.1), it suffices to show that $(q+) \circ |a\rangle = h_p \circ (q+)$. This is implied by (3.1) and additivity of $|a\rangle$ :

$$q + |a\rangle x = p + |a\rangle q + |a\rangle x = p + |a\rangle(q + x).$$

(3) By Corollary 4.3(2) we have $\nu h = 0$ and the claim follows by (2).
(4) Assuming uniqueness, by (2) we have for all $p$ that $\mu h_p = \mu h_p + \nu|a\rangle$, i.e., that $\nu|a\rangle \leq \mu h_p$. Setting $p = 0$ yields the claim by strictness of diamonds. □

A similar result for regular algebras appears in [3]. Our setting is more general in that we do not require completeness of the lattice induced by the natural order.

We now collect some useful algebraic properties of $\mathsf{max}$.

**Lemma 4.5.** *Let $S$ be a modal semiring. Let $a, b \in S$ and $p \in \mathsf{test}(S)$.*
(1) $\mathsf{max}_{a+b} = \mathsf{max}_a \sqcap \mathsf{max}_b$.
(2) $\mathsf{max}_0 = 1$.
(3) $\mathsf{max}_1 = 0$.
(4) $\mathsf{max}_a |a\rangle \leq |a\rangle \mathsf{max}_a$.
(5) *If $S$ is a modal Kleene algebra then* $\mathsf{max}_a |a\rangle^* \leq |a\rangle^* \mathsf{max}_{a^*}$.
(6) $a \leq b \Rightarrow \mathsf{max}_b \leq \mathsf{max}_a$.
(7) *For $m = \mathsf{max}_a 1$ we have $m = \neg\mathsf{dom}\, a = |a]0$. Hence $ma = 0$ and $ma^* = m$.*

*Proof.*
(1) By Boolean algebra

$$\mathsf{max}_{a+b} = 1 - (|a\rangle + |b\rangle) = (1 - |a\rangle) \sqcap (1 - |b\rangle) = \mathsf{max}_a \sqcap \mathsf{max}_b.$$

(2) and (3) follow immediately from the definition of $\mathsf{max}$.
(4) We calculate, using the definition of relative complementation and (2.2),

$$\mathsf{max}_a |a\rangle = (1 - |a\rangle)|a\rangle = |a\rangle - |a\rangle|a\rangle \leq |a\rangle(1 - |a\rangle) = |a\rangle \mathsf{max}_a.$$

(5) The proof is similar to that of (4), but uses the regular identity $aa^* = a^*a$ in the third step.

$$\begin{gathered}\mathsf{max}_a |a\rangle^* = (1 - |a\rangle)|a\rangle^* = |a\rangle^* - |a\rangle|a\rangle^* = \\ |a\rangle^* - |a\rangle^*|a\rangle \leq |a\rangle^*(1 - |a\rangle) = |a\rangle^* \mathsf{max}_{a^*}.\end{gathered}$$

(6) Immediate from (1).
(7) The first claim is immediate from the definitions. Next, $\neg\mathsf{dom}\, a\ a \leq 0$ by (dia1) (set $p = 1$ and $q = \mathsf{dom}\, a = |a\rangle 1$). Finally, by star unfold, $ma^* = m(1 + aa^*) = m + maa^* = m + 0a^* = m + 0 = m$. □

Property (7) will be used in the discussion of normalisation in Section 9; it means that $\mathsf{max}_a 1$ represents the states from which no $a$-transitions are possible, i.e., the normal forms under the transition system represented by $a$. Moreover, Lemma 4.5 is useful for proving some standard properties of Noetherian elements.

**Lemma 4.6.** *Assume a modal semiring $S$.*
(1) *Zero is the only Noetherian test.*
(2) *If a sum is Noetherian then so are its summands.*

(3) *Noetherity is downward closed.*
(4) *If $S$ is a modal Kleene algebra then an element is Noetherian iff its transitive closure is.*

*Proof.* Let $S$ be a modal semiring and let $a, b \in S$ and $p, q \in \mathsf{test}(S)$.
(1) Let $p = 0$. Then, by (2.3), 0 is the only post-fixpoint of $|p\rangle$ and $p$ is Noetherian by Lemma 4.5(2). For the converse direction, let $p \neq 0$. By (2.3) and idempotence of tests, $p$ is a (post-)fixpoint of $|p\rangle$ different from 0, i.e., $p$ is not Noetherian.
(2) Immediate from Lemma 4.5(1).
(3) Immediate from (2).
(4) By (3), Noetherity of $a^+$ implies that of $a$. Let, conversely, $a$ be Noetherian. Assume that $\mathsf{max}_{a^+}\, p \leq 0$. Then antitony of $\mathsf{max}$ and the regular identity $a^+ \leq a^*$ together with isotony and strictness of diamonds imply

$$|a\rangle^*(\mathsf{max}_{a^*}\, p) \leq |a\rangle^*(\mathsf{max}_{a^+}\, p) \leq 0,$$

whence $\mathsf{max}_a(|a\rangle^* p) \leq 0$ by Lemma 4.5(5). Now Noetherity of $a$ implies $|a\rangle^* p \leq 0$ and therefore $p \leq 0$, since $p \leq |a\rangle^* p$. □

Lemma 4.6(1) implies that 1 is not Noetherian. The Noetherian relations $\{(1,2)\}$ and $\{(2,1)\}$ show that the converse direction of Lemma 4.6(2) does not hold; the well-founded union theorem in Section 10 presents conditions that enforce this converse implication. Lemma 4.6(3) implies that Noetherian elements are irreflexive. Finally, if a non-trivial test is below an element then this element cannot be Noetherian. In particular, $a^*$ is not Noetherian, since $1 \leq a^*$.

## 5. Termination via Löb's Formula

We now investigate two alternative equational characterisations of termination. The first one involves transitive closure whereas the second one does not and hence works only for elements with transitive diamonds.

**Definition 5.1.** An element $a$ of a modal semiring is *diamond-transitive* or *d-transitive* if $|a\rangle|a\rangle \leq |a\rangle$.

Obviously, transitivity implies d-transitivity, but not vice versa. Consider, for instance, the path semiring consisting of sets of node sequences in a graph under union and path concatenation via a common intermediate node (also known as fusion product). The natural order in this case is set inclusion. For a set $p$ of graph nodes (each represented as a sequence of length one), the forward diamond $|a\rangle p$ yields the inverse image of $p$ under $a$, i.e., the set of all nodes from which an $a$-path leads to some node of $p$. Now let $a$ consist just of the single path $\langle 1,1\rangle$. Then $a \cdot a = \{\langle 1,1,1\rangle\} \not\subseteq a$, so that $a$ is not transitive. But

$$|a\rangle p = \begin{cases} \{\langle 1\rangle\} & \text{if } \langle 1\rangle \in p, \\ \emptyset & \text{otherwise,} \end{cases}$$

so that $|a\rangle|a\rangle \leq |a\rangle$ and $a$ is d-transitive.

**Definition 5.2.** A modal semiring $S$ is *extensional* if, for all $a, b \in S$,

$$|a\rangle \leq |b\rangle \Rightarrow a \leq b.$$

Equivalently, $S$ is *extensional* if, for all $a, b \in S$,

$$|a\rangle = |b\rangle \Rightarrow a = b.$$

In an extensional modal semiring, d-transitivity implies transitivity. Obviously, the path semiring is not extensional.

We now come to Löb's formula $\Box(\Box p \to p) \to \Box p$ from modal logic (cf. [4]). It expresses well-foundedness of transitive Kripke frames. To represent it algebraically, we first pass to a multi-modal view replacing $\Box$ by $|a]$ and then dualise it, by de Morgan's laws, to a form involving diamonds; in particular, the subformula $|a]p \to p$ turns into $p - |a\rangle p = \mathsf{max}_a\, p$. Finally, the main implication is replaced by the natural order on tests. This gives rise to the following notions.

