Surmounting oscillating barriers: Path-integral approach for weak noise

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(Received 14 June 2000)

We consider the thermally activated escape of an overdamped Brownian particle over a potential barrier in the presence of periodic driving. A time-dependent path-integral formalism is developed which allows us to derive asymptotically exact weak-noise expressions for both the *instantaneous* and the *time-averaged* escape rate. Our results comprise a conceptually different, systematic treatment of the *rate prefactor* multiplying the exponentially leading Arrhenius factor. Moreover, an estimate for the deviations at finite noise strengths is provided and a supersymmetry-type property of the time-averaged escape rate is verified. For piecewise parabolic potentials, the rate expression can be evaluated in closed analytical form, while in more general cases, as exemplified by a cubic potential, an action-integral remains to be minimized numerically. Our comparison with very accurate numerical results demonstrates an excellent agreement with the theoretical predictions over a wide range of driving strengths and driving frequencies.

PACS number(s): 05.40.−a, 82.20.Mj, 82.20.Pm

I. INTRODUCTION

The thermally activated escape over a potential barrier is a recurrent theme in a large variety of physical, chemical, and biological contexts [1–3]. In the case of foremost practical relevance, the characteristic strength of the thermal noise (the thermal energy $k_B T$) is much smaller than the potential barrier. As a consequence, successful barrier crossings constitute rare events and the escape statistics verifies with very high accuracy an exponential decay as a function of time. In other words, a meaningful escape rate can be defined which completely characterizes the decay process. A seminal contribution to the theory of escape rates represents the work by Kramers in 1940 [4], which subsequently has been refined, modified, and generalized in various important directions [1–3].

A particularly challenging direction are systems far away from thermal equilibrium, either due to nonthermal noise or external deterministic forces [1]. In such a case, the relevant probability distribution strongly deviates from the Boltzmann form in the entire state space and its determination becomes a highly nontrivial problem. *Mutatis mutandis*, this very same basic difficulty resurfaces again in all known theoretical methods of calculating escape rates in far from equilibrium systems [5–12].

The subject of our present paper is one of the simplest nonequilibrium descendants of the original problem of Kramers: namely the thermally activated escape of an overdamped Brownian particle over a potential barrier in the presence of a periodic driving (details are given in Sec. II). This is a prototypical setup in the sense that investigating the behavior of a system under the influence of a periodic forcing represents a particularly natural and straightforward experimental situation. Examples arise in the context of laser driven semiconductor heterostructures [13], stochastic resonance [14], directed transport in rocked Brownian motors [15–17], or periodically driven ''resonant activation'' processes [18,19] like ac-driven biochemical reactions in protein membranes [20], to name only a few.

Despite its experimental importance, the theory of oscillating barrier crossing in the regime of weak thermal noise is still at its beginning. Previous quantitative, analytical investigations have been restricted to weak [21–23], slow [24,25], or fast [21,24,26] driving. In this paper we continue our recent study [27] of the most challenging intermediate regime of *moderately strong* and *moderately fast* driving by means of path-integral methods. The general framework of this approach is derived from scratch in Sec. III, thus collecting, streamlining, and partially extending previously known material. The evaluation of the escape rate is worked out in Sec. IV with the central results (116) for the time-averaged and (108) for the instantaneous escape rate. Especially, these results comprise a conceptually different, systematic treatment of the *rate prefactor* multiplying the exponentially leading Arrhenius factor. They become asymptotically exact for any finite amplitude and period of the driving as the noise strength tends to zero. On the other hand, for any fixed (small) noise strength, we have to exclude extremely small driving amplitudes and extremely long or short driving periods since this would lead us effectively back to an undriven escape problem, which is not covered by our present approach. Another situation which is excluded in our theory is the case of extremely strong driving such that escape events become possible even in the absence of the thermal noise [28,29]. Closest in spirit to our methodology is the recent work [23], which is restricted, however, to the linear-response regime (weak driving) for the exponentially leading part (Arrhenius factor) and treats the prefactor by means of a matching procedure, involving the barrier region only. The approximation adopted in that work is complementary to ours in that it admits, for a fixed (small) noise strength, arbitrarily small driving amplitudes.

In Sec. V our analytical predictions are verified for the case of sinusoidally rocked metastable potentials against very precise numerical results. A first example consists of a piecewise parabolic potential, for which our general rate expressions can be evaluated in closed analytical form. In more general cases, exemplified in Sec. V by a cubic potential, a few elementary numerically tasks remain before actual num-

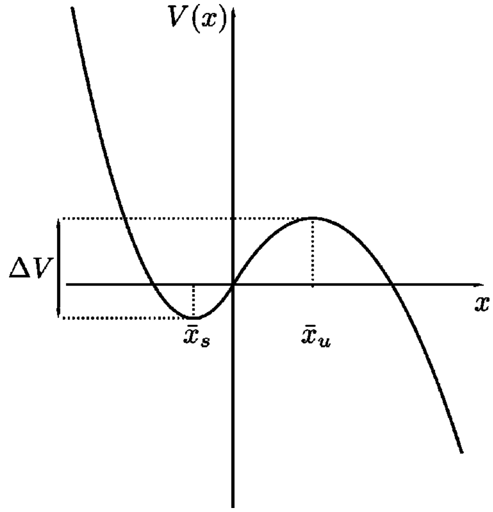


FIG. 1. Sketch of a typical metastable potential $V(x)$ in Eq. (5). Plotted is the piecewise parabolic example (134) with parameters $\Delta V = 0.9$, $\lambda_s = -0.6$, and $\lambda_u = 0.3$ in arbitrary, dimensionless units.

bers can be obtained from our rate expressions. The final conclusions are presented in Sec. VI.

II. ESCAPE PROBLEM

A. Model

We consider the following model for the one-dimensional Brownian motion of a particle with coordinate $x(t)$ and mass m under the influence of a time-dependent force field $F(x,t)$:

$$m\ddot{x}(t) - F(x(t),t) = -\eta\dot{x}(t) + \sqrt{2D}\xi(t). \tag{1}$$

While the left-hand side accounts for the dynamics of the isolated particle, the right-hand side models the influence of its thermal environment [1] with a viscous friction coefficient η and a randomly fluctuating force $\xi(t)$, which is assumed to be unbiased Gaussian white noise with correlation

$$\langle \xi(t)\xi(t')\rangle = \delta(t-t'). \tag{2}$$

At thermal equilibrium, the intensity D of the noise is related to the temperature T according to the Einstein relation $D = \eta k_B T$, where k_B is Boltzmann's constant [1]. Throughout this paper we will restrict ourselves to the overdamped motion such that inertia effects $m\ddot{x}(t)$ in Eq. (1) are negligible [30]. Choosing the time unit such that $\eta = 1$, the stochastic dynamics takes the form

$$\dot{x}(t) = F(x(t),t) + \sqrt{2D}\xi(t). \tag{3}$$

The force field $F(x,t)$ in Eq. (3) is assumed to derive from a metastable potential which undergoes an arbitrary periodic modulation in time with period $\mathcal{T}$,

$$F(x,t+\mathcal{T}) = F(x,t). \tag{4}$$

An example is a metastable static potential $V(x)$ as cartooned in Fig. 1, supplemented by an additive sinusoidal driving

$$F(x,t) = -V'(x) + A\sin(\Omega t), \tag{5}$$

$$\Omega = 2\pi/\mathcal{T}. \tag{6}$$

Our next assumption is that the deterministic dynamics in Eq. (3) with $D = 0$ exhibits a stable periodic orbit $x_s(t)$ and an unstable periodic orbit $x_u(t)$ [31], satisfying

$$\dot{x}_{s,u}(t) = F(x_{s,u}(t),t), \tag{7}$$

$$x_{s,u}(t+\mathcal{T}) = x_{s,u}(t), \tag{8}$$

where ''s,u'' means that the index may be either ''s'' or ''u.'' Moreover, every deterministic trajectory is assumed to approach in the long-time limit either the attractor $x_s(t)$ or to diverge towards $x = \infty$, except if it starts exactly at the separatrix $x_u(t)$ between those two basins of attraction. In other words, the metastable potential is required not to be rocked too violently such that particles cannot escape deterministically, i.e., without the assistance of the random fluctuations in Eq. (3). It is clear that $x_s(t)$ and $x_u(t)$ must be disjoint and by assuming a second ''attractor'' at $x = \infty$ we have, without loss of generality, implicitly restricted ourselves to the case that

$$x_u(t) > x_s(t) \tag{9}$$

for all t. Note that the above requirements do not necessarily exclude the possibility that for certain times t the ''instantaneous potential,'' from which the force field $F(x,t)$ derives, does no longer exhibit a potential barrier.

B. Escape rates

Next, we return to the stochastic dynamics (3) with a finite but very small noise strength D such that a particle $x(t)$ is able to leave the domain of attraction of the stable periodic orbit $x_s(t)$ and subsequently disappear towards $x = \infty$ but the typical waiting time before such an event occurs is much longer than all characteristic time scales of the deterministic dynamics (separation of time scales [1,22,32]). For a quantitative characterization of such escape events, our starting point is the probability distribution $p(x,t)$ of particles which is governed by the Fokker-Planck equation [33]

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}\left\{-F(x,t) + D\frac{\partial}{\partial x}\right\}p(x,t). \tag{10}$$

Once $p(x,t)$ is known, the population $P_{\mathcal{A}}(t)$ of the time-dependent basin of attraction $\mathcal{A}(t) := (-\infty, x_u(t)]$ of the stable periodic orbit $x_s(t)$ follows as

$$P_{\mathcal{A}}(t) = \int_{-\infty}^{x_u(t)} p(x,t)\,dx. \tag{11}$$

A suggestive definition of the ''instantaneous rate'' $\Gamma(t)$ is then provided by the relative decrease of this population per time unit

$$\Gamma(t) := -\dot{P}_{\mathcal{A}}(t)/P_{\mathcal{A}}(t). \tag{12}$$

We note that particles which leave the domain of attraction give rise to a positive contribution to $\Gamma(t)$. There is also a certain probability that particles from outside this domain recross the separatrix $x_u(t)$, giving rise to a negative contribution to $\Gamma(t)$. In other words, Eq. (12) is the net flux of particles (outgoing flux minus back flux) across the separatrix $x_u(t)$ in units of the remaining population $P_{\mathcal{A}}(t)$. By exploiting the deterministic dynamics (7) for $x_u(t)$, the Fokker-Planck equation (10) for $p(x,t)$, and the definition (11) of $P_{\mathcal{A}}(t)$, we can rewrite the instantaneous rate (12) as

$$\Gamma(t) = -\frac{D}{P_{\mathcal{A}}(t)} \frac{\partial p(x_u(t),t)}{\partial x_u(t)}. \tag{13}$$

Inside the metastable state $x < x_u(t)$ the particle distribution is governed by intrawell relaxation processes. For small noise strengths D, their characteristic time scales are well separated from the typical escape time itself [1,22,32]. On this time scale of the intrawell relaxation, transients die out and the distribution $p(x,t)$ approaches a quasiperiodic dependence on time t. More precisely, $p(x,t)/\int_{-\infty}^{x_u(t)} p(x,t)dx$ tends, at least for $x \leqslant x_u(t)$, towards a time-periodic function as t grows. The same carries over to the escape probability (13) and thus the time-averaged escape rate

$$\overline{\Gamma} := \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} \Gamma(t')dt' \tag{14}$$

becomes independent of the time t.

Our assumption of weak noise guarantees that the loss of population $P_{\mathcal{A}}(t)$ is negligible on the time scale of the intrawell relaxation for any initial distribution $p(x,t_0)$ that is negligibly small in the vicinity and beyond the instantaneous separatrix $x_u(t_0)$. The denominator in Eq. (13) can thus be approximated by 1 for all times $t-t_0$ much smaller than the characteristic escape time $1/\overline{\Gamma}$ itself. Further, we can restrict ourselves to δ-distributed initial conditions of the form $p(x,t_0) = \delta(x-x_0)$ with x_0 inside the basin of attraction $\mathcal{A}(t)$ of $x_s(t)$ such that the overwhelming majority of realizations (3) will first relax towards a close neighborhood of the attractor $x_s(t)$ before they escape. The behavior of more general initial distributions then readily follows by way of linear superposition. Moreover, one expects [1,22,32] that after transients (intrawell relaxation processes) have died out, the time-dependent escape rate (13) will actually become independent of the initial conditions x_0 and t_0. Denoting by $p(x,t|x_0,t_0)$ the conditional probability associated with an initial δ-peak at x_0 we thus can rewrite Eq. (13) as

$$\Gamma(t) = -D \frac{\partial p(x_u(t),t|x_0,t_0)}{\partial x_u(t)}. \tag{15}$$

We recall that this expression is valid even if $t-t_0$ is not large, but then $\Gamma(t)$ still depends on x_0 and t_0. On the other hand, $t-t_0$ has been assumed to be much smaller than the typical escape time $1/\overline{\Gamma}$. However, on this time scale the rate $\Gamma(t)$ has practically converged to its asymptotically periodic behavior and thus the extrapolation of $\Gamma(t)$ to arbitrarily large $t-t_0$ is trivial.

C. Supersymmetry

Given a time-periodic force field $F(x,t)$, we define its supersymmetric partner field $\tilde{F}(x,t)$ [22,34,35] according to

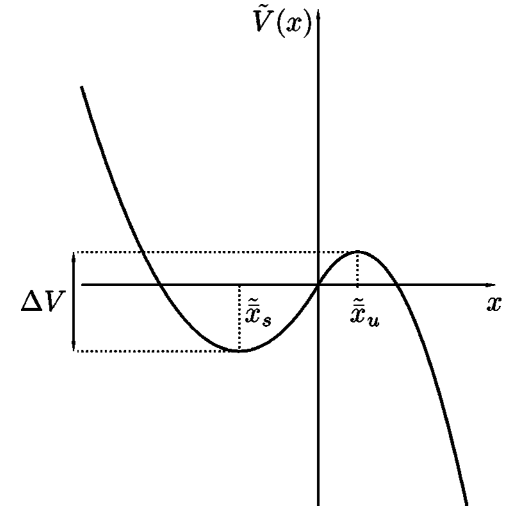


FIG. 2. The supersymmetric partner potential $\tilde{V}(x) := -V(-x)$ of the potential $V(x)$ from Fig. 1 in arbitrary, dimensionless units.

$$\tilde{F}(x,t) := F(-x,-t). \tag{16}$$

For instance, if the force field $F(x,t)$ derives from a periodically rocked potential $V(x)$ like in Eq. (5), then its supersymmetric partner is obtained by turning $V(x)$ upside down, followed by an inversion of the x axis, i.e., $\tilde{V}(x) = -V(-x)$, see Fig. 2, while the driving $A\sin(\Omega t)$ in Eq. (5) remains invariant (up to an irrelevant phase). In such a supersymmetric partner field $\tilde{F}(x,t)$, the stable and unstable periodic orbits exchange their roles, thus defining a different escape problem out of the basin of attraction of the new stable orbit $\tilde{x}_s(t) = -x_u(-t)$ across the new separatrix $\tilde{x}_u(t) = -x_s(-t)$.

It has been demonstrated in [22,34] that for force fields $F(x,t)$ of the form (5), the time-averaged rate (14) is invariant under the supersymmetry transformation (16). The same line of reasoning [22,34] can be readily generalized to force fields of the form $F(x,t) = -V'(x) + y(t)$ with an arbitrary periodic driving $y(t)$. In our present paper we will show that for asymptotically weak noise D *the time-averaged escape rate* (14) *is invariant under the general supersymmetry transformation (16) without any further restrictions on* $F(x,t)$. Regarding the notion of supersymmetry and especially its connection with supersymmetric quantum mechanics, we refer to [22,34] and further references therein. We finally remark that the standard definition of the supersymmetric partner force field is $-F(x,-t)$. For our present purposes, the definition (16) is equivalent but more convenient.

III. PATH INTEGRALS: GENERAL FRAMEWORK

In this section the general framework of a path-integral approach to the stochastic dynamics (3) is outlined. Though these concepts are not new [8–12,36–42], we find it worth while to briefly review them here in order to make our paper self-contained. We also note that most of this section remains valid beyond the particular assumptions on the force field $F(x,t)$ from Sec. II.