**Definition 5.3.** An element $a$ of a modal Kleene algebra is
(1) *pre-Löbian* if $|a\rangle \le |a\rangle^+ \mathsf{max}_a$;
(2) *Löbian* if $|a\rangle \le |a\rangle \, \mathsf{max}_a$.

When $a$ is pre-Löbian, every state from which there is an $a$-step admits only $a$-chains that lead to $a$-maximal states. Thus there is no starting state of $a$ that can lead to non-termination.

Of course, every Löbian element of a modal Kleene algebra is pre-Löbian. For the converse direction we have the following result.

**Lemma 5.4.** *A d-transitive element of a modal semiring is Löbian iff it is pre-Löbian.*

*Proof.* By Proposition 3.5 and standard properties of transitive closure the diamond of a d-transitive element is its own transitive closure. $\Box$

The next statements relate Löbian and Noetherian elements.

**Theorem 5.5.** *An element of a modal Kleene algebra is Noetherian iff it is pre-Löbian.*

*Proof.* Consider a modal Kleene algebra $S$ and $a \in S$. Set $f = |a\rangle$ and $g = \mathsf{max}_a$.
($\Leftarrow$) Let $a$ be pre-Löbian, which is equivalent to $f - f^+ g \le \mathsf{0}$. Let $g(p) \le 0$, that is, $p \le f(p)$. We must show that $p \le 0$. We calculate

$$p \le f(p) = f(p) - f^+(0) = f(p) - f^+(g(p)) \le 0.$$

The second step uses strictness of diamonds. The third step uses the assumption on $g$. The fourth step uses the assumption that $a$ is pre-Löbian.
($\Leftarrow$) Let $a$ be Noetherian. This implies that $a$ is pre-Löbian if we can show that $f - f^+ g \le f(f - f^+ g)$. We calculate

$$\begin{aligned} f - f^+ g &= f - f f^* g \\ &\le f(\mathsf{1} - f^* g) \\ &= f(\mathsf{1} - (\mathsf{1} + f^+) g) \\ &= f(\mathsf{1} - (g + f^+ g)) \\ &= f((\mathsf{1} - g) - f^+ g) \\ &\le f(f - f^+ g). \end{aligned}$$

The first step uses the definition of $f^+$. The second step uses the identity (2.2). The fifth step uses the Boolean identity $p - (q + r) = (p - q) - r$. The last step uses isotony and the fact that $\mathsf{1} - g = \mathsf{1} - (\mathsf{1} - f) \le f$. This follows from the Boolean identities $p - (p - q) = pq \le q$. $\Box$

**Corollary 5.6.** *A d-transitive element of a modal semiring is Noetherian iff it is Löbian.*

*Proof.* This is immediate from Theorem 5.5 and Lemma 5.4. As in that lemma, the required transitive closure exists in the operator semiring by the assumption of d-transitivity. □

Let us discuss the intuition behind the proofs of Theorem 5.5 and Corollary 5.6.

If $a$ is pre-Löbian, then $|a\rangle - |a\rangle^+ \mathsf{max}_a \leq \mathsf{0}$. For a given $p$, the application $(|a\rangle - |a\rangle^+ \mathsf{max}_a)(p)$ of the left-hand side of this identity denotes the set of all states that admit $a$-steps leading outside the basin of attraction of termination in $p$. Now if $p$ had no $a$-maximal elements then *every* $a$-step would lead outside the (empty) basin of attraction, unless $p$ itself were empty. The first part of the proof of Theorem 5.5 formally reconstructs this argument.

Now let $a$ be Noetherian and assume that the set of states having $a$-steps leading outside the basin of attraction is non-empty, i.e., $a$ is not pre-Löbian. By Noetherity, this set, however, has an $a$-maximal element: a contradiction. This motivates the second part of the proof.

The general algebraic connection between Noetherity and Löb's formula is not novel. Goldblatt [12] has given a similar calculational proof in the more general setting of strict and additive mappings over a Boolean algebra. In fact, inspection of the proof of Theorem 5.5 shows that no further properties of modal Kleene algebra are needed. Given a strict additive $f : B \to B$ on a Boolean algebra $B$, Goldblatt defines the transitive closure $f^+$ of $f$ by the identities

$$f^+(p) = f(p + f^+(p)), \qquad f^+(p) - f(p) \leq f^+(f(p) - p).$$

While the first identity follows immediately from the operator-level unfold law $1 + ff^* = f^*$ and the definition of $f^+$ in Kleene algebra (Definition 3.2)), the second identity follows from the induction axiom of propositional dynamic logic (3.2) written as $f^* - 1 \leq f^*(f - 1)$.

Since Goldblatt obtains an entirely equational characterisation of Noetherity in his algebras, the corresponding classes, where all elements are Noetherian, are varieties. This is not the case in our setting, since already the unit of the semiring is not Noetherian. Therefore Noetherity cannot be imposed by an equation on the elements of the semiring, which conflicts with closure under homomorphic images.

A main contribution of this section is to show that Goldblatt's proof can be adapted to Kleene algebra. The richer structure of Kleene algebra simplifies certain arguments and, while Goldblatt's approach is essentially mono-modal, Kleene algebra provides an extension to a poly-modal setting and, as the following sections show, covers a wider range of applications.

The relation between Löb's formula and Noetherity, as expressed in Corollary 5.6, is interesting for the correspondence theory of modal logic. In this view, Noetherity expresses a frame property, which is part of semantics, whereas Löb's formula is a modal formula and hence a part of syntax. In Kleene algebra with domain, we are able to express syntax and semantics in one and the same formalism. Moreover, while the traditional proof of the correspondence uses model-theoretic semantic arguments based on infinite chains, the algebraic proof is entirely calculational and avoids infinity. This is quite beneficial for mechanisation.

## 6. Termination via Absence of Infinite Iteration

Cohen has extended Kleene algebra with an operator for infinite iteration [6] and presented applications of this omega algebra in concurrency control. His approach has been adapted to reasoning about program refinement in [29]. Omega algebra has also been used for proving theorems about rewriting systems that depend on termination [26, 27]. This section compares the notion of Noetherity induced by infinite iteration with the standard one. It turns out that the former behaves in a rather undesirable way for some applications. Section 7 presents an alternative approach that still is very similar to omega algebra, but captures the standard notion.

The omega operator is defined dually to the Kleene star as a greatest post-fixpoint.

**Definition 6.1.** An *$\omega$-algebra* is a structure $(S, \omega)$ such that $S$ is a Kleene algebra and, for all $a, b, c \in S$, the omega operator ${}^{\omega} : S \to S$ satisfies the *unfold axiom* and *co-induction axiom*

$$a^{\omega} \leq aa^{\omega}, \qquad c \leq ac + b \Rightarrow c \leq a^{\omega} + a^{*}b.$$

Thus, $a^{\omega} = \nu x.ax$ is a greatest fixpoint; therefore $\omega$ is isotone with respect to the natural ordering. The Kleene algebra of relations can be extended to an $\omega$-algebra in the standard way (see e.g. [22]).

The natural notion of termination for $\omega$-algebra is of course absence of infinite iteration.

**Definition 6.2.** An element $a$ of an $\omega$-algebra is *$\omega$-Noetherian* if $a^{\omega} \leq 0$.

Like in Section 2 for the Kleene star, it seems interesting to lift the axioms of $\omega$-algebra to the operator-level. This is very simple for the unfold axiom. The lifting of the induction axiom of Kleene algebra uses the demodalisation axiom (dia1) to eliminate a diamond from the left-hand side of an identity. In the co-induction axiom of $\omega$-algebra, however, the diamond of interest occurs at a right-hand side and there is no law like demodalisation to handle it. Therefore, the lifting seems to require additional assumptions.

**Lemma 6.3.** *The diamonds over an extensional modal $\omega$-algebra form an $\omega$-algebra.*

*Proof.* We show that $|a\rangle^{\omega} = |a^{\omega}\rangle$ satisfies the unfold and co-induction axiom of $\omega$-algebra.

For the unfold axiom, $|a\rangle^{\omega} = |a^{\omega}\rangle \leq |aa^{\omega}\rangle = |a\rangle|a^{\omega}\rangle = |a\rangle|a\rangle^{\omega}$, using isotony of diamonds.

For the co-induction axiom, assume $|c\rangle \leq |a\rangle|c\rangle + |b\rangle = |ac + b\rangle$, whence $c \leq ac + b$ by extensionality. Then $c \leq a^{\omega} + a^{*}b$ follows from the co-induction axiom and therefore $|c\rangle \leq |a^{\omega}\rangle + |a^{*}b\rangle = |a\rangle^{\omega} + |a\rangle^{*}|b\rangle$ from isotony of diamonds. $\square$

The following lemma compares Noetherity and $\omega$-Noetherity. In particular, it shows that their interrelation does not depend on extensionality of the modal semiring.

**Lemma 6.4.** *Over modal $\omega$-algebras we have the following results.*

(1) *Noetherian elements are $\omega$-Noetherian.*
(2) *$\omega$-Noetherian elements can, but need not be, Noetherian,*
(3) *not even if extensionality is assumed.*

*Proof.* (1) Let $a$ be Noetherian. Then $|a^{\omega}\rangle \leq |a\rangle|a^{\omega}\rangle$ implies that $|a^{\omega}\rangle p \leq 0$ for all $p \in \mathsf{test}(S)$. Setting $p = 1$ and $q = 0$ in (dia1) shows $a^{\omega} \leq 0$.

(2) In the $\omega$-algebra of languages of finite words, $a^{\omega} = 0$ if $1 \sqcap a \leq 0$, but also $1 = |a\rangle 1$, whenever $a \neq 0$. Thus every $a$ satisfying these conditions is $\omega$-Noetherian, but not Noetherian. Moreover, 0 is $\omega$-Noetherian and Noetherian.

(3) Consider the standard ordering $\leq$ on $\mathbb{N}$ and let $S$ consist of all subrelations of $\leq$ under the usual relational operations. In particular, the identity relation $1 = id$ is the multiplicative unit. Since $S$ forms a complete lattice, by the Knaster-Tarski theorem the

star and omega operators exist for all elements and the structure is an $\omega$-algebra. Also, as a relational structure, it is extensional. Now the successor function $\sigma$ on $\mathbb{N}$ is an element of $S$ and $\sigma^\omega = \nu x\,.\,\sigma \cdot x$. Thus we are searching for a solution of the identity $x = \sigma \cdot x$. Obviously, the empty set is the only one, since every solution of this identity must also be a solution of $x = \sigma^k \cdot x$ for all $k \in \mathbb{N}$. But for each pair $m \leq n$ there is a unique $i \in \mathbb{N}$ such that $(m, n) \in \sigma^i$, so that choosing $k > i$ shows that $(m, n)$ cannot be a member of any solution. Therefore $\sigma^\omega = 0$ and $\sigma$ is $\omega$-Noetherian.

However, $\sigma$ is a total function on $\mathbb{N}$ and therefore $|\sigma\rangle 1 = \mathsf{dom}\,\sigma = 1 \neq 0$. Consequently, $\max_\sigma 1 = 1 - |\sigma\rangle 1 = 0$, but $1 \neq 0$, i.e., $\sigma$ is not Noetherian. $\square$

This lemma is a first indication that Noetherity characterises non-termination more precisely than $\omega$-Noetherity. A more thorough discussion will be provided in the next section.

## 7. Termination via Absence of Divergence

We now introduce an alternative view of infinite iteration on a test algebra that handles the problems with $\omega$-algebra and also seems interesting for the analysis of infinite processes and reactive systems in general.

**Definition 7.1.** Let $S$ be a modal semiring and $a \in S$.

(1) A test $\nabla a \in \mathsf{test}(S)$ is called *divergence* of $a$ if it satisfies, for all $a \in S$ and $p, q \in \mathsf{test}(S)$, the *unfold axiom* and the *co-induction axiom*

$$\nabla a \leq |a\rangle \nabla a, \qquad p \leq |a\rangle p \Rightarrow p \leq \nabla a.$$

(2) When $\nabla a$ exists, we call $a$ *convergent* if $\nabla a = 0$ and *divergent* otherwise.
(3) $(S, \nabla)$ is a *divergence semiring* ($\nabla$-semiring) if $\nabla a$ exists for all $a \in S$ and hence $\nabla : S \to \mathsf{test}(S)$ is a total function.
(4) $(S, \nabla)$ is a *divergence Kleene algebra* ($\nabla$-Kleene algebra) if it is a divergence semiring and $S$ is also a Kleene algebra.

The above axioms characterise $\nabla a$ as the greatest fixpoint of $|a\rangle$; so the divergence of $a$ is unique if it exists.

Similar axioms have been used in [12] for defining mono-modal *foundational algebras*. Here, we use a different name to emphasise the fact that our structure is based on modal semirings and therefore poly-modal. Since $|a\rangle p = \neg|a]\neg p$, existence of $\nabla a$ also implies existence of the least fixpoint $\neg\nabla a$ of $|a]$; this is the *halting predicate* of the modal $\mu$-calculus (cf. [14]).

$\nabla$-Kleene algebras behave analogously to $\omega$-algebras.

**Lemma 7.2.** *Let $S$ be a $\nabla$-Kleene algebra, let $a \in S$ and $p, q \in \mathsf{test}(S)$. The $\nabla$-co-induction axiom is equivalent to*

$$p \leq |a\rangle p + q \Rightarrow p \leq \nabla a + |a^*\rangle q. \tag{7.1}$$

*Proof.* By Lemma 4.4(2), $\nu x.(|a\rangle x + q) = |a^*\rangle q + \nu|a\rangle$ with $\nu|a\rangle = \nabla(a)$. This yields (7.1) from the co-induction axiom. For the converse direction set $q = 0$ in (7.1). $\square$

The law (7.1) is often more suitable for computations than the co-induction axiom. This will become obvious later.

Existence of divergences can be guaranteed under additional assumptions.

**Lemma 7.3.** *Every modal semiring with complete test algebra is a $\nabla$-semiring. Every modal Kleene algebra with complete test algebra is a $\nabla$-Kleene algebra.*

*Proof.* For every element $a$ of a modal semiring with complete test algebra, $|a\rangle$ is isotone and hence, by the Knaster-Tarski theorem, has a greatest fixpoint that clearly satisfies the axioms of Definition 7.1. The claim about modal Kleene algebras follows from the one about modal semirings and the definitions. □

The co-induction axiom for $\nabla$-semirings comprises Noetherity as a special case.

**Lemma 7.4.**

(1) *Every Noetherian element of a modal semiring converges.*
(2) *Every convergent element of a $\nabla$-semiring is Noetherian.*

Thus, for Noetherian elements we can do without divergence and hence without the presuppositions for its existence, such as completeness of the test algebra. This is important for our applications in Section 10.

The following statement shows that the situation for $\omega$-Noetherian elements is different; it is a corollary to Lemma 6.4 (the language counterexample) and Lemma 7.4.

**Corollary 7.5.** *$\omega$-Noetherian elements of divergence $\omega$-algebras may be divergent.*

Therefore divergence, which corresponds to the standard notion of Noetherity, provides a more refined view of termination than $\omega$-Noetherity: the divergence characterises those states from which infinite paths can emanate, while omega iteration tells whether the algebra can represent these infinite paths in some way.