A. Time-discretized path integrals

Much like in quantum mechanics, also in the present context of stochastic processes, path-integral concepts have a tangible meaning only when considered as the limiting case of appropriate discrete-time approximations. Our first step is therefore a discretization in time of the overdamped stochastic dynamics (3). Denoting the initial and final times by t_0 and t_f, we introduce the definitions

$$\Delta t := [t_f - t_0]/N, \tag{17}$$

$$t_n := t_0 + n\Delta t, \tag{18}$$

$$x_n := x(t_n), \tag{19}$$

where $n=0,1,\ldots,N$. The integer N is considered as large but finite and will ultimately be sent to infinity (continuous-time limit). The discretized dynamics (3) then takes the form

$$x_{n+1} - x_n = F(x_n, t_n)\Delta t + \sqrt{2D\Delta t}\,\xi_n \tag{20}$$

where the ξ_n are independent, identically distributed Gaussian random numbers with probability distribution

$$P(\xi_n) = (2\pi)^{-1/2} \exp\{-\xi_n^2/2\}. \tag{21}$$

As a side remark we notice that the so-called ''prepoint discretization scheme'' [40–42] [not to be confused with the Ito scheme in the stochastic dynamics (3)] has been implicitly adopted in Eq. (20) for the sake of later convenience. Other ''discretization schemes'' [40–42] would give rise to a somewhat modified path-integral formalism but would, of course, lead to identical results as far as the actual stochastic dynamics (3) is concerned. In passing we further note that our treatment for Eq. (3) can be generalized to multiplicative noise $g(x)\xi(t)$, with $g(x)\neq 0$, without encountering additional difficulties.

For the conditional probability $p_N(x_{n+1}, t_{n+1}|x_n, t_n)$ to reach the point x_{n+1} at time t_{n+1} when starting out from x_n at the previous time step t_n we find from the discretized dynamics (20) and the noise distribution (21) that

$$\begin{aligned} p_N(x_{n+1}, t_{n+1}|x_n, t_n) &= \int \delta(x_{n+1} - x_n - F(x_n, t_n)\Delta t - \sqrt{2D\Delta t}\,\xi_n) P(\xi_n)\, d\xi_n \\ &= \frac{1}{(4\pi D\Delta t)^{1/2}} \exp\left\{ -\frac{[x_{n+1} - x_n - F(x_n, t_n)\Delta t]^2}{4D\Delta t} \right\}. \end{aligned} \tag{22}$$

Here and in the following, integrals over the entire real axis are written without the integration limits $\pm\infty$. Further, the mutual independence of the random numbers ξ_n in Eq. (20) (Markov property) implies for the conditional probability the Chapman-Kolmogorov relation

$$p_N(x_{n+2}, t_{n+2}|x_n, t_n) = \int p_N(x_{n+2}, t_{n+2}|x_{n+1}, t_{n+1}) \times p_N(x_{n+1}, t_{n+1}|x_n, t_n)\, dx_{n+1}.$$

Upon iteration of this relation in combination with Eq. (22) one finds for the conditional probability the time-discretized path-integral representation

$$p_N(x_f, t_f|x_0, t_0) = \int \frac{dx_1 \cdots dx_{N-1}}{(4\pi D\Delta t)^{N/2}} \exp\left\{ -\frac{S_N(x_0, \ldots, x_N)}{D} \right\}, \tag{23}$$

where

$$S_N(x_0, \ldots, x_N) := \sum_{n=0}^{N-1} \frac{\Delta t}{4} \left[\frac{x_{n+1} - x_n}{\Delta t} - F(x_n, t_n) \right]^2 \tag{24}$$

is the discrete-time ''action'' or ''Onsager-Machlup functional.'' While $x_1, \ldots, x_{N-1}$ are integration variables in Eq. (23), the initial and end points are fixed by the prescribed x_0 and by the additional constraint $x_N = x_f$, see Eqs. (17)–(19).

B. Saddle-point approximation

For small noise strengths D the path integral (23) is dominated by the minima of the action $S_N(x_0, \ldots, x_N)$. The existence of at least one (global) minimum can be readily inferred from the general structure of the action in Eq. (24). To keep things simple we assume for the moment that besides this global minimum no additional (local) minima play a role in Eq. (23). Denoting the global minimum by $\mathbf{x}^* := (x_0^*, \ldots, x_N^*)$ it follows that it satisfies the extremality conditions

$$\frac{\partial S_N(\mathbf{x}^*)}{\partial x_n^*} = 0 \tag{25}$$

for $n=1,\ldots,N-1$, supplemented by the boundary conditions for $n=0,N$,

$$x_0^* = x_0, \quad x_N^* = x_f. \tag{26}$$

Under the assumption that the noise strength D is small, the path integral in Eq. (23) can be evaluated by means of a saddle-point approximation about the minimizing path $\mathbf{x}^*$ with the result

$$p_N(x_f, t_f|x_0, t_0) = Z_N(\mathbf{x}^*) e^{-S_N(\mathbf{x}^*)/D} [1 + \mathcal{O}(D)], \tag{27}$$

where the prefactor $Z_N(\mathbf{x}^*)$ is given by a Gaussian integral of the form

$$Z_N(\mathbf{x}^*) := \int \frac{dy_1 \cdots dy_{N-1}}{(4\pi D\Delta t)^{N/2}} \times \exp\left\{ -\frac{1}{2D} \sum_{n,m=1}^{N-1} y_n \frac{\partial^2 S(\mathbf{x}^*)}{\partial x_n^* \partial x_m^*} y_m \right\}, \tag{28}$$

and where in the order of magnitude expression $\mathcal{O}(D)$ only the dependence on the noise-strength D is being kept. A more detailed quantitative estimate of this correction $\mathcal{O}(D)$ is a difficult, and to our knowledge unsolved task.

The Gaussian integral in Eq. (28) is readily evaluated to yield

$$Z_N(\mathbf{x}^*) := \left[4\pi D\Delta t \det\left(2\Delta t \frac{\partial^2 S(\mathbf{x}^*)}{\partial x_n^* \partial x_m^*} \right) \right]^{-1/2}, \tag{29}$$

where $\det(A_{nm})$ indicates the determinant of an $N-1 \times N-1$ matrix with elements A_{nm}. As demonstrated in Appendix A, the determinant appearing in Eq. (29) can be rewritten in the form of a two-step (second-order) linear recursion

$$\frac{Q_{n+1}^*-2Q_n^*-Q_{n-1}^*}{\Delta t^2} = 2\frac{Q_n^*F'(x_n^*,t_n)-Q_{n-1}^*F'(x_{n-1}^*,t_{n-1})}{\Delta t} - Q_n^*\left[\frac{x_{n+1}^*-x_n^*}{\Delta t}-F(x_n^*,t_n)\right]F''(x_n^*,t_n) + Q_n^*F'(x_n^*,t_n)^2-Q_{n-1}^*F'(x_{n-1}^*,t_{n-1})^2 \quad (30)$$

with initial conditions

$$Q_1^*=\Delta t, \quad \frac{Q_2^*-Q_1^*}{\Delta t}=1+\mathcal{O}(\Delta t), \quad (31)$$

from which the prefactor $Z_N(\mathbf{x}^*)$ in Eq. (29) follows as

$$Z_N(\mathbf{x}^*)=[4\pi D Q_N^*]^{-1/2}. \quad (32)$$

The fact that $\mathbf{x}^*$ is a minimum of the action (24) guarantees that $Q_N^*>0$. Here and in the following we use the abbreviations:

$$F'(x,t):=\frac{\partial F(x,t)}{\partial x}, \quad \dot F(x,t):=\frac{\partial F(x,t)}{\partial t}, \quad (33)$$

and bracket-saving expressions like $f(x)^2$ are understood as $[f(x)]^2$.

As we shall see later, we have to leave room for the possibility that even for small noise strengths D more than one (global or local) minimum of the action (24) notably contributes to the path-integral expression (23). We label those various non-negligible minima $\mathbf{x}_k^*$ by the discrete index k but leave for the moment the precise set of k values unspecified. Each of the minimizing paths $\mathbf{x}_k^*$ thus satisfies an extremality condition of the form (25). Under the assumption that those minima $\mathbf{x}_k^*$ are well separated in the $N-1$-dimensional space of all paths $(x_0,\ldots.x_N)$ appearing in Eq. (23), the saddle-point approximation (27) simply acquires an extra sum over k with a corresponding extra index k in Eqs. (28)–(32). Combining Eqs. (27) and (32) we thus arrive at

$$p_N(x_f,t_f|x_0,t_0)=\sum_k \frac{e^{-S_N(\mathbf{x}_k^*)/D}}{(4\pi D Q_{N,k}^*)^{1/2}}[1+\mathcal{O}(D)]. \quad (34)$$

C. Continuous-time limit

Next we turn to the continuous-time limit $N\to\infty$, $\Delta t\to 0$ in Eq. (17). The continuous-time conditional probability $p(x_f,t_f|x_0,t_0)$ when $N\to\infty$ in Eq. (23) is symbolically indicated by the path-integral expression [36]

$$p(x_f,t_f|x_0,t_0)=\int_{x(t_0)=x_0}^{x(t_f)=x_f}\mathcal{D}x(t)\,e^{-S[x(t)]/D}, \quad (35)$$

where

$$S[x(t)]:=\int_{t_0}^{t_f}L(x(t),\dot x(t),t)\,dt \quad (36)$$

is the continuous-time limit of the action (24) with

$$L(x,\dot x,t):=\tfrac{1}{4}[\dot x-F(x,t)]^2 \quad (37)$$

as Lagrangian. The extremality conditions for the minimizing paths $x_k^*(t)$ in the continuous-time limit are obtained from Eqs. (24) and (25) by letting $\Delta t\to 0$ as

$$\ddot x_k^*(t)=\dot F(x_k^*(t),t)+F(x_k^*(t),t)F'(x_k^*(t),t) \quad (38)$$

with boundary conditions [cf. (26)]

$$x_k^*(t_0)=x_0, \quad x_k^*(t_f)=x_f. \quad (39)$$

The same result (38) can also be recovered as the Euler-Lagrange equation corresponding to the Lagrangian (37).

Equivalent to this Lagrangian dynamics is the following Hamiltonian counterpart:

$$H(x,p,t):=p\dot x-L=p^2+pF(x,t), \quad (40)$$

$$\dot p_k^*(t)=-p_k^*(t)F'(x_k^*(t),t), \quad (41)$$

$$\dot x_k^*(t)=2p_k^*(t)+F(x_k^*(t),t). \quad (42)$$

The last equation (42) may also be considered as the definition of the momentum $p_k^*(t)$ in terms of $x_k^*(t)$ and $\dot x_k^*(t)$. With Eqs. (37) and (42) the action (36) along a minimizing path $x_k^*(t)$ follows as

$$\phi_k(x_f,t_f):=S[x_k^*(t)]=\int_{t_0}^{t_f}p_k^*(t)^2\,dt, \quad (43)$$

where the dependence of the action $\phi_k(x_f,t_f)$ on the initial condition x_0 at time t_0 has been dropped. For later use we also recall the well-known result from classical mechanics that the derivative of the extremizing action with respect to its end point equals the canonical conjugate momentum, i.e.,

$$\frac{\partial\phi_k(x_f,t_f)}{\partial x_f}=p_k^*(t_f). \quad (44)$$

Finally, the continuous-time limit for the conditional probability (34) in combination with Eq. (43) takes the form

$$p(x_f,t_f|x_0,t_0)=\sum_k \frac{e^{-\phi_k(x_f,t_f)/D}}{[4\pi D Q_k^*(t_f)]^{1/2}}[1+\mathcal{O}(D)], \quad (45)$$

where $Q_k^*(t)$ is governed by the second-order homogeneous linear differential equation [27,39,43] that follows in the limit $\Delta t\to 0$ from Eqs. (30) and (42),

$$\frac{1}{2}\ddot Q_k^*(t)-\frac{d}{dt}[Q_k^*(t)F'(x_k^*(t),t)]+Q_k^*(t)p_k^*(t)F''(x_k^*(t),t) = 0. \quad (46)$$

Similarly, the initial conditions (31) go over for $\Delta t\to 0$ into

$$Q_k^*(t_0)=0, \quad \dot{Q}_k^*(t_0)=1. \tag{47}$$

We remark that according to Eqs. (38) and (39) the minimizing paths $x_k^*(t)$ are independent of the noise strength D. Consequently, neither $\phi_k(x_f,t_f)$ from Eqs. (38), (39), and (43) nor $Q_k^*(t)$ from Eqs. (46) and (47) depend on the noise strength, i.e., no implicit additional D dependences are hidden in Eq. (45). We further note that by means of the substitution

$$g_k^*(t):=\frac{\dot{Q}_k^*(t)}{2Q_k^*(t)}-F'(x_k^*(t),t) \tag{48}$$

the linear homogeneous second-order equation (46) goes over into the nonlinear first-order Riccati equation

$$\dot{g}_k^*(t)+2g_k^*(t)^2+2g_k^*(t)F'(x_k^*(t),t)=-p_k^*(t)F''(x_k^*(t),t). \tag{49}$$

Since Eq. (47) does not lead to a meaningful initial condition for $g_k^*(t)$ in Eq. (48), the Riccati equation (49) can only be used for times $t>t_0$. For this reason and also from the viewpoint of calculational efficiency we found that for practical purposes the linear second-order equation (46) is often superior to the Riccati equation (49).

To establish contact with previously known results we finally remark that one can identify

$$\frac{\partial^2\phi_k(x_f,t_f)}{\partial x_f^2}=g_k^*(t_f). \tag{50}$$

This relation (50) and the associated Riccati equation (49) are usually derived by introducing a WKB-type ansatz into the Fokker-Planck equation for the conditional probability distribution [cf. Eq. (10)] and then comparing powers of the noise strength D. Since a direct derivation by means of path-integral methods is not known to us, we have included such a derivation of the relation (50) in Appendix B.

IV. PATH-INTEGRAL SOLUTION OF THE ESCAPE PROBLEM

By introducing the path-integral expression (45) for the conditional probability into the formula (15) for the instantaneous rate $\Gamma(t)$ at time $t=t_f$ and taking into account Eq. (44), we obtain *our first main result* [27], namely,

$$\Gamma(t_f)=\sum_k \frac{p_k^*(t_f)e^{-\phi_k(x_u(t_f),t_f)/D}}{[4\pi D Q_k^*(t_f)]^{1/2}}[1+\mathcal{O}(D)]. \tag{51}$$

In view of Eq. (45), the instantaneous rate (51) has the suggestive structure of ''probability at the separatrix times velocity.''

As already mentioned in Sec. II, for sufficiently large times t_f-t_0 the instantaneous rate (51) is expected to become independent of the initial position x_0 as long as x_0 is located inside the domain of attraction of the stable periodic orbit $x_s(t)$. A more detailed discussion of this point will be given in Sec. IV G. To keep things as simple as possible we focus in Secs. IV A–IV F on the particular case that x_0 is located *at* the stable periodic orbit, i.e.,

$$x_0=x_s(t_0). \tag{52}$$

A. Minimizing paths

Our next goal is the characterization of all the minimizing paths $x_k^*(t)$ which significantly contribute to the sum in Eq. (51). Our first observation is that for any finite t_0 and t_f the action (36) exhibits in the generic case a unique global minimum respecting the boundary conditions

$$x_k^*(t_0)=x_s(t_0), \quad x_k^*(t_f)=x_u(t_f), \tag{53}$$

according to Eqs. (39) and Eqs. (51) and (52). To be specific, we denote this globally minimizing path as $x_{k_0}^*(t)$. From the explicit form of the Lagrangian (37) we can infer that for large values of t_f-t_0 the minimal path $x_{k_0}^*(t)$ follows most of the time rather closely a deterministic trajectory, i.e., $\dot{x}_{k_0}^*(t)\simeq F(x_{k_0}^*(t),t)$, in order not to accumulate a too large amount of action (36). In view of Eqs. (7) and (53) it is thus suggestive that $x_{k_0}^*(t)$ starts at $x_{k_0}^*(t_0)=x_s(t_0)$ and then continues to closely follow the stable periodic orbit $x_s(t)$ for quite some time. At a certain moment, $x_{k_0}^*(t)$ leaves this neighborhood and travels in a comparatively short time into the vicinity of the unstable periodic orbit $x_u(t)$, where it remains for the rest of its time and ends at $x_{k_0}^*(t_f)=x_u(t_f)$. Only during the crossover from the neighborhood of $x_s(t)$ into that of $x_u(t)$ does the path $x_{k_0}^*(t)$ substantially deviate from a deterministic behavior and so gives rise to the main contribution to the action (36). We desist from a more rigorous derivation of these basic qualitative features since they are quite similar to the well-known barrier-crossing problem in a static potential (time-independent force field) [37,38,44]. Especially, the relatively short ''crossover segment'' of $x_{k_0}^*(t)$ between the long sojourns close to the stable and unstable orbits has lead to the name ''instanton'' for such a path.

As we will see in more detail later, a meaningful limit of $x_{k_0}^*(t)$ exists for $t_0\to-\infty$ and $t_f\to\infty$ (henceforth abbreviated as $t_f-t_0\to\infty$) in the sense that $x_{k_0}^*(t)$ follows closer and closer the periodic orbits $x_{s,u}(t)$ over longer and longer time intervals, while the crossover-segment does practically not change its shape any more. Also the associated minimal action $S[x_{k_0}^*(t)]$ from Eq. (36) tends to a finite limit. In fact, one can readily show that the minimal action cannot increase upon increasing t_f and/or decreasing t_0. Furthermore, since it is bounded from below, the existence of the limit follows for the action as well as for the minimizing path itself. More importantly, from the time periodicity of the force field (4) one can infer that in the limit $t_f-t_0\to\infty$ the action $S[x_{k_0}^*(t+n\mathcal{T})]$ has the same value for any integer n. In other words, for infinitely large t_f-t_0 the action no longer exhibits a unique absolute minimum, rather each path $x_{k_0}^*(t+n\mathcal{T})$ globally minimizes the action. However, these *degenerate* absolute minima are still well separated in the space of all paths $x(t)$ appearing in Eq. (35). This feature is the salient difference between our present problem and its time-independent counterpart [37,38,44–46], which exhibits a *continuous* degeneracy (Goldstone mode) in the limit $t_f - t_0 \to \infty$. Put differently, the time-periodic force field reduces the continuous time-translation symmetry into a discrete one. Since the rate formula (51) assumes well separated minima $x_k^*(t)$ of the action, it is quite clear that the *time-independent case must be excluded in the following*.

We emphasize that the minimizing paths $x_k^*(t)$ remain well separated and thus the rate formula (51) becomes asymptotically exact for any (arbitrary but fixed) finite values of the driving amplitude and period as the noise strength D tends to zero. Apart from this fact that *in the limit* $D \to 0$ the $\mathcal{O}(D)$ correction in the saddle-point approximation (27) and thus in Eq. (51) vanishes, a more detailed quantitative statement seems difficult. On the other hand, for a given (small) noise strength D, we have to exclude extremely small driving amplitudes and extremely long or short driving periods since this would lead us effectively back to the static (undriven) escape problem, which requires a completely different treatment (especially of the (quasi-) Goldstone mode [23,37,38,44–46]) than in Eq. (27). Put differently, in any of these three asymptotic regimes, the error $\mathcal{O}(D)$ from Eqs. (27) and (51) becomes very large.

For later reference we denote the minimizing path $x_{k_0}^*(t)$ when $t_f - t_0 \to \infty$ by $x_{\mathrm{opt}}^*(t)$, keeping in mind that we are still free to shift its time argument by an arbitrary multiple of $\mathcal{T}$. The corresponding action is

$$\phi_{\mathrm{opt}} := S[x_{\mathrm{opt}}^*(t)] = \lim_{\substack{t_0 \to -\infty \\ t_f \to \infty}} \min_{\substack{x(t) \\ x(t_0) = x_s(t_0) \\ x(t_f) = x_u(t_f)}} S[x(t)], \tag{54}$$

where the second identity may also be considered as an implicit definition of $x_{\mathrm{opt}}^*(t)$. Similarly, any other quantity associated with $x_{\mathrm{opt}}^*(t)$ will be marked by an index ''opt,'' for instance $p_{\mathrm{opt}}^*(t)$ [see Eq. (42)], $Q_{\mathrm{opt}}^*(t)$ [see Eq. (46)], and $g_{\mathrm{opt}}^*(t)$ [see Eq. (48)].

In principle, besides the absolute minimum $x_{\mathrm{opt}}^*(t)$ of the action there may coexist further (absolute or relative) minima which cannot be identified with each other after a time shift by an appropriate multiple of $\mathcal{T}$. While the coexistence of further absolute minima is nongeneric, coexisting relative minima are irrelevant for sufficiently small noise strengths D as far as the sum in Eq. (51) is concerned. Though both cases could be easily taken into account in the following discussion, we will restrict ourselves to the simplest and most common case that $x_{\mathrm{opt}}^*(t + n\mathcal{T})$ are the only (relevant) minima of the action (36) in the limit $t_f - t_0 \to \infty$.

Returning to finite but large values of $t_f - t_0$, we expect—as a precursor of the $t_f - t_0 \to \infty$ limit—that besides the unique absolute minimum $x_{k_0}^*(t)$ there will coexist many additional relative minima $x_k^*(t)$ with an only slightly larger action. All those minima $x_k^*(t)$ possess a limit when $t_f - t_0 \to \infty$ in the same sense as for the case $k = k_0$ described above (quantitative details will be given later). Moreover, when $t_f - t_0 \to \infty$ then each $x_k^*(t)$ approaches $x_{\mathrm{opt}}^*(t + n(k)\mathcal{T})$ for a suitable choice of $n(k)$ and without loss of generality we can assume a (re-)labeling of the $x_k^*(t)$ such that $n(k) = k$. In other words, to each $x_k^*(t)$ belongs a very similarly looking ''*master path*'' $x_{\mathrm{opt}}^*(t + k\mathcal{T})$, see Fig. 3. Since $t_f - t_0$ is finite,

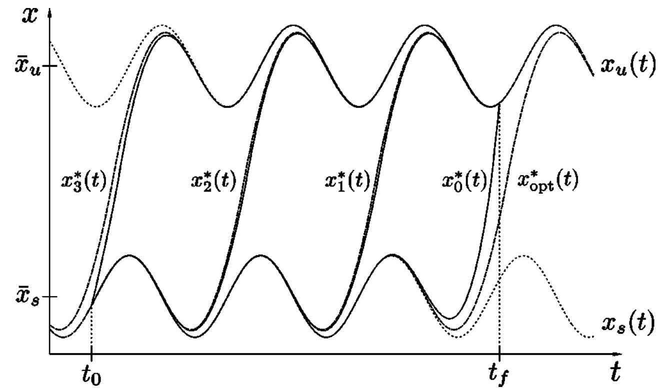


FIG. 3. Solid: The paths $x_k^*(t)$, $k = 0, \ldots, K(t_f, t_0) = 3$ which minimize the action [Eqs. (36) and (37)] with boundary conditions (53). Dashed: The associated ''master paths'' $x_{\mathrm{opt}}^*(t + k\mathcal{T})$, implicitly defined via Eq. (54). Dotted: Stable and unstable periodic orbits $x_s(t)$ and $x_u(t)$ from Eq. (7). In this plot, $t_f - t_0$ has been chosen rather small. As $t_f - t_0$ increases, more and more intermediate paths $x_k^*(t)$ appear which better and better agree with their associated master paths $x_{\mathrm{opt}}^*(t + k\mathcal{T})$. The depicted curves have been obtained for the addictively driven piecewise parabolic potential [Eqs. (5) and (134)] with parameters $A = 0.5$, $\Omega = 1$, $\lambda_s = -1$, $\lambda_u = 1$, $\Delta V = 1$, $t_0 = -12$, $t_f = 7.5$ (dimensionless units).

there is a finite number [of the order $(t_f - t_0)/\mathcal{T}$] of minimizing paths $x_k^*(t)$ and without loss of generality we can assume that the indices in the sum (51) start at $k = 0$ and run until a certain maximal value $K(t_f, t_0)$:

$$0 \leq k \leq K(t_f, t_0) = \mathcal{O}((t_f - t_0)/\mathcal{T}). \tag{55}$$

Thus $x_0^*(t)$ is that minimizing path which closely follows $x_s(t)$ as long as possible and crosses over to the neighborhood of $x_u(t)$ at ''the latest possible moment'' (see Fig. 3), and similarly for $x_{K(t_f, t_0)}^*(t)$. Note that all the general qualitative features discussed above are nicely illustrated by the explicit example in Sec. V A.

B. Neighborhood of periodic orbits

Our final goal is to approximate the action $\phi_k(x_f, t_f)$ and the prefactor $p_k^*(t_f)/[Q_k^*(t_f)]^{1/2}$ for all minimizing paths $x_k^*(t)$ that play a non-negligible role in the sum [Eqs. (51) and (55)] solely in terms of the master path $x_{\mathrm{opt}}^*(t)$ and its descendants $p_{\mathrm{opt}}^*(t), Q_{\mathrm{opt}}^*(t)$, etc. To this end we first address the behavior of a path $x_k^*(t)$ within the neighborhood of either the stable periodic orbit $x_s(t)$ or of the unstable one $x_u(t)$. Within these regions, the time-dependent force field can be approximately written as

$$F(x,t) = F(x_{s,u}(t),t) + (x - x_{s,u}(t))F'(x_{s,u}(t),t). \tag{56}$$

Note that these approximations are valid not only if $x - x_{s,u}(t)$ is small but also if $F''(y,t)$ is small for all y between x and $x_{s,u}(t)$. An immediate consequence of Eq. (56) are the relations

$$F'(x,t) = F'(x_{s,u}(t),t), \quad F''(x,t) = 0. \tag{57}$$

As long as a minimizing path $x_k^*(t)$ remains in a region where these approximations apply, the Hamiltonian equations (41) and (42) take the form

$$\dot{p}_k^*(t) = -p_k^*(t) F'(x_{s,u}(t),t), \tag{58}$$

$$\Delta \dot{x}_k^*(t) = 2p_k^*(t) + \Delta x_k^*(t) F'(x_{s,u}(t),t), \tag{59}$$

where we have introduced

$$\Delta x_k^*(t) := x_k^*(t) - x_{s,u}(t). \tag{60}$$

Their solutions are

$$p_k^*(t) = p_k^*(t_1) e^{-\Lambda_{s,u}(t,t_1)}, \tag{61}$$

$$\Delta x_k^*(t) = \Delta x_k^*(t_1) e^{\Lambda_{s,u}(t,t_1)} + p_k^*(t) I_{s,u}(t,t_1), \tag{62}$$

where t_1 is an arbitrary reference time [within our assumption that Eq. (56) applies for all the considered times t] and where

$$\Lambda_{s,u}(t,t_1) := \int_{t_1}^{t} F'(x_{s,u}(t'),t') dt', \tag{63}$$

$$I_{s,u}(t,t_1) := 2\int_{t_1}^{t} e^{2\Lambda_{s,u}(t,t')} dt'. \tag{64}$$

Obvious properties of the functions $\Lambda_{s,u}(t,t_1)$ from Eq. (63) are

$$\Lambda_{s,u}(t,t_2) = \Lambda_{s,u}(t,t_1) + \Lambda_{s,u}(t_1,t_2), \tag{65}$$

$$\Lambda_{s,u}(t_1,t) = -\Lambda_{s,u}(t,t_1), \tag{66}$$

$$\Lambda_{s,u}(t+\mathcal{T},t_1+\mathcal{T}) = \Lambda_{s,u}(t,t_1). \tag{67}$$

Further, one readily sees that the quantities

$$\lambda_{s,u} := \Lambda_{s,u}(t+\mathcal{T},t)/\mathcal{T} \tag{68}$$

are indeed t independent and characterize the stability or instability (''Lyapunov exponents'') of the periodic orbits, namely,

$$\lambda_s < 0, \quad \lambda_u > 0. \tag{69}$$

One even expects that $\Lambda_s(t,t_1) < 0$ and $\Lambda_u(t,t_1) > 0$ not only for $t - t_1 = \mathcal{T}, 2\mathcal{T}, \ldots$ [cf. (68)] but in fact for all $t - t_1 > 0$; however, exceptions cannot be excluded for not too large $t - t_1$. From Eqs. (63) and (68) it follows that $\Lambda_{s,u}(t,t_1)$ can be written as the sum of a linear function $\lambda_{s,u} \cdot (t - t_1)$ and a periodic function of t. As a consequence, we obtain

$$\Lambda_{s,u}(t,t_1) \sim \lambda_{s,u} \cdot (t - t_1) \tag{70}$$

for asymptotically large positive and negative times $t - t_1$.

Turning to the discussion of $I_{s,u}(t,t_1)$ from Eq. (64), we first note that

$$I_s(t,t_1) = I_s(t) - e^{2\Lambda_s(t,t_1)} I_s(t_1), \tag{71}$$

$$I_s(t) := \lim_{t_1 \to -\infty} I_s(t,t_1) = 2\int_{-\infty}^{t} e^{2\Lambda_s(t,t')} dt', \tag{72}$$

and similarly

$$I_u(t,t_1) = -I_u(t) + e^{2\Lambda_u(t,t_1)} I_u(t_1), \tag{73}$$

$$I_u(t) := -\lim_{t_1 \to \infty} I_u(t,t_1) = 2\int_{t}^{\infty} e^{2\Lambda_u(t,t')} dt'. \tag{74}$$

Thus, $I_{s,u}(t)$ are positive and finite for all t and obey

$$I_{s,u}(t+\mathcal{T}) = I_{s,u}(t). \tag{75}$$

It follows that $I_s(t,t_1)$ in Eq. (71) is given by a periodic function of t minus the product of another periodic function times an exponentially decreasing factor $\exp\{\lambda_s \cdot (t - t_1)\}$, and analogously for $I_u(t,t_1)$ in Eq. (73).

Choosing as reference time $t_1 = t_0$ in Eq. (62) and taking into account that $\Delta x_k^*(t_0) = 0$ according to Eqs. (52) and (60) implies that in the neighborhood of $x_s(t)$ we have

$$\Delta x_k^*(t) = p_k^*(t) I_s(t,t_0). \tag{76}$$

Dividing this result by the same identity evaluated at a different reference time $t_s > t_0$ and taking into account Eq. (61), we obtain

$$\Delta x_k^*(t) = \Delta x_k^*(t_s) e^{-\Lambda_s(t,t_s)} \frac{I_s(t,t_0)}{I_s(t_s,t_0)}, \tag{77}$$

$$p_k^*(t) = \Delta x_k^*(t_s) e^{-\Lambda_s(t,t_s)} / I_s(t_s,t_0). \tag{78}$$

Both these expressions consist of an exponentially *increasing* factor $\exp\{-\lambda_s \cdot (t - t_s)\}$ times some periodic function of t. In Eq. (77) one has in addition a quickly decreasing correction. The corresponding behavior in the neighborhood of $x_u(t)$ is given by

$$\Delta x_k^*(t) = p_k^*(t) I_u(t,t_f), \tag{79}$$

$$\Delta x_k^*(t) = \Delta x_k^*(t_u) e^{-\Lambda_u(t,t_u)} \frac{I_u(t,t_f)}{I_u(t_u,t_f)}, \tag{80}$$

$$p_k^*(t) = \Delta x_k^*(t_u) e^{-\Lambda_u(t,t_u)} / I_u(t_u,t_f), \tag{81}$$

where t_u is some reference time with $t_u < t_f$. As expected, Eqs. (80) and (81) are now dominated by an exponentially *decreasing* behavior $\exp\{-\lambda_u \cdot (t - t_u)\}$. We further remark that for the master path $x_{\rm opt}^*(t)$ we have $t_0 \to -\infty$ and $t_f \to \infty$, thus $I_{s,u}(t,t_{0,f})$ in Eqs. (76)–(81) go over into $I_{s,u}(t)$ according to Eqs. (71) and (73) and so all four equations (77), (78), (80), and (81) are *exactly* given by $\exp\{-\lambda_{s,u} \cdot (t - t_{s,u})\}$ times certain periodic functions of t.

Within our above local analysis of the neighborhoods of $x_{s,u}(t)$, the reference times $t_{s,u}$ are still arbitrary and the corresponding parameters $\Delta x_k^*(t_{s,u})$ remain undetermined. They can only be fixed through the global behavior of $x_k^*(t)$. It is instructive to reconsider the same thing from a somewhat different viewpoint. From Eqs. (71), (72), and (76) we conclude that

$$\frac{\Delta x_k^*(t_s)}{p_k^*(t_s)} = I_s(t_s) - 2\int_{-\infty}^{t_0} e^{2\Lambda_s(t_s,t')}dt' \tag{82}$$

and similarly

$$\frac{\Delta x_k^*(t_u)}{p_k^*(t_u)} = -I_u(t_u) + 2\int_{t_f}^{\infty} e^{2\Lambda_u(t_u,t')}dt'. \tag{83}$$

Let us consider $t_s > t_0$ as fixed and such that the approximation Eq. (56) is valid for all $t \in [t_0, t_s]$. Within the same restriction, we now consider the quantity $\Delta x_k^*(t_s)$ as a parameter. For any value of $\Delta x_k^*(t_s)$, Eq. (82) thus fixes $p_k^*(t_s)$. With these initial conditions for $x_k^*(t)$ and $p_k^*(t)$ at time $t = t_s$ on may then propagate the Hamiltonian equations (41) and (42) up to the time $t = t_f$. It is clear that for a typical choice of $\Delta x_k^*(t_s)$ such a "shooting procedure" does not lead to the desired end result $\Delta x_k^*(t_f) = 0$. But we also know from the mere existence of the minimizing paths that there must be specific $\Delta x_k^*(t_s)$ values which do the job. Furthermore, Eq. (83) tells us that it is not necessary to proceed until $t = t_f$, rather it is sufficient to take any time t_u at which $x_k^*(t)$ has reached the $x_u(t)$ neighborhood and then check whether Eq. (83) is satisfied.