Let us illustrate this with the examples from the proof of Lemma 6.4. In the language semiring all elements $a \neq 0$ with $a \sqcap 1 = 0$ are non-Noetherian but $\omega$-Noetherian. The distinction vanishes in the encompassing algebra of languages over finite and infinite words, since it explicitly contains the infinite words as limits of iterated compositions of non-empty finite words. In the algebra of relations presented in the proof of Lemma 6.4(3), the successor relation $\sigma$ on $\mathbb{N}$ was shown to be non-Noetherian but $\omega$-Noetherian. This is caused by the restriction to relations that are subrelations of the standard order $\leq$ on $\mathbb{N}$. The analysis there shows that in a relation $a$ satisfying $a \leq \sigma a$ the inverse image of every number needs to be closed under $\sigma^* =\leq$, which is not possible for subrelations of $\leq$. In the encompassing full relation algebra over $\mathbb{N}$, however, such relations do exist; in particular, there $\sigma^\omega$ is the universal relation.

We now give a sufficient criterion for the coincidence of $\omega$-Noetherity and Noetherity. It uses the fact that in each $\omega$-algebra $1^\omega$ is the greatest element. This follows from setting $a = 1$ and $b = 0$ in the co-induction axiom. We define $\top = 1^\omega$. In particular, $\mathsf{dom}\,\top = 1$ since $\mathsf{dom}\,1 = 1$ and $\mathsf{dom}$ is isotone.

**Lemma 7.6.** *Let $S$ be an $\omega$-algebra.*

(1) $\mathsf{dom}\,a^\omega \leq \nabla a$ *holds for all* $a \in S$.
(2) $\forall a.a\top = \mathsf{dom}\,a\,\top \Rightarrow \forall a.\nabla a \leq \mathsf{dom}\,a^\omega$, *i.e., under this assumption $\omega$-Noetherity and Noetherity coincide.*

*Proof.*

(1) The operator level unfold law of $\omega$-algebra matches the antecedent of the $\nabla$-co-induction axiom. The claim then follows by modus ponens.

(2) First note that every test $p$ satisfies

$$\mathsf{dom}(p\top) = \mathsf{dom}(p\,\mathsf{dom}\,\top) = \mathsf{dom}(p1) = \mathsf{dom}\,p = p. \qquad (\dagger)$$

Now, by $\nabla$-unfold and the assumption,

$$\nabla a\top \leq (|a\rangle\nabla a)\top = \mathsf{dom}\,(a\,\nabla a\,)\top = a\,\nabla a\,\top.$$

Therefore $\nabla a\,\top \;\leq a^\omega$ by $\omega$ coinduction, and the claim follows by ($\dagger$) and isotony of domain.

$\square$

The condition in (2) is equivalent to the explicit domain representation

$$\mathsf{dom}\,a = a\top \sqcap 1,$$

which holds in full relation algebras but not in the subalgebra in the proof of Lemma 6.4(3).

Section 10 will provide examples where proofs from omega algebra can faithfully be translated from $\omega$-algebra to $\nabla$-Kleene algebra. And even beyond termination analysis, we believe that $\nabla$-Kleene algebras are interesting for modelling infinite behaviour of programs, transition systems and reactive systems. This surely deserves a deeper investigation.

## 8. Basic Divergence Calculus

The unfold and co-induction axioms of $\nabla$-Kleene algebras lead to properties that are analogous to those of $\omega$-algebras. However, because of the difference between the two notions already in the case of termination, we cannot transfer them without further argument. Here we collect only some that are needed in a later section.

**Lemma 8.1.** *Let $S$ be a $\nabla$-Kleene algebra and let $a, b \in S$.*

(1) $\nabla 0 = 0$ *and* $\nabla 1 = 1$,
(2) $\nabla a = |a\rangle\nabla a$,
(3) $\nabla a = |a\rangle^*\,\nabla a$,
(4) $a \leq b \Rightarrow \nabla a \leq \nabla b$,
(5) $\nabla a = \nabla(a^+)$,
(6) $\nabla(a+b) = \nabla(a^*b) + |a^*b\rangle^*\,\nabla a$,
(7) $|b^*\rangle(\nabla(b^*a)) = \nabla(b^*a)$.

*Proof.*

(1) The first property follows by $\nabla$-unfold, the second one by $\nabla$-co-induction.

(2) ($\leq$) is just the unfold axiom. ($\geq$) reduces, by co-induction, to $|a\rangle\nabla a \leq |a\rangle|a\rangle\nabla a$, which follows from the unfold axiom and isotony.

(3) ($\leq$) follows from the regular identity $1 \leq a^*$ and isotony. ($\geq$) reduces, by the unfold axiom, to $|a^*\rangle\nabla a \leq |a\rangle|a^*\rangle\nabla a$. But $|a^*\rangle\nabla a = |a^*\rangle|a\rangle\nabla a = |a\rangle|a^*\rangle\nabla a$ holds by (2) and the regular identity $aa^* = a^*a$.

(4) Let $a \leq b$. For $\nabla a \leq \nabla b$ it suffices by co-induction to show that $\nabla a \leq |b\rangle\nabla a$. But $\nabla a \leq |a\rangle\nabla a \leq |b\rangle\nabla a$ holds by unfold and isotony.

(5) ($\leq$) follows from the isotony of $\nabla$ (4) and the regular identity $a \leq a^+$. ($\geq$) reduces, by co-induction, to $\nabla(a^+) \leq |a\rangle\nabla(a^+)$. We calculate

$$\nabla(a^+) \leq |a^+\rangle\nabla(a^+) = |a\rangle|a^*\rangle\nabla(a^+) = |a\rangle|(a^+)^*\rangle\nabla(a^+) = |a\rangle\nabla(a^+).$$

The third step follows by the regular identity $a^* = (a^+)^*$. The last step uses (3).

(6) ($\leq$) reduces by co-induction (7.1) to

$$\nabla(a+b) \leq \nabla a + |a^*b\rangle\nabla(a+b) = \nabla a + |a^*\rangle(|b\rangle\nabla(a+b)),$$

which, again by co-induction (7.1), reduces to

$$\nabla(a+b) \leq |a\rangle\nabla(a+b) + |b\rangle\nabla(a+b) = |a+b\rangle\nabla(a+b).$$

But this holds by the unfold axiom.

($\geq$) We calculate

$$\begin{aligned} \nabla(a^*b) + |(a^*b)^*\rangle\nabla a &\leq \nabla((a+b)^+) + |(a+b)^*\rangle\nabla(a+b) \\ &= \nabla(a+b) + \nabla(a+b) \\ &= \nabla(a+b). \end{aligned}$$

The first step follows from the regular identities $a^*b \leq (a+b)^+$ and $(a^*b)^* \leq (a+b)^*$ and isotony. The second step follows from (5) and (3).

(7) We calculate

$$\nabla(b^*a) = |b^*a\rangle\nabla(b^*a) = |b^*\rangle|b^*a\rangle\nabla(b^*a) = |b^*\rangle\nabla(b^*a).$$

The first and last steps use (2). The second step uses the regular identity $b^*b^* = b^*$. □

## 9. Termination via Normalisation

After this introduction to the divergence calculus, we now resume the connection between semiring elements and transition systems. Remember from Lemma 4.5(7) that, for transition system $a$, the test $\mathsf{max}_a 1 = \neg\mathsf{dom}\, a$ can be viewed as an abstract representation of the *normal forms* w.r.t. $a$-transitions, i.e., the states from which no (further) $a$-transitions are possible. The process of normalisation, i.e., repeated $a$-transitions until a normal form has been reached (if there is one) is then described by the following notion.