If $t_s - t_0$ and $t_f - t_u$ are already large then the integrands in Eqs. (82) and (83) are extremely small. Thus, tiny changes of $\Delta x_k^*(t_s)$ and $p_k^*(t_s)$ will lead to huge changes of t_0 and t_f. Especially, by letting $t_0 \to -\infty$ and $t_f \to \infty$ those integrals vanish and one recovers the master path $x_{\rm opt}^*(x + k\mathcal{T})$ associated with $x_k^*(t)$. This confirms our conclusion from Sec. IV A that a meaningful limit of $x_k^*(t)$ for $t_0 \to \infty$ and $t_f \to \infty$ exists and that for finite but large $t_f - t_0$ the difference between $x_k^*(t)$ and the associated master path $x_{\rm opt}^*(t + k\mathcal{T})$ is extremely small for all $t \in [t_0, t_f]$.

An example for which $t_f - t_u$ is *not* large is the path $x_0^*(t)$, i.e., the one which crosses over from the neighborhood of $x_s(t)$ into that of $x_u(t)$ at the latest possible moment, see Fig. 3. For this path $x_0^*(t)$, the time t_u at which it enters the neighborhood of $x_u(t)$ is already rather close to t_f and so the integral in Eq. (83) is not any more small. As a consequence, the deviation of $x_0^*(t)$ from $x_{\rm opt}^*(t)$ is no longer small as t approaches t_f. In particular, for $t = t_f$ it follows that $x_0^*(t_f) - x_{\rm opt}^*(t_f) = -\Delta x_{\rm opt}^*(t_f)$ is no longer small and with Eq. (79) we conclude that the same is true for the momentum $p_{\rm opt}^*(t_f)$, i.e.,

$$p_{\rm opt}^*(t_f) \quad \text{not small.} \tag{84}$$

With increasing k values, the deviations $-\Delta x_{\rm opt}^*(t + k\mathcal{T})$ between $x_k^*(t)$ and the associated master path $x_{\rm opt}^*(t + k\mathcal{T})$ in the vicinity of t_f are rapidly decreasing, essentially like $\exp\{-\lambda_u k\mathcal{T}\}$, see Eqs. (68) and (80). In the same way, for the largest possible k values, $k \simeq K(t_f, t_0)$ [see Eq. (55)], corresponding to paths $x_k^*(t)$ with only a very short initial time segment close to $x_s(t)$, the deviations from $x_{\rm opt}^*(t + k\mathcal{T})$ are no longer small for t close to t_0. As we will see later, paths $x_k^*(t)$ with such large k values are negligible in the sum (51). For this reason, we will henceforth neglect deviations between $x_k^*(t)$ and $x_{\rm opt}^*(t + k\mathcal{T})$ and between $p_k^*(t)$ and $p_{\rm opt}^*(t + k\mathcal{T})$ for times t near the starting point t_0. Formally, this approximation is equivalent to letting

$$t_0 \to -\infty. \tag{85}$$

C. Approximations in terms of the master path

Our next objective is to express the action (43) of the path $x_k^*(t)$ in terms of the associated master path $x_{\rm opt}^*(t + k\mathcal{T})$. We recall that while $x_k^*(t)$ satisfies the boundary conditions (53), those of $x_{\rm opt}^*(t)$ are $x_{\rm opt}^*(t) - x_s(t) \to 0$ for $t \to -\infty$ and $x_{\rm opt}^*(t) - x_u(t) \to 0$ for $t \to \infty$. We now modify the latter boundary condition and require instead that

$$t_k := t_f + k\mathcal{T} \tag{86}$$

is the final time and $x_k := x_{\rm opt}^*(t_k)$ the final position. In other words, we simply truncate the master path $x_{\rm opt}^*(t)$ at the time t_k, associated with the final time t_f of $x_k^*(t)$. Since this "new" path $x_{\rm opt}^*(t)$ with $t \in [-\infty, t_k]$ obviously still satisfies the Hamiltonian equations (41) and (42) it is again an extremizing path. The value of the action for this path follows like in Eq. (43) as

$$\phi_{\rm opt}(x_k, t_k) := \int_{-\infty}^{t_k} p_{\rm opt}^*(t)^2 dt \tag{87}$$

and the relations (44) and (50) take the form

$$\frac{\partial \phi_{\rm opt}(x_k, t_k)}{\partial x_k} = p_{\rm opt}^*(t_k), \tag{88}$$

$$\frac{\partial^2 \phi_{\rm opt}(x_k, t_k)}{\partial x_k^2} = g_{\rm opt}^*(t_k). \tag{89}$$

With Eqs. (43) and (54) we can rewrite Eq. (87) as

$$\phi_{\rm opt}(x_k, t_k) = \phi_{\rm opt} - \int_{t_k}^{\infty} p_{\rm opt}^*(t)^2 dt. \tag{90}$$

Next we express the action (36) of the path $x_k^*(t)$ by expanding the one belonging to the associated master path $x_{\rm opt}^*(t + k\mathcal{T})$ in powers of the difference $-\Delta x_{\rm opt}^*(t_k)$ between the end points $x_k^*(t_f) = x_u(t_f)$ and $x_{\rm opt}^*(t_f + k\mathcal{T}) = x_{\rm opt}^*(t_k)$,

$$\phi_k(x_k^*(t_f), t_f) = \phi_{\rm opt}(x_k, t_k) - \Delta x_{\rm opt}^*(t_k)\frac{\partial \phi_{\rm opt}(x_k, t_k)}{\partial x_k} + \frac{\Delta x_{\rm opt}^*(t_k)^2}{2}\frac{\partial^2 \phi_{\rm opt}(x_k, t_k)}{\partial x_k^2} + \cdots. \tag{91}$$

As justified above Eq. (85), the analogous contribution in powers of $\Delta x_{\rm opt}^*(t_0 + k\mathcal{T})$ is negligible on the right-hand side of Eq. (91). By exploiting Eqs. (88)–(90) and the counterparts of Eqs. (79)–(81) for $x_{\rm opt}^*(t + k\mathcal{T})$, one arrives after a short calculation at

$$\phi_k(x_k^*(t_f), t_f) = \phi_{\rm opt} + \int_{t_k}^{\infty} p_k^*(t)^2 dt \times [1 + g_{\rm opt}^*(t_k) I_u(t_k) + \cdots]. \tag{92}$$

A similar expansion of $p^*_{\rm opt}(t_k)$ from Eq. (88) yields for $p^*_k(t_f)$ the approximation

$$p^*_k(t_f)=p^*_{\rm opt}(t_k)+\Delta^*_{\rm opt}(t_k)\,\frac{\partial^2\phi_{\rm opt}(x_k,t_k)}{\partial x_k^2}+\cdots$$
$$=p^*_{\rm opt}(t_k)[1+g^*_{\rm opt}(t_k)I_u(t_k)+\cdots]. \tag{93}$$

We now turn to the prefactor terms $Q^*_k(t)$ in Eq. (51). Within the neighborhoods of $x_{s,u}(t)$ for which the approximations (56) and thus (57) are valid, we can infer from Eq. (46) that

$$\dot Q^*_k(t)/2-Q^*_k(t)F'(x_{s,u}(t),t)=\text{const}=:\mu_{s,u}. \tag{94}$$

By comparison with Eq. (48) we further see that

$$g^*_k(t)Q^*_k(t)=\mu_{s,u}. \tag{95}$$

The constant μ_s, which is connected with the neighborhood of $x_s(t)$ and is typically different form μ_u, follows from the initial conditions (47) as $\mu_s=1/2$. Hence, the solution of Eq. (94) takes the form

$$Q^*_k(t)=I_s(t,t_0)/2. \tag{96}$$

As a by-product we find from Eq. (50), evaluated for an arbitrary final condition $t_f=t$ and $x_f=x$ in combination with Eqs. (95) and (96) that

$$\frac{\partial^2\phi_k(x,t)}{\partial x^2}=\frac{1}{I_s(t,t_0)}. \tag{97}$$

Within the linearization (56), closer inspection shows that only a single summand appears in the conditional probability (45) and one recovers the expected Gaussian result for x close to the stable periodic orbit $x_s(t)$:

$$p(x,t|x_s(t_0),t_0)=\left(\frac{1}{2\pi DI_s(t,t_0)}\right)^{1/2}\exp\left\{-\frac{[x-x_s(t)]^2}{2DI_s(t,t_0)}\right\}. \tag{98}$$

Returning to Eq. (96), it is remarkable that besides the initial time t_0 no further details of the path $x^*_k(t)$ play a role. Especially, if $x^*_k(t)$ remains for a long time in the neighborhood of $x_s(t)$ where Eq. (96) is valid, then by the time it leaves this neighborhood, say $t=t_s$, the quantity $I_s(t,t_0)$ is practically equal to $I_s(t)$ from Eq. (72) and thus $Q^*_k(t)$ equal to the associated master prefactor $Q^*_{\rm opt}(t+k\mathcal{T})$. Within our usual approximation (85) we thus have

$$Q^*_k(t_s)=Q^*_{\rm opt}(t_s+k\mathcal{T})=I_s(t_s)/2, \tag{99}$$

$$\dot Q^*_k(t_s)=\dot Q^*_{\rm opt}(t_s+k\mathcal{T})=\dot I_s(t_s)/2. \tag{100}$$

These relations are then used as initial conditions in Eq. (46) in order to propagate $Q^*_k(t_s)$ and $Q^*_{\rm opt}(t_s+k\mathcal{T})$ through the crossover segments of the corresponding paths $x^*_k(t_s)$ and $x^*_{\rm opt}(t_s+k\mathcal{T})$ up to a certain time point, say $t=t_u$, beyond which the linearization (56) about $x_u(t)$ and thus Eq. (94) can be applied.

Once the neighborhood of $x_u(t)$ is reached, i.e., for $t\geqslant t_u$, the solution of Eqs. (94) and (95) can be written with Eq. (64) as

$$Q^*_k(t)=Q^*_k(t_u)e^{2\Lambda_u(t,t_u)}[1-g^*_k(t_u)I_u(t_u,t)]. \tag{101}$$

In view of Eq. (61) this yields, furthermore,

$$Q^*_{\rm opt}(t)p^*_{\rm opt}(t)^2=Q^*_{\rm opt}(t_u)p^*_{\rm opt}(t_u)^2[1-g^*_{\rm opt}(t_u)I_u(t_u,t)]. \tag{102}$$

Due to Eq. (73), the factor $I_u(t_u,t)$ approaches $-I_u(t_u)$ as $t-t_u$ becomes large. It follows that the left-hand side of Eq. (102) tends towards a finite limit as $t\to\infty$,

$$q_{\rm opt}:=\lim_{t\to\infty}Q^*_{\rm opt}(t)p^*_{\rm opt}(t)^2. \tag{103}$$

Since t_u is an arbitrary reference time in Eq. (102), we can first let $t\to\infty$ and then rename t_u as t with the result

$$Q^*_{\rm opt}(t)=\frac{q_{\rm opt}}{p^*_{\rm opt}(t)^2}-\mu_{\rm opt}I_u(t), \tag{104}$$

where the (finite) constant $\mu_{\rm opt}$ is defined analogously to Eq. (95) as

$$\mu_{\rm opt}:=\lim_{t\to\infty}g^*_{\rm opt}(t)Q^*_{\rm opt}(t). \tag{105}$$

Exploiting once more Eqs. (95) and (105), we can eliminate $Q^*_{\rm opt}(t)$ in Eq. (104) in favor of $g^*_{\rm opt}(t)$ with the result

$$g^*_{\rm opt}(t)=\frac{p^*_{\rm opt}(t)^2}{q_{\rm opt}/\mu_{\rm opt}-p^*_{\rm opt}(t)^2I_u(t)}. \tag{106}$$

As discussed below (83), the deviations of $x^*_k(t)$ from the associated master path $x^*_{\rm opt}(t+k\mathcal{T})$ become smaller and smaller as k increases and in view of Eqs. (99), (100), and (46) we expect a similar convergence of $Q^*_k(t)$ towards $Q^*_{\rm opt}(t+k\mathcal{T})$. In Appendix C, the following quantitative estimate for this convergence is established for all times $t\in[t_u,t_f]$:

$$Q^*_k(t)=Q^*_{\rm opt}(t+k\mathcal{T})[1+O(p^*_{\rm opt}(t_k)^2)], \tag{107}$$

where the order of magnitude is meant with respect to the dependence on k.

From the technical viewpoint, Eq. (104) in combination with Eq. (107) is a central and highly nontrivial result of our present work. Since $I_u(t)$ is periodic in t and since $p^*_{\rm opt}(t)$ decreases exponentially according to Eq. (61), we see from Eqs. (104) and (107) that the prefactor $Q^*_k(t)$ diverges exponentially with the time which the path $x^*_k(t)$ spends in the neighborhood of $x_u(t)$, in striking contrast to the behavior (96) close to $x_s(t)$. The basic physical reason for this divergence of $Q^*_k(t)$ is that the probability of a stochastic process (3) to permanently remain close to the unstable periodic orbit $x_u(t)$ decreases exponentially with increasing time. Since typically the process closely follows a deterministic trajectory, the action barely grows and thus it is the prefactor $1/Q^*_k(t_f)^{1/2}$ in Eq. (45) which has to account for the exponential decrease in time.

Since $p^*_{\rm opt}(t_k)$ decreases exponentially with k we see from Eq. (106) that $g^*_{\rm opt}(t_k)$ tends to zero like $p^*_{\rm opt}(t_k)^2$. In view of Eqs. (88) and (89) we therefore conjecture that also higher derivatives of $\phi_{\rm opt}(x_k,t_k)$ continue to scale like the corresponding powers of $p^*_{\rm opt}(t_k)$. The terms indicated by the dots in Eqs. (11)–(93) are then indeed negligibly small.

D. Evaluation and discussion of the rate

We are now in the position to evaluate the rate formula (51) in terms of the master path $x^*_{\rm opt}(t)$. To this end we approximate in Eqs. (92) and (93) the square brackets by 1, neglect in Eq. (107) the term of order $p^*_{\rm opt}(t_k)^2$, and in Eq. (104) the last term [being also a correction of order $p^*_{\rm opt}(t_k)^2$]. By dropping the index of t_f we then can infer from Eqs. (51) and (55) our central result for the *instantaneous rate* [27]

$$\Gamma(t)=\sqrt{D}\,\alpha_{\rm opt}e^{-\phi_{\rm opt}/D}\kappa_{\rm opt}(t,D)[1+\mathcal{O}(D^{\gamma})], \quad (108)$$

$$\alpha_{\rm opt}:=[4\pi\mathcal{T}^2\lim_{t\to\infty}p^*_{\rm opt}(t)^2Q^*_{\rm opt}(t)]^{-1/2}, \quad (109)$$

$$\kappa_{\rm opt}(t,D):=\mathcal{T}\sum_{k=0}^{K(t,t_0)}\frac{p^*_{\rm opt}(t+k\mathcal{T})^2}{D}\times\exp\left\{-\frac{1}{D}\int_t^{\infty}p^*_{\rm opt}(t'+k\mathcal{T})^2dt'\right\}. \quad (110)$$

The effect of our various approximations in deriving this result together with the corresponding ''accuracy exponent'' $\gamma>0$ in Eq. (108) will be discussed in Sec. IV E. Next, we analyze in more detail the properties of $\kappa_{\rm opt}(t,D)$. By means of Eqs. (61), (68), and (74) we rewrite Eq. (110) as

$$\kappa_{\rm opt}(t,D)=\mathcal{T}\sum_{k=0}^{K(t,t_0)}\frac{p^*_{\rm opt}(t)^2C^k}{D}\exp\left\{-\frac{p^*_{\rm opt}(t)^2C^kI_u(t)}{2D}\right\}, \quad (111)$$

$$C:=e^{-2\lambda_u\mathcal{T}}. \quad (112)$$

Since $0<C<1$ there is a competition in the sum (111) between the exponential terms which increase with k and the pre-exponential factors which decrease with k. One readily sees that the dominant contribution to the sum stems from a few k values around the real number $\hat{k}$, implicitly defined via

$$p^*_{\rm opt}(t)^2C^{\hat{k}}I_u(t)=2D. \quad (113)$$

Recalling that t stands here for t_f and since neither $I_u(t=t_f)$ nor $p^*_{\rm opt}(t=t_f)$ [cf. Eqs. (74) and (84)] are small quantities, it follows that $\hat{k}$ is, for small noise strengths D, much larger than 0 but, for sufficiently large t_f-t_0, according to Eq. (55), also much smaller than $K(t=t_f,t_0)$. Therefore the sum in Eq. (111) and thus in Eq. (110) can be extended to arbitrary integers k at the price of an error which is *exponentially* small in D, i.e., without actually affecting the accuracy exponent γ in Eq. (108). As a further consequence of the fact that the dominant k values are much smaller than the upper limit $K(t,t_0)$ for large $t-t_0$, we see that our formal approximation (85) is indeed self-consistently satisfied.

Next we notice that under the sum in Eq. (110), the pre-exponential term is nothing else than the time derivative of the expression in the exponential. By extending the sum over all integer k values as justified above we obtain

$$\frac{1}{\mathcal{T}}\int_t^{t+\mathcal{T}}\kappa_{\rm opt}(t',D)dt' =\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{1}{D}\int_t^{\infty}p^*_{\rm opt}(t')^2dt'\right\}\Bigg|_{t=k\mathcal{T}}^{(k+1)\mathcal{T}} =1-\exp\left\{-\frac{1}{D}\int_{-\infty}^{\infty}p^*_{\rm opt}(t')^2dt'\right\}. \quad (114)$$

Neglecting as usual errors exponentially small in D this leads us to the remarkable conclusion that

$$\frac{1}{\mathcal{T}}\int_t^{t+\mathcal{T}}\kappa_{\rm opt}(t',D)dt'=1 \quad (115)$$

for all t and all (small) D. For the *time-averaged* rate (14) we thus obtain from Eqs. (108) and (115) our central result [27]

$$\overline{\Gamma}=\sqrt{D}\,\alpha_{\rm opt}e^{-\phi_{\rm opt}/D}[1+\mathcal{O}(D^{\gamma})]. \quad (116)$$

It consists of an Arrhenius-type exponentially leading part with an ''effective potential barrier'' $\phi_{\rm opt}$ and a nontrivial pre-exponential D dependence. The two quantities $\alpha_{\rm opt}$ and $\phi_{\rm opt}$ follow from the master path $x^*_{\rm opt}(t)$ according to Eqs. (54) and (109). Thus they are independent of D but depend in a highly nontrivial way on various global properties of the deterministic force field $F(x,t)$ in Eq. (3). In general, their explicit value can only be determined numerically or by means of approximations. An exactly analytically solvable special case will be presented in Sec. V A.

We recall that for equilibrium systems, characterized by a time-independent force field $F(x)=-V'(x)$ in Eq. (3), the escape rate exhibits an exponentially leading Arrhenius factor, which involves simply the barrier against the escape of the static potential $V(x)$, and a D-independent pre-exponential factor which depends only on local properties of the potential at the barrier and the well [1], see also Eq. (161) below. The different structure of Eq. (116) is thus a consequence of the far from equilibrium situation created by the time dependence of the deterministic force field $F(x,t)$.

As announced in Sec. II C, the time-averaged escape rate for the periodic force field $F(x,t)$ can be identified with that of its supersymmetric partner force field (16) for asymptotically weak noise D without any further restrictions on $F(x,t)$. The detailed proof of this highly nontrivial invariance property of Eq. (116) is carried out in Appendix D.

Returning to the instantaneous rate (108), we see that it exhibits in comparison with the time averaged rate (116) the additional time-dependent factor $\kappa_{\rm opt}(t,D)$. The explicit evaluation of this factor requires the knowledge of one more global quantity, for instance of

$$\beta_{\rm opt}(t):=\lim_{\hat{t}\to\infty}p^*_{\rm opt}(\hat{t})e^{\Lambda_u(\hat{t},t)}. \quad (117)$$

Note that due to relation (65) the t dependency of this quantity is actually quite simple. According to Eq. (61), this definition (117) allows us to rewrite Eq. (111)—with the range of k extended to arbitrary integers—as

$$\kappa_{\text{opt}}(t,D)=\mathcal{T}\sum_{k=-\infty}^{\infty}\frac{\beta_{\text{opt}}(t)^2C^k}{D}\exp\left\{-\frac{\beta_{\text{opt}}(t)^2C^kI_u(t)}{2D}\right\}. \tag{118}$$

Besides $\beta_{\text{opt}}(t)$ all other quantities in this expression are determined by local properties of the force field $F(x_u(t),t)$ along the unstable periodic orbit. By exploiting Eqs. (68), (75), and (112) it follows that

$$\kappa_{\text{opt}}(t+\mathcal{T},D)=\kappa_{\text{opt}}(t,D), \tag{119}$$

$$\kappa_{\text{opt}}(t,CD)=\kappa_{\text{opt}}(t,D). \tag{120}$$

Together with Eq. (113) and the obvious property $0<\kappa_{\text{opt}}(t,D)<\infty$ this completes our qualitative picture of the way in which $\Gamma(t)$ oscillates around its average value $\bar{\Gamma}$.

E. The accuracy exponent γ

In the following we come to the determination of the accuracy exponent γ in Eqs. (108) and (116). We will not elaborate here all the details of the rather involved calculations but restrict ourselves to the main steps.

First of all, we recall that a contribution $\mathcal{O}(D)$ is inherited right away from formula (51). Next we have approximated the square brackets in Eq. (92) by 1. For those k values which mainly contribute to the rate it can be inferred from Eq. (113) together with $t=t_f$ and Eq. (86) that

$$p^*_{\text{opt}}(t_k)^2=\mathcal{O}(D) \tag{121}$$

and hence with Eq. (106) that

$$g^*_{\text{opt}}(t_k)I_u(t_k)=\mathcal{O}(D). \tag{122}$$

Since the integral in Eq. (92) is of the same order of magnitude as $p^*_{\text{opt}}(t_k)$ from Eq. (121) we conclude that the total error we committed in Eq. (92) is of the order $\mathcal{O}(D^2)$, thus contributing once more a term of the order $\mathcal{O}(D)$ in the rate formulas (108) and (116). The same conclusion can be drawn with respect to our approximating the square brackets by 1 in Eq. (93) and neglecting the $\mathcal{O}(p^*_{\text{opt}}(t_k)^2)$ term in Eq. (107) as well as the last term in Eq. (104). In other words, the relative error induced by all our so far made approximations is of the order $\mathcal{O}(D)$.

What remains is a closer inspection of the approximation (56) for $F(x,t)$ in the neighborhood of $x_{s,u}(t)$. One readily sees that actually only the approximation in the neighborhood of the unstable periodic orbit $x_u(t)$ matters in our quantitative evaluation of the rate; the basic reason for this is once more our assumption $t_0\to-\infty$ in Eq. (85). In case Eq. (56) happens to be an exact identity in this neighborhood of $x_u(t)$, then the total error committed in the rate formulas (108) and (116) is thus of the order $\mathcal{O}(D)$. Otherwise, a closer analysis of the relevant perturbative corrections shows that the error introduced via the approximation (56) amounts to corrections of the order $\mathcal{O}(p^*_{\text{opt}}(t_k))$ in the rate formula, i.e., of the order $\mathcal{O}(\sqrt{D})$ according to Eq. (121). In other words, we can conclude that

$$\gamma=\begin{cases}1 & \text{if } F''(x_u(t),t)\equiv 0\\ 1/2 & \text{otherwise.}\end{cases} \tag{123}$$

In the case $\gamma=1/2$ it is important that in the global quantities $\phi_{\text{opt}},\alpha_{\text{opt}},\beta_{\text{opt}}(t)$ from Eqs. (54), (109), and (117) the long-time limits are made and the exact master path is utilized without any further approximations. If instead in these definitions any finite reference time in combination with relations based on the approximation (56) were used, then this would introduce a possibly very small but nevertheless D-independent error and so ruin the asymptotically exact predictions (108) and (116) in the weak noise limit $D\to 0$.

In cases for which Eq. (56) is not exactly satisfied in the neighborhood of the unstable periodic orbit $x_u(t)$ and hence $\gamma=1/2$, it is in principle possible to calculate perturbatively the corresponding corrections such as to arrive again at a reduced relative error $\mathcal{O}(D)$ in the so improved rate formulas, though the actual calculations and the resulting expressions become very complicated. On the other hand, further reducing the $\mathcal{O}(D)$ error is even in principle rather problematic since it would require going beyond the saddle-point approximation in the path-integral approach from Sec. III.

At this point it may also be worth recalling from Sec. IV A that for any fixed (however small) D value, the error $\mathcal{O}(D)$ in Eq. (51), which is inherited by the final rate formula, diverges as the amplitude of the time dependency of $F(x,t)$ tends to zero, but also if its period $\mathcal{T}$ either tends to zero or to infinity. Thus, neither of these limits commutes with the limit $D\to 0$.

F. The limits $t\to\infty$ and $D\to 0$

In the derivation of the rate formula (108) we have assumed that all paths $x^*_k(t)$ which notably contribute in Eq. (51) sojourn for a very long initial time interval close to the stable periodic orbit $x_s(t)$, see Eq. (85). On the other hand, Eq. (113) tells us that the amount of time which those dominant paths spend in the neighborhood of the unstable periodic orbit $x_u(t)$ is roughly speaking of the order $\mathcal{O}(\ln 1/D)$. Both these conditions are compatible only if $t-t_0$ substantially exceeds in order of magnitude $\ln 1/D$. In the physically relevant case, the noise strength D is small but finite and this condition is well satisfied after a comparatively short ''transient'' time period. Thus, strictly speaking, in Eqs. (108) and (116), with decreasing D values also the lower limit of the admitted times $t-t_0$ is tacitly assumed to slowly increase in proportion to $\ln 1/D$.

We remark that our result (108) obviously remains periodic in t for arbitrarily large $t-t_0$, see Eq. (119). Therefore, the restriction of the utilized basic formula (14) to values of $t-t_0$ much smaller than $1/\bar{\Gamma}$ no longer applies to the final result (108); see also the discussion below Eq. (15).

In the physically less relevant case that $t-t_0$ is kept at an arbitrary but fixed value and then D is made smaller and smaller, things become different. As pointed out in Sec. IV A, for any finite initial and final times t_0 and $t=t_f$, there exists generically a unique absolute minimum $x^*_{k_0}(t)$ of the action. For sufficiently small D the k_0 term will thus completely dominate the sum in Eq. (51), i.e.,

$$\Gamma(t=t_f)=\frac{p^*_{k_0}(t_f)e^{-\phi_{k_0}(x_u(t_f),t_f)/D}}{[4\pi D Q^*_{k_0}(t_f)]^{1/2}}. \tag{124}$$

While most of the quantities on the right-hand side of this result (including the index k_0) still depend in a very complicated way on the time $t=t_f$, no additional implicit D dependence is hidden. The most striking feature is the $1/\sqrt{D}$ pre-exponential behavior in comparison with the $\sqrt{D}$ scaling in Eq. (108).

Qualitatively, the crossover from Eq. (124) to Eq. (108), either as t increases or as D decreases, is clear: At some point the k dependence of the pre-exponential factors in Eq. (51) is no longer negligible in comparison with the exponentially leading contributions and so the dominant k value moves away from $k_0=k_0(t,t_0)$ towards smaller values $k\simeq\hat{k}$, cf. Eq. (113). At the same time, more that one term in the sum (51) starts to notably contribute.

Quantitatively, a leading-order approximation follows along the same line of reasoning as in the derivation of Eq. (108) from Eq. (51), except that in the approximation for the action (91), also contributions due to the deviations between $x^*_k(t)$ and its associated master path $x^*_{\rm opt}(t+k\mathcal{T})$ at times close to t_0 have to be included, that is, the approximation (85) should be abandoned. The final result is again of the same form as in Eq. (108) but with a larger error than $\mathcal{O}(D^\gamma)$ and instead of Eq. (110) with

$$\kappa_{\rm opt}(t,D):=T\sum_{k=0}^{K(t,t_0)}\frac{p^*_{\rm opt}(t+k\mathcal{T})^2}{D}\times\exp\left\{-\frac{1}{D}\left[\int_t^\infty+\int_{-\infty}^{t_0}\right]p^*_{\rm opt}(t'+k\mathcal{T})^2dt'\right\}. \tag{125}$$

For moderate $t-t_0$ or extremely small D the exponential in Eq. (125) depends very strongly on k and therefore the sum is dominated by a single term $k=k_0(t,t_0)$. Upon increasing $t-t_0$ or D this strong k dependence of the exponential and hence the dominance of the k_0 contribution is softened and the already discussed qualitative crossover behavior is recovered.

G. More general seeds x_0

So far, our rate formula (108) is restricted to the case (52) that the initial position x_0 at time t_0 coincides with the stable periodic orbit $x_s(t_0)$. As pointed out in Sec. II, one expects that for large enough times $t-t_0$ the initial position x_0 should not matter, provided it is chosen inside the domain of attraction of $x_s(t)$. For sufficiently small noise strengths D this is the case whenever

$$x_0<x_u(t_0). \tag{126}$$

In the following, we analyze this intuitive expectation in some more quantitative detail.

For arbitrary but fixed x_0 satisfying Eq. (126), the observation from Sec. IV A remains true, namely that only minimizing paths $x^*_k(t)$ play a role in the rate (51) which closely follow a deterministic behavior $\dot{x}^*_k(t)\simeq F(x^*_k(t),t)$ for most of the time. This requirement can be fulfilled in two basic ways and appropriate compromises thereof. The first possibility is that the path $x^*_k(t)$ closely approximates a deterministic trajectory for a very long initial time interval. During this time, $x^*_k(t)$ approaches the periodic attractor $x_s(t)$ very closely and practically does not accumulate any action [Eqs. (36) and (37)]. Consequently, one is back to the case (52) after an appropriate redefinition of the initial time t_0. Regarding the prefactor $Q^*_k(t)$, one can, according to Eq. (42), approximate $p^*_k(t)$ in Eq. (46) by zero. With the initial conditions (47) one then recovers the same solution as in Eq. (96) except that in Eqs. (63) and (64) the function $\Lambda_s(t,t_0)$ is now defined as $\int_{t_0}^t F'(x^*_k(t'),t')dt'$. Since $x^*_k(t)$ practically agrees with $x_s(t)$ during a very long time interval, one sees that also with respect to the prefactor $Q^*_k(t)$ we are back to the case (52). As before, we may label such paths by low k values and their contributions to the rate (51) are identical to those of the low k values in Eqs. (108)–(110).

The second possibility is that the minimizing path $x^*_k(t)$ travels from its starting point x_0 immediately into the neighborhood of the unstable periodic orbit $x_u(t)$ and then very closely follows this deterministic trajectory $x_u(t)$ for the rest of its time. If x_0 is already close to $x_u(t_0)$, such paths lead to a very small value of the action in Eqs. (36) and (37) and thus will ultimately dominate the rate (51) if $t-t_0$ is kept fixed and D becomes asymptotically small. This puzzling observation has lead to some amount of confusion in the recent literature [47,48]. The resolution is that, much like in Sec. IV F, things become very different for a small but fixed D in combination with larger and larger times $t-t_0$. The salient point is that the price to be paid for a long sojourn close to $x_u(t)$ is a very small prefactor $p^*_k(t_f)/[Q^*_k(t_f)]^{1/2}$ in Eq. (51), as discussed below (107), namely of the order $\exp\{-2\lambda_u\cdot(t-t_0)\}$. As a consequence, the paths with low k-values, as discussed in the preceding paragraph, will dominate in spite of their unfavorable action. Therefore, the rate formula (108) applies for any x_0 satisfying Eq. (126) on condition that

$$t-t_0\gg\phi_{\rm opt}/(2D\lambda_u). \tag{127}$$

This condition characterizes the asymptotic time regime for which the rate formula (108) is valid in the case of a general initial condition. Even for rather small D, the preceding transient regime is typically confined to a few driving periods $\mathcal{T}$, as illustrated by the examples in Sec. V. Note that Eq. (127) comprises the condition from Sec. IV F that $t-t_0$ has to substantially exceed in order of magnitude $\ln 1/D$. In other words, for a generic initial condition, Eq. (127) is the only restriction for the rate formula (108), apart from the exclusion of vanishing driving amplitudes and vanishing or diverging periods $\mathcal{T}$.

H. Summary from the practical viewpoint

Given an arbitrary time-periodic force field $F(x,t)$ that satisfies the condition in Eq. (9), what are the necessary practical (numerical or analytical) steps for an explicit quantitative evaluation of the rate (108)?