**Definition 9.1.** The *normaliser* of an element $a$ of a modal Kleene algebra is

$$\mathsf{nml}\, a = a^*\,(\mathsf{max}_a\, 1) = a^*\,\neg\mathsf{dom}\, a.$$

In the relation semiring, $\mathsf{nml}\, a$ relates every element to all its normal forms under iterated $a$-transitions (if any). A first property is that normalisation is idempotent.

**Lemma 9.2.** $(\mathsf{nml}\, a)(\mathsf{nml}\, a) = \mathsf{nml}\, a$.

*Proof.* We calculate, using Lemma 4.5(7) and the multiplicative idempotence of tests,

$$a^*\,(\mathsf{max}_a\, 1)\; a^*\,(\mathsf{max}_a\, 1) = a^*\,(\mathsf{max}_a\, 1)\,(\mathsf{max}_a\, 1) = a^*\,(\mathsf{max}_a\, 1).$$

□

Second, Noetherity implies existence of normal forms for all starting elements.

**Lemma 9.3.** *For every Noetherian element $a$ of a modal Kleene algebra,* $\mathsf{dom}\,\mathsf{nml}\,a = 1$.

*Proof.* By Theorem 5.5 $a$ is pre-Löbian. Now we calculate, setting $m = \max_a 1 = \neg\mathsf{dom}\,a$,

$$\begin{aligned}\mathsf{dom}\,\mathsf{nml}\ a &= \mathsf{dom}(a^* m) = |a^*\rangle m = |1 + a^+\rangle m = m + |a^+\rangle(\max_a 1)\\ &\geq m + |a\rangle 1 = \neg\mathsf{dom}\,a + \mathsf{dom}\,a = 1.\end{aligned}$$

The decisive step is the last but two; it uses the defining property of pre-Löbian elements from Definition 5.3(1). $\square$

The converse of this statement does not hold.

**Example 9.4.** Consider the relation semiring over a two-element set $\{A, B\}$ and let $a = \{(A, A), (A, B)\}$. Then $\mathsf{nml}\ a = \{(A, B), (B, B)\}$ and $\mathsf{dom}\,\mathsf{nml}\ a = \{(A, A), (B, B)\} = 1$. But $\{(A, A)\} \subseteq a$ is not Noetherian and therefore by Lemma 4.6(3), neither is $a$. $\square$

The following example compares normalisation and $\omega$-Noetherity.

**Example 9.5.** The algebra of formal languages is both an $\omega$-algebra and a modal Kleene algebra. Its test set is $\{0, 1\}$. We have already shown that $|a\rangle 1 = \mathsf{dom}\,a = 1 \neq 0$ when $a \neq 0$ and hence $a$ is Noetherian iff $a = 0$. Moreover, distinguishing the cases $a = 0$ and $a \neq 0$, easy calculations using Lemma 4.5(2)/(3) show that $\mathsf{nml}\,a = \max_a 1$ (and hence also $\mathsf{dom}\,\mathsf{nml}\,a = \neg\mathsf{dom}\,a$). This mirrors the fact that, by totality of concatenation, a non-empty language can be iterated indefinitely without reaching a normal form. But we also have $a^\omega = 0$ whenever $1 \sqcap a = 0$. Therefore, $a^\omega = 0$ does not imply that $\mathsf{dom}\,\mathsf{nml}\,a = 1$, while $\nabla a = 0$ still implies this fact. $\square$

Again, this shows that $\omega$-algebra models non-termination less finely than the notions of Noetherity or divergence.

## 10. Additivity of Termination

We now turn to transition systems induced by term rewriting or reduction rules. Abstract reduction is that part of rewriting theory that disregards the term structure. It is essentially relational. Many statements of abstract reduction that depend on termination can be proved in $\omega$-algebra [26, 27], among them a variant of the well-founded union theorem of Bachmair and Dershowitz [2]. Since we have seen that termination is characterised in $\omega$-algebra less sharply than in $\nabla$-Kleene algebra, it is interesting and necessary to reconsider that proof. We will see that our new proofs again yield precise reconstructions of the standard diagrammatic arguments, but in a generalised setting. Thus modal Kleene algebra also admits an algebraic semantics for abstract reduction systems.

The connection between Kleene algebra and rewriting is as follows. An *abstract reduction system* (cf. [28]) is simply a set endowed with a family of binary relations. The operations on relations considered in rewriting are composition, union, conversion and symmetric, transitive and reflexive transitive closure. Therefore, properties of abstract rewrite systems can be expressed in modal Kleene algebra (conversion is obtained via the backward modal operators).

**Definition 10.1.** Let $S$ be a Kleene algebra and let $a, b \in S$.

(1) $a$ *locally semi-commutes* over $b$ if $ba \leq a^+ b^*$.
(2) $a$ *semi-commutes* over $b$ if $b^* a \leq a^+ b^*$.
(3) $a$ *quasi-commutes* over $b$ if $ba \leq a(a + b)^*$.

Semi-commutation and quasi-commutation state conditions for shifting certain steps to the left of others. In general, sequences of $a$-steps and $b$-steps can be split into a "good" part with all $a$-steps occurring to the left of $b$-steps and into a "bad" part where both kinds of steps are mixed.

For working with $\nabla$-Kleene algebras, we lift these properties to the operator level. As in Section 5 for transitivity, we introduce notions of diamond-commutation.

**Definition 10.2.** We say that *$a$ locally d-semi-commutes* over $b$ if $|b\rangle|a\rangle \leq |a\rangle^+|b\rangle^*$, and likewise for the other notions.

Again, the d-commutation properties are more general than the respective commutation properties; they are equivalent when the modal Kleene algebra is extensional. In order to avoid extensionality we will henceforth base our statements and proofs on d-commutation.

But first, we mention two auxiliary properties used to relate semi-commutation and quasi-commutation. The first one has been shown in [27], the second one lifts corresponding properties in [16].

**Lemma 10.3.**

(1) *For all elements $a$ and $b$ of a Kleene algebra,*

$$(a+b)^* = a^*b^* + a^*b^+a(a+b)^*. \tag{10.1}$$

(2) *For all $a$, $b$ and $c$ of a modal Kleene algebra,*

$$|ba\rangle \leq |ac\rangle \Rightarrow |b\rangle^*|a\rangle \leq |a\rangle|c\rangle^*, \qquad |ba\rangle \leq |ac\rangle \Rightarrow |b\rangle^+|a\rangle \leq |a\rangle|c\rangle^+. \tag{10.2}$$

The following lemma relates semi-commutation and quasi-commutation. A proof in $\omega$-algebra has been given in [27]. Here, we show that it translates to modal Kleene algebra. Remember that by Lemma 7.4(1) we can freely use the calculus of $\nabla$-Kleene algebra for Noetherian elements already in modal Kleene algebra.

**Lemma 10.4.** *Let $S$ be a modal Kleene algebra and let $a, b \in S$ with $a$ Noetherian. The following properties are equivalent.*

(1) *$a$ locally d-semi-commutes over $b$.*
(2) *$a$ d-semi-commutes over $b$.*
(3) *$a$ d-quasi-commutes over $b$.*

*Proof.* We only show equivalence between local semi-commutation and quasi-commutation. The proof for semi-commutation is similar. We set $f = |a\rangle$ and $g = |b\rangle$.

Let $a$ locally d-semi-commute over $b$. By pure Kleene algebra and without any Noetherity assumptions, $gf \leq f^+g^* = ff^*g^* \leq f(f+g)^*$.

Let now $a$ d-quasi-commute over $b$ and let $a$ be Noetherian. First, as in [27], we show that $h = f(f+g)^*$ satisfies $h \leq f^+(g^* + h)$:

$$\begin{aligned} f(f+g)^* &= f(f^*g^* + f^*g^+f(f+g)^*) \\ &= f^+(g^* + g^+f(f+g)^*) \\ &\leq f^+(g^* + f(f+g)^{*+}(f+g)^*) \\ &\leq f^+(g^* + f(f+g)^*). \end{aligned}$$

The first step uses (10.1). The second step uses distributivity and the definition of $f^+$. The third step uses the assumption of d-quasi-commutation and (10.2). The fourth step uses the regular identity $c^{*+}c^* \leq c^*$.

The above identity written point-wise means that, for all $p \in \mathsf{test}(S)$,

$$h(p) \le f^+(h(p)) + f^+(g^*(p)).$$

This matches the left-hand side of the co-induction rule (7.1) of $\nabla$-Kleene algebra for $\nabla(a^+)$. Since $a$ is Noetherian, so is $a^+$ by Lemma 4.6(4). Therefore $\nabla(a^+)$ exists by Lemma 7.4(1), viz. $\nabla(a^+) = 0$. Hence

$$g(f(p)) \le h(p) \le \nabla(a^+) + (f^+)^*(f^+(g^*(p))) = f^+(g^*(p)),$$

as required, where the first step uses the assumption of d-quasi-commutation. □

The proof of Lemma 10.4 simulates a previous one in $\omega$-algebra at the operator level. In [27] it has been argued that the latter formally reconstructs the previous diagrammatic proof from [25]. Therefore the new proof shares this property. However, our other formal notions of Noetherity provide the flexibility to use different techniques, when necessary. An alternative proof that directly uses Noetherity is given in [8].

**Lemma 10.5.** *Let $S$ be a modal Kleene algebra. Let $a, b \in S$ and let $a$ d-quasi-commute over $b$. Then Noetherity of $a$ implies Noetherity of $b^*a$.*

*Proof.* Using Lemma 7.4(1), we show $\nabla a \le 0 \Rightarrow \nabla(b^*a) \le 0$. Let $\nabla a \le 0$ and let $|ba\rangle \le |a(a+b)^*\rangle$. Then $|b^*a\rangle \le |a^+b^*\rangle$ by Lemma 10.4. Therefore

$$\nabla(b^*a) = |b^*a\rangle\nabla(b^*a) \le |a^+b^*\rangle\nabla(b^*a) = |a^+\rangle\nabla(b^*a).$$

The first step uses Lemma 8.1(2). The second step uses the assumption. The third step uses Lemma 8.1(7).

Now $\nabla(b^*a) \le |a^+\rangle\nabla(b^*a)$ implies $\nabla(b^*a) \le \nabla(a^+)$ by co-induction, from which the claim $\nabla(b^*a) \le 0$ follows by Lemma 8.1(5) and Noetherity. □

Lemma 10.5 generalises Lemma 2 of [2]. Again, its proof simulates a previous calculation in $\omega$-algebra at the operator level and directly corresponds to a diagrammatic proof [11, 27].

We now generalise the quasi-commutation theorem of Bachmair and Dershowitz (Theorem 1 of [2]).

**Theorem 10.6.** *Let $S$ be a modal Kleene algebra. Let $a, b \in S$ be such that $a$ d-quasi-commutes over $b$. Then $a + b$ is Noetherian iff $a$ and $b$ are Noetherian:*

$$\nabla(a+b) \le 0 \Leftrightarrow \nabla a + \nabla b \le 0.$$

*Proof.* By Lemma 4.6(2), Noetherity of a sum is inherited by its summands. So it remains to show the converse direction. Let $\nabla a + \nabla b \le 0$. First, denesting $\nabla(a+b)$ using Lemma 8.1(6) yields

$$\nabla(a+b) = \nabla(b^*a) + |b^*a\rangle^* \, \nabla b.$$

Now $\nabla(b^*a)$ vanishes by Lemma 10.5, using the assumption of d-quasi-commutation and Noetherity of $a$, and $|b^*a\rangle^*\nabla b$ vanishes by Noetherity of $b$ and strictness of diamonds. Thus also $\nabla(a+b) \le 0$. □

These results show that proofs for abstract reduction systems in modal Kleene algebra are as simple as those in $\omega$-algebra. The original proofs in [2] are rather informal, while also previous diagrammatic proofs (e.g. [25]) suppress many steps. Contrarily, the algebraic proofs are complete, formal and still simple. An extensive discussion of the relation between the proofs in $\omega$-algebra and their diagrammatic counterparts can be found in [11, 27]. In particular, the algebraic proofs mirror precisely the diagrammatic ones and follow essentially the line of reasoning from [2]. While this also holds for the modal proofs, it is not true for a relational proof of a similar, but somewhat more general theorem in [10] that uses the weaker condition $ba \leq a(a+b)^* + b$ instead of quasi-commutation. $\omega$-algebra has been used for proving further statements from concurrency control [6] and abstract rewriting [27] in a simple calculational way. We conjecture that they all translate to modal Kleene algebra.

## 11. Newman's Lemma

We now turn from quasi-commutation and semi-commutation to commutation and confluence. In rewriting theory, the generalisation from confluence to commutation has led to a theory of term rewriting for non-symmetric transitive relations and pre-congruences that comprises the traditional equational case [24, 25]. In particular, it introduces commutation-based variants of Church-Rosser theorems and of Newman's lemma. While the former can be proved in plain Kleene algebra [26, 27], it has been conjectured in [27] that a proof of Newman's lemma in pure $\omega$-algebra is impossible; that approach seems to cover only the regular fragment of abstract reduction, i.e, working at one end of a derivation expression, whereas proofs of Newman's lemma seem to require a context-free setting, since they also have to work in the interior of such expressions.

We reconstruct a previous diagrammatic proof of a variant of Newman's lemma for non-symmetric rewriting in modal Kleene algebra. Independently, the same statement has been obtained by purely syntactic considerations in [10]. There, it has been proven in a relation algebra without complementation that is more expressive than the algebras considered here. A relation-algebraic proof of the equational variant of Newman's lemma (cf. [28]) has been given in [22]. This proof, however, depends on normal forms which are not present in the non-symmetric case. In general, the results from [24, 25] show that confluence properties should be conceptually separated from such normal forms.

A straightforward relational specification of commutation and confluence requires the operation of relational conversion, which is not present in Kleene algebra. In [10], residuals (or factors) are used as a restricted form of conversion. We simulate conversion in modal Kleene algebra by semiring opposition, i.e., by switching between forward and backward modalities.

**Definition 11.1.** Let $S$ be a modal Kleene algebra and let $a, b \in S$.

(1) $a$ and $b$ *d-commute* if $\langle b^*||a^*\rangle \leq |a^*\rangle\langle b^*|$.
(2) $a$ and $b$ *locally d-commute* if $\langle b||a\rangle \leq |a^*\rangle\langle b^*|$.
(3) An element is *(locally) d-confluent* if it (locally) d-commutes with itself.

As in the case of transitivity and semi-commutation, the d-variants are strictly more general than the "classical" diamond-free ones (like that $a$ and $b$ commute if $(b^{\smile})^* a^* \leq a^*(b^{\smile})^*$, where $^{\smile}$ is the conversion operator).

Alternatively, if forward and backward modalities are not both available, commutation can be expressed by an algebraic variant of the Geach formula $|b\rangle|d] \leq |a]|c\rangle$ from modal logic (cf. [5]). The equivalences

$$|b\rangle|d] \le |a]|c\rangle \Leftrightarrow \langle a||b\rangle|d] \le |c\rangle \Leftrightarrow \langle a||b\rangle \le |c\rangle\langle d|$$

follow from the Galois and co-Galois connections.

We now prove the following variant of Newman's lemma.