The first step is the determination of the stable and unstable periodic orbits $x_s(t)$ and $x_u(t)$. An efficient way to do this is by evolving the deterministic dynamics forward and backward over a long time, respectively, with a reasonably well chosen initial condition. Once $x_{s,u}(t)$ is known, the functions $\Lambda_{s,u}(t,t_1)$ from Eq. (63) and $I_{s,u}(t,t_1)$ from Eqs. (71) and (73) follow readily, with t_1 being an arbitrary reference time.

The next step is the determination of the master path $x^*_{\rm opt}(t)$. To this end, we chose an arbitrary but fixed time t_s and a very small but finite positive number $\Delta x_{\min}$, characterizing the neighborhood of $x_s(t)$ within which we are willing to accept the errors introduced by the approximation (56). We now consider the quantity $\Delta x^*_{\rm opt}(t_s)$ as a parameter that may take values in the interval $[\Delta x_{\min}, \Delta x_{\min} e^{-\lambda_s \mathcal{T}}]$. Each such parameter value $\Delta x^*_{\rm opt}(t_s)$ yields a set of initial conditions

$$x^*_{\rm opt}(t_s) = x_s(t_s) + \Delta x^*_{\rm opt}(t_s), \tag{128}$$

$$p^*_{\rm opt}(t_s) = \Delta x^*_{\rm opt}(t_s)/I_s(t_s), \tag{129}$$

see Eqs. (60) and (82). With these initial conditions one then evolves $x^*_{\rm opt}(t)$ and $p^*_{\rm opt}(t)$ according to Eqs. (41) and (42). For a generic value of the parameter $\Delta x^*_{\rm opt}(t_s)$, the path $x^*_{\rm opt}(t)$ will either reach $x_u(t)$ after a finite time and then proceed towards $x=\infty$ or never reach $x_u(t)$ and instead return into the vicinity of $x_s(t)$ as t grows. By fine tuning $\Delta x^*_{\rm opt}(t_s)$ one has to find a path $x^*_{\rm opt}(t)$ which remains close to $x_u(t)$ as long as possible, say until $t=t_{\max}$. Upon varying $\Delta x^*_{\rm opt}(t_s)$ within $[\Delta x_{\min}, \Delta x_{\min} e^{-\lambda_s \mathcal{T}}]$ the existence of at least one such path is guaranteed by the theory. A second solution, corresponding to a saddle point instead of a minimum of the action, is also to be expected [see Sec. V A, below Eq. (158)]. Further local extrema may coexist as well. Among them, the desired solution $x^*_{\rm opt}(t)$ is the one with the smallest value of the action

$$\phi_{\rm opt} = \frac{\Delta x^*_{\rm opt}(t_s)^2}{2 I_s(t_s)} + \int_{t_s}^{t_{\max}} p^*_{\rm opt}(t)^2 dt, \tag{130}$$

see Eqs. (43), (54), (61), and (76). By approximating $\hat{t}$ in Eq. (117) by $t_{\max}$ we obtain

$$\beta_{\rm opt}(t) = p^*_{\rm opt}(t_{\max}) e^{\Lambda_u(t_{\max},t)}, \tag{131}$$

whence $\kappa_{\rm opt}(t,D)$ from Eq. (118) follows with C from Eq. (112). Finally, one chooses the initial conditions

$$Q^*_{\rm opt}(t_s) = I_s(t_s)/2, \quad \dot{Q}^*_{\rm opt}(t_s) = \dot{I}_s(t_s)/2, \tag{132}$$

see Eqs. (99) and (100), and then propagates $Q^*_{\rm opt}(t)$ according to Eq. (46) until $t=t_{\max}$, to obtain

$$\alpha_{\rm opt} = [4\pi \mathcal{T}^2 p^*_{\rm opt}(t_{\max})^2 Q^*_{\rm opt}(t_{\max})]^{-1/2}, \tag{133}$$

see Eq. (109).

The accuracy of $\phi_{\rm opt}, \beta_{\rm opt}(t), \alpha_{\rm opt}$ from Eqs. (130), (131), and (133) can be estimated by observing how little these quantities change if $t_{\max}$ is varied and if $\Delta x_{\min}$ is changed by a factor $e^{\lambda_s \mathcal{T}}$.

We finally note that the association of $x^*_{\rm opt}(t+k\mathcal{T})$ with the specific path $x^*_k(t)$ as in Secs. IV A–IV G does not play a role any more in the above described practical procedure.

V. EXAMPLES

In general, the explicit quantitative evaluation of $\phi_{\rm opt}$, $\alpha_{\rm opt}$, and $\kappa_{\rm opt}(t,D)$ in the rate formula (108) is not possible in closed analytical form. Exceptions are piecewise parabolic potentials $V(x)$ in conjunction with an additive sinusoidal driving (5). In Sec. V A the simplest example [27] with two parabolic pieces will be worked out and compared with accurate numerical results and with the approximation for small driving amplitudes from [23]. In Sec. V B we elaborate as a second example the case of a force field (5) deriving from a cubic potential $V(x)$ along the lines of the numerical recipe from Sec. IV H.

A. Piecewise parabolic potential

We consider the force field from Eq. (5) with a piecewise parabolic potential of the form

$$V(x\leq 0) = \frac{\lambda_s}{2}[\bar{x}_s^2 - (x-\bar{x}_s)^2],$$
$$V(x\geq 0) = \frac{\lambda_u}{2}[\bar{x}_u^2 - (x-\bar{x}_u)^2], \tag{134}$$

where $\bar{x}_s$ denotes the potential well (stable fixed point) and $\bar{x}_u$ the saddle (unstable fixed point), with the properties

$$\bar{x}_s < 0, \quad \bar{x}_u > 0. \tag{135}$$

The parameters

$$\lambda_s < 0, \quad \lambda_u > 0 \tag{136}$$

characterize the piecewise constant curvatures and thus the time scales (Lyapunov exponents) of the deterministic motion near the attractor $\bar{x}_s$ and the repeller $\bar{x}_u$, respectively. The force field (5) then takes the explicit form

$$F(x\leq 0,t) = \lambda_s(x-\bar{x}_s) + A\sin(\Omega t),$$
$$F(x\geq 0,t) = \lambda_u(x-\bar{x}_u) + A\sin(\Omega t). \tag{137}$$

In particular, the quantities $\lambda_{s,u}$ in Eqs. (134) and (137) are identical to those from Eq. (68)

Requiring continuity at $x=0$ we conclude from Eq. (137) that $\lambda_s \bar{x}_s = \lambda_u \bar{x}_u$. Selecting as independent model parameters λ_s, λ_u, and the static potential barrier

$$\Delta V := V(\bar{x}_u) - V(\bar{x}_s), \tag{138}$$

the fixed points $\bar{x}_{s,u}$ can be expressed through

$$\lambda_s \bar{x}_s = \lambda_u \bar{x}_u = \sqrt{\frac{2\Delta V |\lambda_s| \lambda_u}{|\lambda_s| + \lambda_u}}. \tag{139}$$

Turning to the determination of the stable and unstable periodic orbits (7), it is convenient to make a somewhat stronger assumption than in Eq. (9), namely that both periodic orbits $x_{s,u}(t)$ never cross the matching point $x=0$ of the two parabolic pieces of $V(x)$, i.e., we require that

$$x_s(t)<0<x_u(t) \tag{140}$$

for all times t. One finds that this property is granted if and only if the conditions

$$A^2<(\lambda_{s,u}^2+\Omega^2)\bar{x}_{s,u}^2 \tag{141}$$

are satisfied for both the ''s'' and the ''u'' indices, and that the periodic orbits then take the explicit form

$$x_{s,u}(t)=\bar{x}_{s,u}-\frac{A[\lambda_{s,u}\sin(\Omega t)+\Omega\cos(\Omega t)]}{\lambda_{s,u}^2+\Omega^2}. \tag{142}$$

With the definitions (63), (72), and (74) it follows that

$$\Lambda_{s,u}(t,t_1)=\lambda_{s,u}\cdot(t-t_1), \tag{143}$$

$$I_{s,u}(t)=|\lambda_{s,u}|^{-1}. \tag{144}$$

Our next goal is the determination of the master path $x_{\rm opt}^*(t)$. To simplify the analytical calculations we restrict ourselves to the case that the master path $x_{\rm opt}^*(t)$ crosses the point $x=0$ exactly once, say at the time $t=t_1$,

$$x_{\rm opt}^*(t)=0 \Leftrightarrow t=t_1. \tag{145}$$

The self-consistency of this assumption with the final solution for $x_{\rm opt}^*(t)$ remains to be checked later.

From Eqs. (41) and (42) we see that both $x_{\rm opt}^*(t)$ and $p_{\rm opt}^*(t)$ are still continuous at $t=t_1$. For all other times t the relation (56) and hence the following conclusions are not approximations but exact identities since the force field $F(x,t)$ in Eq. (137) is by construction piecewise linear. By introducing Eqs. (144), (145), and (60) into Eqs. (76) and (79) we obtain, by letting $t_0\to\infty$ and $t_f\to\infty$ for the master path, the following two relations (one with index ''s'' and one with ''u''):

$$x_{s,u}(t_1)=p_{\rm opt}^*(t_1)/\lambda_{s,u}. \tag{146}$$

These two equations for the two unknowns t_1 and $p_{\rm opt}^*(t_1)$ imply with Eq. (142) the result

$$\tan(\Omega t_1)=\frac{1}{\Omega}\frac{\lambda_s\lambda_u-\Omega^2}{\lambda_s+\lambda_u}, \tag{147}$$

$$p_{\rm opt}^*(t_1)=\lambda_u\bar{x}_u-\frac{A\lambda_s\lambda_u\cos(\Omega t_1)}{\Omega(\lambda_s+\lambda_u)}. \tag{148}$$

We observe that the solutions t_1 of Eq. (147) are independent of A. Furthermore, there are obviously two solutions t_1 within every time period $\mathcal{T}=2\pi/\Omega$. We anticipate that only one of them corresponds to a minimum of the action, and thus to the master path. Hence we fix t_1 uniquely (up to the usual degeneracy under $t\mapsto t+\mathcal{T}$) by (147) in conjunction with

$$\frac{A\Omega\cos(\Omega t_1)}{\lambda_s+\lambda_u}<0, \tag{149}$$

and show later, that this condition singles out the right solution of (147). [The case $\lambda_s+\lambda_u=0$ has to be treated as limit $\lambda_s+\lambda_u\to 0$.] Combining Eqs. (147)–(149) it follows that

$$p_{\rm opt}^*(t_1)=\lambda_u\bar{x}_u-|A||\lambda_s|\lambda_u/\nu^2>0, \tag{150}$$

where we have introduced the definition

$$\nu^2:=[(\lambda_s^2+\Omega^2)(\lambda_u^2+\Omega^2)]^{1/2}. \tag{151}$$

Note that $\lambda_u\bar{x}_u$ in Eq. (150) may be rewritten in various equivalent forms according to Eq. (139) and that the last relation $p_{\rm opt}^*(t_1)>0$ in Eq. (150) follows as a consequence of Eqs. (136), (140), and (146).

Given t_1 and $p_{\rm opt}^*(t_1)$, the entire time evolution of the master path $x_{\rm opt}^*(t)$ can be readily inferred from Eqs. (60), (61), (72), (74), (76), and (79) and Eqs. (143) and (144) with the result

$$p_{\rm opt}^*(t)=p_{\rm opt}^*(t_1)e^{-\lambda_{s,u}\cdot(t-t_1)}, \tag{152}$$

$$x_{\rm opt}^*=x_{s,u}(t)-p_{\rm opt}^*(t)/\lambda_{s,u}, \tag{153}$$

where ''s'' is associated with times $t\leqslant t_1$ and ''u'' with $t\geqslant t_1$. All the general qualitative features discussed in Sec. IV A are nicely illustrated by this explicit example (152) and (153).

Finally, we have to check the self-consistency of the solution Eqs. (152) and (153) with our initial assumption (145), i.e., we have to verify that $x_{\rm opt}^*(t)$ is strictly positive for $t>t_1$ and negative for $t<t_1$. In general, in doing so, a transcendental equation arises which has to be evaluated numerically. Without going into the details of the proof we further mention that one can show analytically that $A^2<\lambda_{s,u}^2\bar{x}_{s,u}^2$ is a sufficient but not necessary self-consistency criterion for Eq. (145). On the other hand, it is obvious that the assumption $x_s(t)<0<x_u(t)$ in Eq. (140) is automatically covered by the stronger requirement (145). Thus, Eqs. (140) and (141) are a necessary but not sufficient self-consistency criterion for Eq. (145).

Introducing the above relations (152) and (153) into Eqs. (43) and (54), we obtain for the action of the master path

$$\phi_{\rm opt}=\Delta V\left[1-\left|\frac{A^2\lambda_s\lambda_u(|\lambda_s|+\lambda_u)}{2\Delta V\nu^4}\right|^{1/2}\right]^2. \tag{154}$$

For $A\to 0$ or $\Omega\to\infty$ we thus recover the static (undriven) potential barrier ΔV from Eq. (138). The leading-order corrections for small A decrease like $|A|$ [23]. For any finite amplitude A and driving period $\mathcal{T}=2\pi/\Omega$ the ''effective potential barrier'' $\phi_{\rm opt}$ is smaller than the static barrier ΔV and is monotonically decreasing both with increasing A and increasing $\mathcal{T}$. Invoking the necessary but not sufficient self-consistency criterion (141) for Eq. (145), one can explicitly confirm that $\phi_{\rm opt}$ can never become zero [see Eqs. (36), (37), and (54)] by demonstrating that the argument in the square brackets in Eq. (154) is always positive. If we had chosen the solution of Eq. (147) with the opposite inequality than in Eq. (149), then a plus instead of the minus sign in Eq. (154)

would have been the consequence. Thus Eq. (149) is indeed the pertinent condition for singling out the solution which minimizes the action.

By using Eqs. (143) and (152) in the definition (117) of $\beta_{\rm opt}(t)$ we obtain

$$\beta_{\rm opt}(t) = p^*_{\rm opt}(t_1)e^{-\lambda_u\cdot(t-t_1)}. \qquad (155)$$

Turning to the prefactor $Q^*_{\rm opt}(t)$, we see from Eqs. (96) and (144) that

$$Q^*_{\rm opt}(t) = \frac{1}{2|\lambda_s|} \qquad (156)$$

for all times $t \leqslant t_1$. Since $F''(x,t) = (\lambda_u - \lambda_s)\delta(x)$ according to Eq. (137), we can infer from Eq. (46) that the prefactor $Q^*_{\rm opt}(t)$ is continuous at $t = t_1$ while its derivative jumps from $\dot{Q}^*_{\rm opt}(t_1^-) = 0$ to the value

$$\dot{Q}^*_{\rm opt}(t_1^+) = \frac{|\lambda_s| + \lambda_u}{|\lambda_s|} \frac{\dot{x}^*_{\rm opt}(t_1) - p^*_{\rm opt}(t_1)}{\dot{x}^*_{\rm opt}(t_1)}, \qquad (157)$$

where t_1^+ indicates the limit $t \to t_1$ from above and t_1^- from below. With these initial conditions, the solution of Eq. (94) in the domain $t > t_1$ is straightforward, yielding

$$\lim_{t\to\infty} Q^*_{\rm opt}(t) p^*_{\rm opt}(t)^2 = \frac{p^*_{\rm opt}(t_1)^2 \dot{Q}^*_{\rm opt}(t_1^+)}{2\lambda_u}. \qquad (158)$$

Using Eqs. (148), (152), and (153) one can show that $p^*_{\rm opt}(t_1) - \dot{x}^*_{\rm opt}(t_1)$ is identical to the expression on the left-hand side of Eq. (149), so that Eqs. (157) and thus (158) are positive quantities. With the opposite inequality in Eq. (149) they would be negative, confirming once more that the latter case corresponds to a saddle point rather than a minimum of the action. Collecting everything, we are finally in the position to evaluate Eq. (109) with the result

$$\alpha_{\rm opt} = \left[\frac{|A|(\Omega^2 + \lambda_s\lambda_u) + \sqrt{\dfrac{2\Delta V \nu^4}{|\lambda_s|^{-1} + \lambda_u^{-1}}}}{16\pi^3 |A| \phi_{\rm opt}} \right]^{1/2}. \qquad (159)$$

Once again, the fact that the argument in the square root is positive can be explicitly verified by exploiting the necessary but not sufficient self-consistency criterion (141) for (145).