**Theorem 11.2.** *Let $S$ be a modal Kleene algebra with complete test algebra. If $a+b$ is Noetherian and $a$ and $b$ locally d-commute then $a$ and $b$ d-commute.*

*Proof.* We express by a predicate $dc$ that two elements $a$ and $b$ d-commute when restricted to a set $p$ of starting states:

$$dc(p,a,b) \Leftrightarrow \langle b^*|\,\langle p\rangle\,|a^*\rangle \le |a^*\rangle\langle b^*|.$$

The notation $\langle p\rangle$ indicates that, since $p$ is a test, it does not matter whether we use the forward or backward diamond. Then $a$ and $b$ d-commute iff $dc(1,a,b)$ holds. By isotony of diamonds, $dc$ is downward closed with respect to its first argument, i.e., $dc(p,a,b)$ and $q \le p$ imply $dc(q,a,b)$. Moreover, by completeness of the test algebra,

$$r = \mathsf{sup}\,\{p : dc(p,a,b)\}$$

exists. It represents the set of all states on which $a$ and $b$ d-commute. In particular, $r$ itself satisfies $dc(r,a,b)$. This holds since diamonds and, by (2.1), also meets in a Boolean algebra are completely additive.

Together with downward closure of $dc$ this implies

$$p \le r \Leftrightarrow dc(p,a,b). \tag{11.1}$$

We use the dual variant $|a+b]q \le q \Rightarrow 1 \le q$ of Noetherity of $a+b$ to show that $r=1$, which, by the above remark, establishes d-commutation.

To obtain a suitable sufficient condition, we calculate

$$\begin{aligned}
|a+b]r \le r &\Leftrightarrow \forall p.(p \le |a+b]r \Rightarrow p \le r)\\
&\Leftrightarrow \forall p.(\langle a+b|p \le r \Rightarrow p \le r)\\
&\Leftrightarrow \forall p.(\langle a|p \le r \wedge \langle b|p \le r \Rightarrow p \le r)\\
&\Leftrightarrow \forall p.(dc(p_a,a,b) \wedge dc(p_b,a,b) \Rightarrow dc(p,a,b)),
\end{aligned}$$

where, for $x \in \{a,b\}$, $p_x$ abbreviates $\langle x|p = \mathsf{cod}(px)$. The second step uses the Galois connection for modalities. The third step uses additivity of diamonds and Boolean algebra. The fourth step uses (11.1).

So, assuming $dc(p_a,a,b) \wedge dc(p_b,a,b)$, we must now show $dc(p,a,b)$. By the star unfold law and distributivities,

$$\langle b^*|\langle p\rangle|a^*\rangle \le \langle b^*|\langle p\rangle + \langle b^*|\langle b|\langle p\rangle|a\rangle|a^*\rangle + \langle p\rangle|a^*\rangle.$$

The outer two of these summands are below $|a^*\rangle\langle b^*|$ by isotony of diamonds and Kleene algebra. For the middle summand we first show

$$\langle p\rangle|a\rangle \le |a\rangle\langle p_a\rangle, \qquad \langle b|\langle p\rangle \le \langle p_b\rangle\langle b|. \tag{11.2}$$

For the left identity, we calculate

$$\langle p\rangle|a\rangle = |pa\rangle = |pa\;\mathsf{cod}(pa)\rangle \le |a\;\mathsf{cod}(pa)\rangle = |a\rangle\langle p_a\rangle.$$

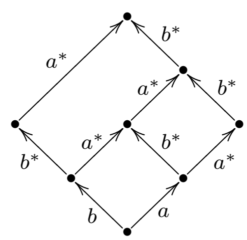
The proof of the right identity is dual.

Now the main claim is shown by the following calculation

$$\begin{aligned}
\langle b^*|\langle b|\langle p\rangle|a\rangle|a^*\rangle &\leq \langle b^*|\langle p_b\rangle\langle b||a\rangle\langle p_a\rangle|a^*\rangle \\
&\leq \langle b^*|\langle p_b\rangle|a^*\rangle\langle b^*|\langle p_a\rangle|a^*\rangle \\
&\leq \langle b^*|\langle p_b\rangle|a^*\rangle|a^*\rangle\langle b^*| \\
&\leq \langle b^*|\langle p_b\rangle|a^*\rangle\langle b^*| \\
&\leq |a^*\rangle\langle b^*|\langle b^*| \\
&\leq |a^*\rangle\langle b^*|.
\end{aligned}$$

The first step uses idempotence of $\langle p\rangle$, (11.2) twice and isotony of diamonds. The second step uses the local d-commutation of $a$ and $b$. The third step uses the assumption $dc(p_a, a, b)$. The fourth step uses the regular identity $c^*c^* = c^*$ lifted to diamonds. The fifth step uses the assumption $dc(p_b, a, b)$. The sixth step uses again the above regular identity. □

The last calculation in the proof can be visualised by the following diagram in which the bottom point is in $p$ and the two points in the next higher layer are in $p_b$ and $p_a$, respectively.




We conclude by noting that the assumption of Noetherity of $a + b$ cannot be weakened to separate Noetherity of $a$ and of $b$.

**Example 11.3** ([24])**.** Consider the following relations $a$ and $b$.

$$\begin{tikzcd}
1 & 2 \arrow[l, "b"'] \arrow[r, bend left, "a"] & 3 \arrow[l, bend left, "b"] \arrow[r, "a"] & 4
\end{tikzcd}$$

Relations $a$ and $b$ locally commute:

- $\langle b||a\rangle\{1\} = \langle b|a\rangle\{2\} = \emptyset$.
- $\langle b||a\rangle\{3\} = \{1\} \leq \{1, 2, 3\} = |a\rangle^*\langle b|^*\{3\}$.
- $\langle b||a\rangle\{4\} = \{2\} \leq \{2, 3, 4\} = |a\rangle^*\langle b|^*\{4\}$.
- The remaining cases follow from the atomic ones by additivity.

However, $a$ and $b$ do not commute, even though both are Noetherian:

- $\langle b||a\rangle|a\rangle\{4\} = \{1\} \not\leq \{2, 3, 4\} = |a\rangle^*\langle b|^*\{4\}$.

$a + b$ is not Noetherian: An infinite $a + b$-chain alternates between 2 and 3. □

## 12. Confluence and Unique Normal Forms

From the relational setting it is well known that confluence implies uniqueness of normal forms. This means that there the normaliser $\mathsf{nml}\, a = a^*\, (\mathsf{max}_a\, 1)$ (cf. Section 9) is a (partial) function, i.e., a deterministic relation. A relation $a$ is a partial function iff $a^{\smile} a \leq 1$ [22]. Again, this property can be abstracted to the level of modal operators.

**Definition 12.1.** An element $a$ of a modal semiring is *d-deterministic* if

$$\langle a||a\rangle \leq 1,$$

equivalently, if $|a\rangle \leq |a]$.

Of course, d-determinism is a special case of local d-confluence or d-commutation. It is also an interesting concept in the operational semantics of programming languages. From the definition it is immediate that every test is d-deterministic. The analogue to the above-mentioned relational property can be stated as follows.

**Lemma 12.2.** *The normaliser of a d-confluent element of a modal Kleene algebra is d-deterministic.*

*Proof.* Set $m = \mathsf{max}_a\, 1 = \neg\mathsf{dom}\, a$.

Now we calculate, using confluence of $a$ in the third step, Lemma 4.5(7) in the sixth step and the fact that $|p\rangle = \langle p|$ for all tests $p$ in the fourth and last steps,

$$\begin{aligned}
\langle \mathsf{nml}\, a||\mathsf{nml}\, a\rangle &= \langle a^*m||a^*m\rangle \\
&= \langle m|\langle a^*||a^*\rangle|m\rangle \\
&\leq \langle m||a^*\rangle\langle a^*||m\rangle \\
&= |m\rangle|a^*\rangle\langle a^*|\langle m| \\
&= |ma^*\rangle\langle ma^*| \\
&= |m\rangle\langle m| \\
&= \langle m||m\rangle \\
&\leq 1.
\end{aligned}$$

□

This statement is independent of termination properties. It has been added to further demonstrate the applicability of modal Kleene algebra in rewriting theory.