B. Comparison of analytical and numerical results

We have compared the above analytical predictions for the instantaneous rate (108) with very accurate numerical results in Fig. 4. To this end, we have computed the solution of the Fokker-Planck equation (10) and then evaluated the rate according to Eq. (13), starting with a narrow Gaussian initial distribution $p(x,t_0)$ about the potential well $\overline{x_s}$ and then waiting until transients have died out, i.e., until $\Gamma(t)$ has reached its $\mathcal{T}$-periodic asymptotic behavior. In order to numerically evolve the one-dimensional time-dependent Fokker-Planck equation (10) one can employ standard parabolic partial-differential equation solving procedures in one spatial variable. We have adopted a Chebyshev collocation method [49] to reduce the problem to a coupled system of ordinary differential equations, which is then solved by standard numerical methods. By changing the various parameters of the numerical procedure, the typical relative errors of the numerical rates $\Gamma(t)$ in our figures are estimated to be at most of the order of 10^{-4} for rates down to about 10^{-100} and of the order of 10^{-3} for rates down to about 10^{-200}.

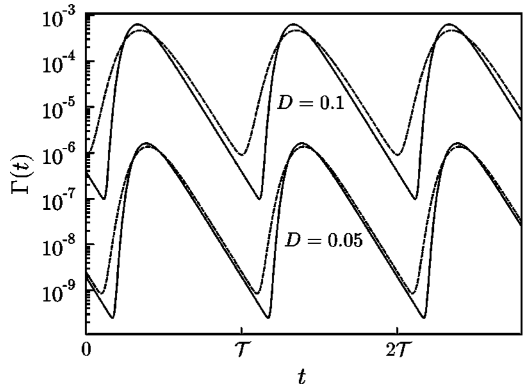


FIG. 4. Instantaneous rate $\Gamma(t)$ versus time t for the piecewise linear force field (137) in dimensionless units with parameters $x_s = \lambda_s = -1$, $x_u = \lambda_u = 1$, $\Omega = 1$ ($\mathcal{T} = 2\pi$), and $A = 0.5$, corresponding to a static ($A = 0$) potential barrier $\Delta V = 1$ in Eqs. (138) and (139). Solid line: Analytical prediction [Eqs. (108), (118), (150), (151), (154), (155), and (159)] by neglecting the $\mathcal{O}(D^\gamma)$ term in Eq. (108). Dashed line: High-precision numerical results, obtained as described in Sec. V B.

The results in Fig. 4 confirm for a representative set of parameter values that the agreement between the analytical predictions and the practically exact numerical results for the instantaneous rate $\Gamma(t)$ indeed improves with decreasing noise strength D. While the absolute values of $\Gamma(t)$ and the location of the extrema strongly depend on D, the overall shape changes very little and does *not* develop singularities as $D \to 0$.

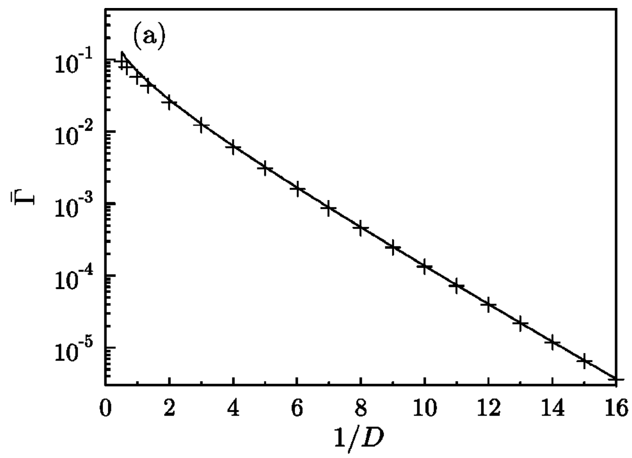
The corresponding time-averaged rates (116) are depicted in Fig. 5(a), exhibiting excellent agreement between theory and numerics even for relatively large D. Figure 5(b) confirms our analytical prediction that the relative error in Eq. (116) decreases asymptotically like D, see Eq. (123).

Finally, Fig. 6 illustrates the dependence of the time-averaged rate $\overline{\Gamma}$ upon the amplitude A of the periodic driving force. As expected, our theoretical prediction compares very well with the (numerically) exact rate, except for very small driving amplitudes A. The latter discrepancy is in accordance with our discussion in Sec. IV A and Sec. IV E.

We have, furthermore, included in Fig. 6 a comparison with the analytical approximation for the time-averaged rate $\overline{\Gamma}$ from Ref. [23]. By way of a matching procedure, involving the barrier region only, it is predicted [23] that

$$\overline{\Gamma} = \Gamma_0 \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-s(\phi)/D}, \qquad (160)$$

where Γ_0 is the well-known Kramers-Smoluchowsky rate in the absence of the periodic driving force [1]



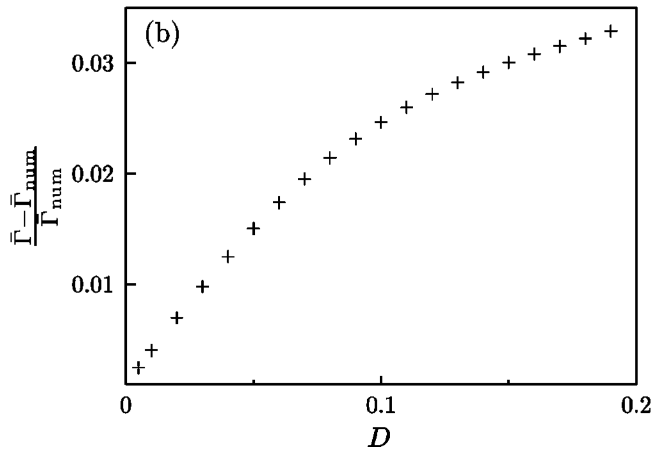


FIG. 5. (a) Arrhenius plot of the time-averaged rate $\bar{\Gamma}$ for the piecewise linear force field (137) in dimensionless units with the same parameters as in Fig. 4. Solid line: Analytical prediction [Eqs. (116), (154), and (159)] by neglecting the $\mathcal{O}(D^{\gamma})$ term in Eq. (116). Crosses: High-precision numerical results, obtained as described in Sec. V B. (b) Relative difference between the analytical ($\bar{\Gamma}$) and numerical ($\bar{\Gamma}_{\mathrm{num}}$) rate.

$$\Gamma_0=\frac{|V''(\bar{x}_s)V''(\bar{x}_u)|^{1/2}}{2\pi}e^{-\Delta V/D}. \qquad (161)$$

The leading-order effect of an additive sinusoidal driving (5), such that the associated periodic modulations of the potential barrier are small in comparison with the unperturbed barrier ΔV, but not necessarily in comparison with the noise strength D, are captured by the function $s(\phi)$ in Eq. (160). It can be written for a general metastable potential $V(x)$ in Eq. (5) under the form [23]

$$s(\phi)=A(S\sin\phi-C\cos\phi), \qquad (162)$$

$$S=S(x_1):=\int_{\bar{x}_s}^{\bar{x}_u}dx\,\sin\left(\Omega\int_{x_1}^{x}\frac{dy}{V'(y)}\right),$$

$$C=C(x_1):=\int_{\bar{x}_s}^{\bar{x}_u}dx\,\cos\left(\Omega\int_{x_1}^{x}\frac{dy}{V'(y)}\right), \qquad (163)$$

with an arbitrary reference point $x_1\in(\bar{x}_s,\bar{x}_u)$. Figure 6 confirms that this approximation from [23] is indeed complementary to ours in that it is very accurate for small driving amplitudes A but develops considerable deviations with increasing A. Those approximations have been omitted in Figs. 4 and 5 since they are not valid in this parameter regime and indeed are way off.

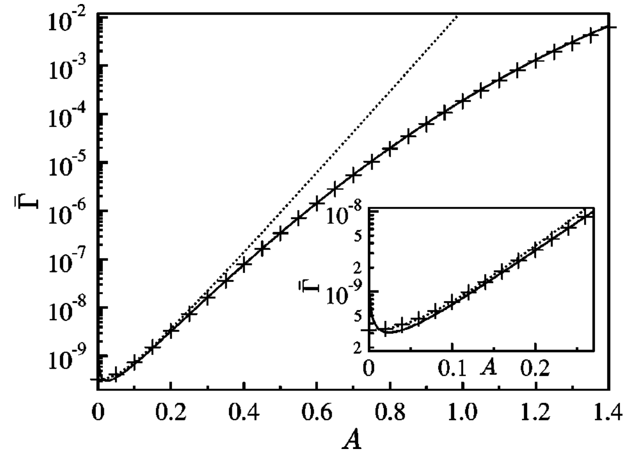


FIG. 6. Time-averaged rate $\bar{\Gamma}$ versus the driving amplitude A for the piecewise linear force field (137) in dimensionless units with parameters $x_s=\lambda_s=-1$, $x_u=\lambda_u=1$, $\Omega=1$, and $D=0.05$. Solid line: Analytical prediction [Eqs. (116), (154), and (159)] by neglecting the $\mathcal{O}(D^{\gamma})$ term in Eq. (116). Dotted line: Theoretical approximation (160)–(163) according to Ref. [23]. Crosses: High-precision numerical results, obtained as described in Sec. V B. Inset: Magnification of the small-A regime.

C. Cubic potential

As a second example we consider a force field (5) with a cubic metastable potential

$$V(x)=-\frac{a}{3}x^3+\frac{b}{2}x^2, \quad a,b>0. \qquad (164)$$

The stable and unstable fixed points $\bar{x}_{s,u}$ of this potential are given by

$$\bar{x}_s=0, \quad \bar{x}_u=\frac{b}{a}, \qquad (165)$$

with curvatures at those fixed points

$$V''(\bar{x}_s)=b, \quad V''(\bar{x}_u)=-b, \qquad (166)$$

and a static potential barrier height

$$\Delta V:=V(\bar{x}_u)-V(\bar{x}_s)=\frac{b^3}{6a^2}. \qquad (167)$$

The time-dependent force field (5) takes the following form:

$$F(x,t)=ax^2-bx+A\sin(\Omega t). \qquad (168)$$

Since already the analytical evaluation of such a force field's periodic orbits is impossible, one has to recourse to numerical methods for the calculation of the quantities ϕ_{opt}, α_{opt}, and $\kappa_{\mathrm{opt}}(t,D)$ appearing in the rate expressions (108) and (116). A convenient numerical strategy for doing so has been discussed in detail already in Sec. IV H. The so obtained predictions for the time-averaged rate (116) are com-

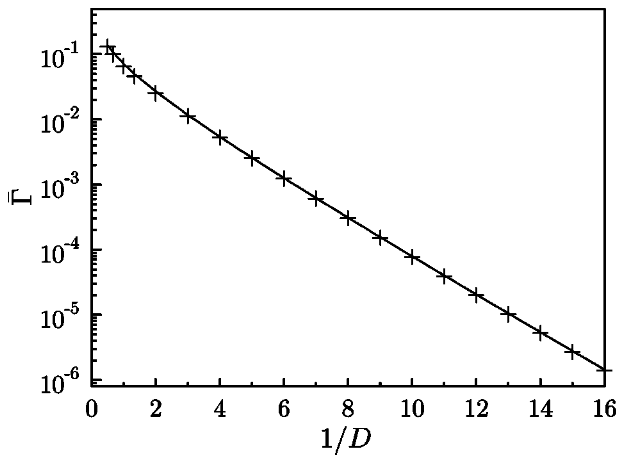


FIG. 7. Arrhenius plot of the time-averaged rate $\bar{\Gamma}$ for the cubic potential [Eqs. (5) and (164)] in dimensionless units with parameters $a=1/\sqrt{6}$, $b=1$, $A=0.5$, and $\Omega=1$, corresponding to a static ($A=0$) potential barrier $\Delta V=1$ in Eq. (167) and curvatures $|V''(\bar{x}_{s,u})|=1$ in Eq. (166). Solid line: Analytical prediction from Eq. (116) without the $\mathcal{O}(D^{\gamma})$ term by adopting the calculational procedure from Sec. IV H. Crosses: High-precision numerical results, obtained as described in Sec. V B.

pared in Fig. 7 against precise numerical results for a representative set of parameter values. Note that these parameter values are quantitatively very similar to those in Fig. 6, hence also the rates as a function of the noise strength D are very similar. The agreement between the theoretical prediction and the practically exact numerical results is again excellent even for relatively large noise strengths D. However, in contrast to the piecewise parabolic case, the numerically accessible D values are still not small enough in order to check the validity of our prediction (123) for the behavior of the relative error in the analytical approximation (116).

VI. CONCLUSIONS

In our present work we have scrutinized by means of path-integral methods the thermally activated escape of an overdamped Brownian particle over a periodically oscillating potential barrier in the most challenging regime of weak thermal noise in combination with moderately strong and moderately fast driving.

A first major result of our path-integral approach is the expression (51) for the instantaneous escape rate, which displays the suggestive general structure of ''probability at the separatrix times velocity.'' The summation appearing in this expression reflects the fact that several local minima of the relevant action in the path-integral formulation of the escape problem notably contribute to the rate. In contrast to the undriven escape problem, giving rise to a (quasi-) Goldstone mode due to the (quasi-) time-translation symmetry, in our present case the paths corresponding to the local minima of the action are well separated and therefore admit a standard saddle-point approximation of the path integral for small noise strengths D. Pictorially speaking, by switching on the periodic driving, the continuous time-translation symmetry of the escape problem is reduced to a time-discrete one.

Our present explorations indicate that from the practical (technical) viewpoint, a path-integral approach which keeps an entire sum of possibly relevant contributions to the rate may be easier to handle than WKB-type or quasipotential-type methods [9,12], which operate with the concept of a single exponentially dominating weak-noise contribution and a single pre-exponential factor, both of them typically of a nonanalytic nature.

The central result of our present paper represents the formula (108) for the instantaneous rate, supplemented by the result (123) for the ''accuracy exponent'' γ. The above discussed summation over the relevant local minima of the action resurfaces in all the equivalent alternative expressions [Eqs. (110), (111), and (118)] for $\kappa_{\mathrm{opt}}(t,D)$ but drops out (can be performed) in the time-averaged rate (116) due to the miraculous identity (115).

The rate expressions (108) and (116) share the general Arrhenius-type structure of the exponentially leading weak-noise contribution with the typical form of an equilibrium (undriven) rate (161). However, both the Arrhenius factor and the pre-exponential contribution to the rates (108) and (116) now depend in a very complicated way on global features of the periodically oscillating potential [in contrast to the purely local properties $\Delta V = V(\bar{x}_u) - V(\bar{x}_s)$ and $V''(\bar{x}_{s,u})$ governing Eq. (161)]. Moreover, a nontrivial $\sqrt{D}$ dependence of the pre-exponential factor on the noise strength D arises.

For the time-averaged rate (116) we have shown in Appendix D that for asymptotically weak noise D an invariance property holds under the supersymmetry transformation (16) without any further restrictions on the force field $F(x,t)$. Such an invariance property can be established rigorously on very general grounds [34] for force fields of the form $F(x,t) = -V'(x) + y(t)$ and arbitrary noise strengths D.

The time-averaged rate (116) displays a remarkable structural similarity with the rate expressions obtained in [50] for one-dimensional discrete-time systems in the presence of weak Gaussian white noise. While a general qualitative connection between these two different types of escape problems via some kind of stroboscopic mapping is quite suggestive, the quantitative details are not so simple. Especially, the Gaussianity of the resulting noise after the stroboscopic mapping is crucial [51] but is far from obvious [52] for the rare but strong fluctuations (large deviations) which govern the escape events.

The condition for the validity of our rate formulas is Eq. (127) and that for a fixed (small) noise strength D, extremely weak, fast, and slow periodic driving forces should be excluded. Especially, the weak noise limit $D \to 0$ displays a rather intriguing noninterchangeability with the long-time limit $t \to \infty$ (see Sec. IV F), and with the limits of asymptotically weak, fast, or slow driving.

In general, an action integral remains to be minimized numerically and an ordinary linear differential equation of second order for the prefactor to be solved (Sec. IV H) before actual numbers can be extracted from our rate formulas. However, for the special case of a sinusoidally driven, piecewise parabolic metastable potential this entire program can be executed in closed analytical from. This example retains all the typical features of more general setups and exhibits excellent agreement with high-precision numerical results (Sec. V).

Conceptually, our path-integral approach should be of considerable interest for many related problems. Generalizations for higher-dimensional systems and for nonperiodic driving forces are currently under investigation [30]. Also the proper handling of the tantalizing weak, fast, and slow driving limits within a consistent path-integral formalism remains as an open problem for future research.

Finally, the complicated dependence of the rate on the global details of the oscillating potential poses a challenging inverse problem, namely to reconstruct the underlying force field from a given (e.g., measured) behavior of the escape rates.