**Example 12.3.** The relation $a$ from Example 9.4 is confluent but not Noetherian and has normal form 2. The normaliser of $a$ is deterministic, as prescribed by Lemma 12.2. □

## 13. Conclusion

We have shown that modal semirings, modal Kleene algebras and divergence Kleene algebras are versatile tools for termination analysis, introducing and comparing different notions of termination and applying our techniques to examples from rewriting theory. All proofs are abstract, concise and entirely calculational. A particular result of our analysis is a critique of an earlier approach to termination based on omega algebra. Together with previous work [26, 27], our case studies on rewriting, more precisely, on abstract reduction systems, show that large parts of this theory can be reconstructed in modal Kleene algebra and divergence Kleene algebra. Due to its simplicity, the approach has considerable potential for mechanisation and automation. There are strong connections to automata-based decision procedures [20].

The proof of Newman's lemma and the associated diagram show that modal Kleene algebra allows induction in the interior part of an expression. This is not possible in pure Kleene algebra or omega algebra due to the shape of the star induction and omega co-induction axioms. Thus modal Kleene algebra supports "context-free" induction, whereas pure Kleene or omega algebra admits only its "regular" subcase. To achieve the same purpose, in [10] residuals are used to move the locus of induction from the interior of an expression to one of its ends and back.

The results of the present paper contribute to establishing modal Kleene algebra as a formalism that enhances cross-theory reasoning between different calculi for program analysis. While the de Morgan dual of divergence Kleene algebra was recently shown useful for modelling weakest precondition semantics à la Dijkstra [13, 21], we envision three main lines of further work.

First, a revision of the refinement calculus [29], which is based on $\omega$-algebra, in the modal setting and a modal analysis of non-terminating behaviour based on divergence Kleene algebra in general. Some results along these lines are contained in [15, 23].

Second, a further exploration of the connection with the modal $\mu$-calculus mentioned after Definition 7.1, in particular with a view on complexity.

Third, the mechanisation of our technique, the integration in a formal method like B [1] and applications to the analysis of programs and protocols and reactive systems.


## Acknowledgements

The authors would like to thank Roland Backhouse, Ernie Cohen, Peter Höfner, Gunther Schmidt and Kim Solin for inspiring discussions and the anonymous referees of the IFIP-TCS 2004 conference for helpful comments on an earlier version.


## References


[1] J.-R. Abrial. *The B-Book*. Cambridge University Press, 1996.

[2] L. Bachmair and N. Dershowitz. Commutation, transformation, and termination. In J. H. Siekmann, editor, *8th International Conference on Automated Deduction*, volume 230 of *Lecture Notes in Computer Science*, pages 5–20. Springer, 1986.

[3] R. Backhouse et al. Fixed point calculus. *Information Processing Letters*, 53:31–136, 1995.

[4] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[5] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[6] E. Cohen. Separation and reduction. In R. Backhouse and J. N. Oliveira, editors, *5th International Conference on Mathematics of Program Construction, MPC 2000*, volume 1837 of *Lecture Notes in Computer Science*, pages 45–59. Springer, 2000.

[7] J. Desharnais, B. Möller, and G. Struth. Modal Kleene algebra and applications—a survey. *Journal on Relational Methods in Computer Science*, 1:93–131, 2004.

[8] J. Desharnais, B. Möller, and G. Struth. Termination in modal Kleene algebra. In J.-J. Levy, E. W. Mayr, and J. C. Mitchell, editors, *3rd International Conference on Theoretical Computer Science*, pages 647–660. Kluwer, 2004.

[9] J. Desharnais, B. Möller, and G. Struth. Kleene algebra with domain. *ACM Transactions on Computational Logic*, 2006. To appear.

[10] H. Doornbos, R. C. Backhouse, and J. van der Woude. A calculational approach to mathematical induction. *Theoretical Computer Science*, 179:103–135, 1997.

[11] M. Ebert and G. Struth. Diagram chase in relational system development. In M. Minas, editor, *3rd IEEE workshop on Visual Languages and Formal Methods (VLFM'04 )*, volume 127 of *Electronic Notes in Theoretical Computer Science*, pages 87–105. Elsevier Science Publishers, 2005.

[12] R. Goldblatt. An algebraic study of well-foundedness. *Studia Logica*, 44(4):422–437, 1985.

[13] W. Guttmann and B. Möller. Modal design algebra. In S. Dunne and B. Stoddart, editors, *Unifying Theories of Programming*, volume 4010 of *Lecture Notes in Computer Science*, pages 236–256. Springer, 2006.

[14] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. MIT Press, 2000.
[15] P. Höfner, B. Möller, and K. Solin. Omega algebra, demonic refinement algebra and commands. In R. Schmidt, editor, *RelMiCS/AKA 2006*, volume 4136 of *Lecture Notes in Computer Science*, pages 222–234. Springer, 2006.
[16] D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366–390, 1994.
[17] D. Kozen. Kleene algebra with tests. *ACM Transactions on Programming Languages and Systems*, 19(3):427–443, 1997.
[18] A. Melton, D.A. Schmidt, and G.E. Strecker. Galois connections and computer science applications. In D. Pitt, S. Abramsky, A. Poigné, and D. Rydeheard, editors, *Category Theory and Computer Programming*, volume 240 of *Lecture Notes in Computer Science*, pages 299–312. Verlag, 1986.
[19] B. Möller. Lazy Kleene algebra. In D. Kozen, editor, *Mathematics of Program Construction*, volume 3125 of *Lecture Notes in Computer Science*, pages 252–273. Springer, 2004. Revised version: B. Möller: Kleene getting lazy. Sci. Comput. Prog. To appear.
[20] B. Möller and G. Struth. Algebras of modal operators and partial correctness. *Theoretical Computer Science*, 351:221–239, 2006.
[21] B. Möller and G. Struth. wp is wlp. In W. MacCaull, M. Winter, and I. Düntsch, editors, *Relational Methods in Computer Science*, volume 3929 of *Lecture Notes in Computer Science*, pages 200–211. Springer, 2006.
[22] G. Schmidt and T. Ströhlein. *Relations and Graphs: Discrete Mathematics for Computer Scientists*. EATCS Monographs on Theoretical Computer Science. Springer, 1993.
[23] K. Solin and J. von Wright. Refinement algebra with operators for enabledness and termination. In T. Uustalu, editor, *Mathematics of Program Construction*, volume 4014 of *Lecture Notes in Computer Science*, pages 397–415. Springer, 2006.
[24] G. Struth. Non-symmetric rewriting. Technical Report MPI-I-96-2-004, Max-Planck-Institut für Informatik, 1996.
[25] G. Struth. *Canonical Transformations in Algebra, Universal Algebra and Logic*. PhD thesis, Institut für Informatik, Universität des Saarlandes, Germany, 1998.
[26] G. Struth. Calculating Church-Rosser proofs in Kleene algebra. In H.C.M. de Swart, editor, *Relational Methods in Computer Science, 6th International Conference*, volume 2561 of *Lecture Notes in Computer Science*, pages 276–290. Springer, 2002.
[27] G. Struth. Abstract abstract reduction. *Journal of Logic and Algebraic Programming*, 66(2):239–270, 2006.
[28] Terese, editor. *Term Rewriting Systems*. Cambridge University Press, 2003.
[29] J. von Wright. From Kleene algebra to refinement algebra. In B. Möller and E. Boiten, editors, *6th International Conference on Mathematics of Program Construction, MPC 2002*, volume 2386 of *Lecture Notes in Computer Science*, pages 233–262. Springer, 2002.