ACKNOWLEDGMENTS

This work was supported by the DFG-Sachbeihilfe HA1517/13-2 (P.H., P.R.), the Sonderforschungsbereich 486 (P.H., J.L.), the Studienstiftung des deutschen Volkes (J.L.), and the Graduiertenkolleg GRK283 (P.H., P.R.).

APPENDIX A

To prove the equivalence of Eq. (29) and Eqs. (30)–(32) one first needs the Hessian of the discrete-time action $S_N(x_0,\ldots,x_N)$ in Eq. (24), which is given by the $(N-1)\times(N-1)$ matrix

$$\left(2\Delta t\,\frac{\partial^2 S(x^*)}{\partial x_n^*\partial x_n^*}\right)_{n,m=1,2,\ldots,N-1} = \begin{pmatrix} a_1 & -b_1 & & & \\ -b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & a_{N-2} & -b_{N-2} \\ & & & -b_{N-2} & a_{N-1} \end{pmatrix}, \tag{A1}$$

where

$$a_n := 2+[2F'(x_n^*,t_n)-(x_{n+1}^*-x_n^*)F''(x_n^*,t_n)]\Delta t + [(F'(x_n^*,t_n))^2+F(x_n^*,t_n)F''(x_n^*,t_n)]\Delta t^2, \tag{A2}$$

$$b_n := 1+F'(x_n^*,t_n)\Delta t. \tag{A3}$$

For the prefactor $Z_N(\mathbf{x}^*)$ in Eq. (29) the determinant of the matrix on the right-hand side of (A1) has to be evaluated. This can be done by a standard procedure (cf. [39]). The result is a linear two-step recursion relation for the principal minor $\tilde{Q}_n^*$, consisting of the first n columns and rows of Eq. (A1), of the form

$$\tilde{Q}_{n+1}^* = a_n\tilde{Q}_n^* - b_{n-1}^2\tilde{Q}_{n-1}^* \tag{A4}$$

for $2\leqslant n\leqslant N$, with initial conditions

$$\tilde{Q}_1^*=1, \quad \tilde{Q}_2^*=a_1. \tag{A5}$$

Comparing Eqs. (29) and (32) with Eq. (A1), we observe that $Q_N^*=\Delta t\tilde{Q}_N^*$, and due to the linearity of Eq. (A4) we can conclude that $Q_n^*:=\Delta t\tilde{Q}_n^*$ also obeys these equations. Therefore, using the definitions (A2) and (A3), it is readily shown that Q_n^* satisfies the required recursion relations (30) together with the initial conditions (31).

As a by-product, needed in Appendix B, we notice that by defining for $1\leqslant n<N$

$$\mu_n := \frac{\tilde{Q}_{n+1}^*}{\tilde{Q}_n^*} = \frac{Q_{n+1}^*}{Q_n^*}, \tag{A6}$$

the linear two-step recursion relation (A4) can be rewritten as an equivalent nonlinear one-step recursion

$$\mu_{n+1} = a_{n+1} - \frac{b_k^2}{\mu_n}, \quad \mu_1 = a_1. \tag{A7}$$

The actual quantity of interest $\tilde{Q}_N^*$ then follows as

$$\tilde{Q}_N^* = \prod_{n=1}^{N-1}\mu_n. \tag{A8}$$

APPENDIX B

In the following we derive Eq. (50) by showing that

$$\frac{\partial^2\phi_k(x_f,t_f)}{\partial x_f^2} = \frac{\dot{Q}_k^*(t_f)}{2Q_k^*(t_f)} - F'(x_f,t_f), \tag{B1}$$

which gives together with Eqs. (39) and (48) for $t=t_f$ the desired result.

We work with the time-discrete version of the quantities in Eq. (B1) and consider in a first step the dependency of a minimizing path $\mathbf{x}^*=\mathbf{x}^*(x_f)$ on the end point x_f for fixed t_0, t_f, and x_0. In order to complicate the notation as little as possible, we have left out the index k labeling the different paths $\mathbf{x}_k^*$.

Since the initial point x_0 is kept fixed, we have that $dx_0^*/dx_f=0$. Further, we know from Eq. (25) that for all $n=1,\ldots,N-1$

$$\left.\frac{\partial S_N}{\partial x_n}\right|_{\mathbf{x}^*(x_f)} = 0, \tag{B2}$$

for any x_f value, which implies, after taking the derivative with respect to x_f, that

$$\sum_{m=1}^{N-1}\frac{dx_m^*}{dx_f}\left.\frac{\partial^2 S_N}{\partial x_m\partial x_n}\right|_{\mathbf{x}^*(x_f)} = 0. \tag{B3}$$

Using Eq. (A1), we thus get

$$a_n\frac{dx_n^*}{dx_f} - b_{n-1}\frac{dx_{n-1}^*}{dx_f} - b_n\frac{dx_{n+1}^*}{dx_f} = 0. \tag{B4}$$

Introducing the new quantities η_n by

$$\eta_n := b_n\frac{dx_{n+1}^*}{dx_n^*}, \tag{B5}$$

one obtains from Eq. (B4) the one-step recursion relation

$$\eta_{n+1}=a_{n+1}-\frac{b_n^2}{\eta_n}. \tag{B6}$$

The corresponding initial condition follows from Eq. (B4) for $n=1$ by taking into account the above-mentioned fact that $dx_0^*/dx_f=0$:

$$\eta_1=a_1. \tag{B7}$$

Comparing Eqs. (A7) with (B6) and (B7) yields $\eta_n=\mu_n$ for $n=1,\ldots,N-1$ and therefore, using the definitions (A3), (A6), and (B5), for $n=N-1$

$$\frac{Q_N^*}{Q_{N-1}^*}=[1+F'(x_{N-1}^*,t_{N-1})]\frac{dx_N^*}{dx_{N-1}^*}. \tag{B8}$$

In a next step an explicit expression for dx_N^*/dx_{N-1}^* in terms of well-known quantities has to be found. This can be achieved by taking the second derivative of the discrete-time action $S_N(\mathbf{x}^*(x_f))$ of the same minimizing path $\mathbf{x}^*$ as above with respect to the end point x_f. With Eqs. (B2) and (B3) and the boundary conditions (26) we find

$$\frac{d^2S_N(\mathbf{x}^*(x_f))}{dx_f^2}=\sum_{n=1}^{N}\frac{dx_n^*}{dx_N^*}\frac{\partial^2 S_N}{\partial x_n\partial x_N}\bigg|_{\mathbf{x}^*(x_f)}, \tag{B9}$$

and using Eq. (24) we can conclude that

$$\frac{dx_N^*}{dx_{N-1}^*}=\frac{1+\Delta tF'(x_{N-1}^*,t_{N-1})}{1-2\Delta t\dfrac{d^2}{dx_f^2}S_N(\mathbf{x}^*(x_f))}. \tag{B10}$$

Together with Eq. (B8) we thus arrive at

$$\frac{Q_N^*}{Q_{N-1}^*}=\frac{[1+\Delta tF'(x_{N-1}^*,t_{N-1})]^2}{1-2\Delta t\dfrac{d^2}{dx_f^2}S_N(\mathbf{x}^*(x_f))}, \tag{B11}$$

which with Eq. (43) yields in the continuous-time limit the searched for relation (B1).

APPENDIX C

In this appendix we derive Eq. (107) for $t\in[t_u,t_f]$, where the order of magnitude refers to the asymptotics with respect to k. As discussed below Eq. (83), the differences

$$\delta x_k^*(t):=x_k^*(t)-x_{\rm opt}^*(t+k\mathcal{T}), \tag{C1}$$

$$\delta p_k^*(t):=p_k^*(t)-p_{\rm opt}^*(t+k\mathcal{T}) \tag{C2}$$

rapidly decrease with increasing index k uniformly on the entire time interval $[t_0,t_f]$. Our first conclusion is that the time point t_u at which $x_u^*(t)$ enters the neighborhood of $x_u(t)$ depends itself on the index k; basically it decreases like $-k\mathcal{T}$, see Fig. 3. On the other hand, the distance $\Delta x_k^*(t_u)$ at which the path enters this neighborhood is, by definition, k independent. The corresponding momentum $p_k^*(t_u)$ is not strictly k independent, but approaches an asymptotic k independence for large k. Similar conclusions apply for the time t_s at which $x_u^*(t)$ leaves the neighborhood of the stable periodic orbit.

Next we can conclude from Eqs. (71) and (76) that

$$\delta x_k^*(t)=\delta p_k^*(t)I_s(t)-p_k^*(t)e^{2\Lambda_s(t,t_0)}I_s(t_0). \tag{C3}$$

Within the approximation (85) it follows that $\delta p_k^*(t_s)=\delta x_k^*(t_s)/I_s(t_s)$. With these initial conditions at $t=t_s$, the small perturbations $\delta x_k^*(t)$ and $\delta p_k^*(t)$ are then propagated according to Eqs. (41) and (42) until $t=t_u$. In linear order of these small perturbations it follows that $\delta p_k^*(t_u)/\delta x_k^*(t_u)$ is an asymptotically k-independent constant, which, however, depends on all the details of the force field $F(x,t)$ along the crossover segment of the master path $x_{\rm opt}^*(t)$.

The counterpart of Eq. (C3) in the neighborhood of $x_u(t)$ follows along the same line of reasoning, reading

$$\delta x_k^*(t)=-\delta p_k^*(t)I_u(t)+p_k^*(t)e^{-2\Lambda_s(t_f,t)}I_u(t_f). \tag{C4}$$

Replacing on the right-hand side $e^{-2\Lambda_s(t_f,t)}$ by $p_{\rm opt}^*(t_f+k\mathcal{T})^2/p_{\rm opt}^*(t+k\mathcal{T})^2$ according to Eqs. (61) and (65), choosing $t=t_u$, and making use of Eqs. (86) and (C2), we can infer that

$$\delta x_k^*(t_u)+\delta p_k^*(t_u)I_u(t_u)=\frac{p_k^*(t_u)I_u(t_f)}{[p_k^*(t_u)-\delta p_k^*(t_u)]^2}p_{\rm opt}^*(t_k)^2. \tag{C5}$$

As we have just pointed out, the quantity $\delta x_k^*(t_u)$ is proportional to $\delta p_k^*(t_u)$ with an asymptotically k-independent proportionality constant that depends on the details of the force field $F(x,t)$ along the crossover segment of the master path $x_{\rm opt}^*(t)$. In the generic case, this proportionality constant is thus not expected to coincide with $-I_u(t_u)$ since the latter depends on the behavior of $F(x,t)$ along the unstable periodic orbit $x_u(t)$ only. Consequently, both $\delta x_k^*(t_u)$ and $\delta p_k^*(t_u)$ on the left-hand side of Eq. (C5) are, with respect to their k dependence, of the same order of magnitude as the right-hand side. Since $p_k^*(t_u)$ is asymptotically k independent and $\delta p_k^*(t_u)$ tends to zero, we can infer from Eq. (C5) that

$$\delta x_k^*(t_u)=\mathcal{O}(p_{\rm opt}^*(t_k)^2),\quad \delta p_k^*(t_u)=O(p_{\rm opt}^*(t_k)^2). \tag{C6}$$

With the initial conditions (99) and (100) it follows from Eq. (46) that the relative difference between $Q_k^*(t_u)$ and $Q_{\rm opt}^*(t_u+k\mathcal{T})$ scales as a function of k like $\delta x_k^*(t_u)$ and $\delta p_k^*(t_u)$. With Eq. (C6) this implies that

$$Q_k^*(t_u)=Q_{\rm opt}^*(t_u+k\mathcal{T})[1+\mathcal{O}(p_{\rm opt}^*(t_k)^2)]. \tag{C7}$$

A similar relation follows for $\dot{Q}_k^*(t_u)$ and thus for $g_k^*(t_u)$ [cf. Eq. (48)], namely,

$$g_k^*(t_u)=g_{\rm opt}^*(t_u+k\mathcal{T})[1+\mathcal{O}(p_{\rm opt}^*(t_k)^2)]. \tag{C8}$$

Finally we conclude from Eq. (101) that

$$\frac{Q_k^*(t)}{Q_{\rm opt}^*(t+k\mathcal{T})}=\frac{Q_k^*(t_u)}{Q_{\rm opt}^*(t_u+k\mathcal{T})}\,\frac{1-g_k^*(t_u)I_u(t_u,t)}{1-g_{\rm opt}^*(t_u+k\mathcal{T})I_u(t_u,t)}. \tag{C9}$$

Like for $\Delta x_k^*(t_u)$ and $p_k^*(t_u)$ [see below Eq. (C2)] one can convince oneself that also $g_k^*(t_u)$ is asymptotically k independent. With Eqs. (C7) and (C8) the result (107) then follows from Eq. (C9).

APPENDIX D

The purpose of this appendix is to verify that our expression (116) for the time-averaged rate is invariant with respect to the supersymmetry transformation (16). To this end, we first note that the path defined via

$$\tilde{x}_{\rm opt}^*(t):=-x_{\rm opt}^*(-t), \tag{D1}$$

$$\tilde{p}_{\rm opt}^*(t):=p_{\rm opt}^*(-t) \tag{D2}$$

satisfies the Hamilton equations (41) and (42) for the supersymmetric partner field $\tilde{F}(x,t)$ from Eq. (16). Since the periodic orbits of this new force field are given by $\tilde{x}_s(t)=-x_u(-t)$ and $\tilde{x}_u(t)=-x_s(-t)$ (see Sec. II C) one can readily see that $\tilde{x}_{\rm opt}^*(t)$ from Eq. (D1) also obeys the boundary conditions (53) in the relevant limit $t_f-t_0\to\infty$. Hence we have found (up to the usual degeneracy with respect to time shifts by arbitrary multiples of $\mathcal{T}$) the unique solution of the supersymmetric partner variational problem (54). Inserting $\tilde{p}_{\rm opt}^*(t)$ from Eq. (D2) into the definitions (43) and (54) then leads to the following result:

$$\tilde{\phi}_{\rm opt}=\phi_{\rm opt}. \tag{D3}$$

Somewhat more elaborate considerations are necessary in order to establish a corresponding identity for the prefactor $\alpha_{\rm opt}$ in Eq. (116). To this end we first consider two arbitrary but linear independent solutions $Q_i(t)$ $(i=1,2)$ of the prefactor equation (46) for $Q_{\rm opt}^*(t)$. One can then easily verify that the prefactor $Q_{\rm opt}^*(t)$ which, moreover, has to fulfill the initial conditions (47) in the limit $t_0\to-\infty$, is given by

$$Q_{\rm opt}^*(t)=\lim_{t_0\to\infty}\frac{Q_1(t)Q_2(t_0)-Q_1(t_0)Q_2(t)}{W(t_0)}, \tag{D4}$$

with the Wronskian

$$W(t):=\dot{Q}_1(t)Q_2(t)-Q_1(t)\dot{Q}_2(t). \tag{D5}$$

Due to Eq. (46) one can infer that

$$\dot{W}(t)=2W(t)F'(x_{\rm opt}^*(t),t). \tag{D6}$$

With help of the Hamilton equation (41) it follows that

$$p_{\rm opt}^*(t)^2W(t)=\text{const}. \tag{D7}$$

Turning now to the supersymmetric partner problem, it is readily seen that one obtains via $\tilde{Q}_i(t):=Q_i(-t)$ two linear independent solutions of the prefactor equation (46) for the supersymmetric partner field (16) and the path given by Eq. (D1). Thus we can use Eqs. (D2) together with (D4) and (D5) (with tildes) to establish the identity

$$\lim_{t\to\infty}\tilde{p}_{\rm opt}^*(t)^2\tilde{Q}_{\rm opt}^*(t)=\lim_{\substack{t_0\to-\infty\\ t\to\infty}}\frac{p_{\rm opt}^*(-t)^2}{-W(-t_0)}\times[Q_1(-t)Q_2(-t_0)-Q_1(-t_0)Q_2(-t)]. \tag{D8}$$

According to Eq. (D7) we can now rewrite $p_{\rm opt}^*(-t)^2/W(-t_0)$ as $p_{\rm opt}^*(-t_0)^2/W(-t)$. Replacing $t\to-t_0$ and vice versa one can then conclude with help of Eq. (D4) that

$$\lim_{t\to\infty}\tilde{p}_{\rm opt}^*(t)^2\tilde{Q}_{\rm opt}^*(t)=\lim_{t\to\infty}p_{\rm opt}^*(t)^2Q_{\rm opt}^*(t). \tag{D9}$$

Hence we finally obtain

$$\alpha_{\rm opt}=\tilde{\alpha}_{\rm opt}. \tag{D10}$$

This, in combination with Eq. (D3), proves our proposition that the time-averaged rate (116) is invariant with respect to the supersymmetric transformation (16).

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