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The sharp interface limit for the stochastic Cahn–Hilliard equation

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Abstract. We study the ε-dependent two and three dimensional stochastic Cahn–Hilliard equation in the sharp interface limit $\varepsilon \to 0$. The parameter ε is positive and measures the width of transition layers generated during phase separation. We also couple the noise strength to this parameter. Using formal asymptotic expansions, we identify the limit. In the right scaling, our results indicate that the stochastic Cahn–Hilliard equation converge to a Hele-Shaw problem with stochastic forcing on the curvature equation. In the case when the noise is sufficiently small, we rigorously prove that the limit is a deterministic Hele-Shaw problem. Finally, we discuss which estimates are necessary in order to extend the rigorous result to larger noise strength.

Résumé. Nous étudions l'équation de Cahn–Hilliard stochastique dépendante en ε, posée en dimensions deux et trois, dans la limite de l'interface nette $\varepsilon \to 0$. Le paramètre ε est positif et mesure la largeur de couches de transition générées pendant la séparation de phase. Nous couplons aussi la puissance de bruit à ce paramètre. Nous déterminons la limite à l'aide de séries asymptotiques formelles. Dans l'échelle appropriée, nos résultats indiquent que l' équation de Cahn–Hilliard stochastique converge vers un problème de Hele-Shaw avec un forçage stochastique dans l'équation de la courbure. Dans le cas d'un bruit suffisamment petit, nous prouvons rigoureusement que la limite est un problème Hele-Shaw déterministe. Finalement, nous discutons des estimations nécessaires afin d'étendre le résultat rigoureux en présence de bruit d'une intensité plus grande.

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Keywords: Multi-dimensional stochastic Cahn–Hilliard equation; Additive Noise; stochastic sharp interface limit; Hele Shaw problem; Interface motion

1. Introduction

In this paper we consider the sharp interface limit of the stochastic Cahn–Hilliard equation

$$\partial_t u = \Delta\big(-\varepsilon\Delta u + \varepsilon^{-1} f'(u)\big) + \varepsilon^{\sigma}\dot{\mathcal{W}}(x,t), \tag{1.1}$$

for times $t \in [0,T]$ subject to Neumann boundary conditions on a bounded domain in $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{2,3\}$

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial\mathcal{D}. \tag{1.2}$$

Here $u : \mathcal{D} \times [0,T] \to \mathbb{R}$ is the scalar concentration field of one of the components in a separation process, for example of binary alloys.

We always assume that the spatial domain $\mathcal{D}$ has a sufficiently piece-wise smooth boundary. The typical example for the bistable nonlinearity is $f'(u) := \partial_u f(u) = u(u^2 - 1)$, where the primitive $f := \frac{1}{4}(u^2 - 1)^2$ is a double well potential with equally deep wells taking its global minimum value 0 at the points $u = \pm 1$. The small parameter ε measures an atomistic interaction length that fixes the length-scale of transition layers between 1 and -1. Obviously, the solution u depends on ε, which we usually suppress in the notation, but wherever needed we denote the solution u by u_ε.

The forcing $\dot{\mathcal{W}}$ denotes a Gaussian noise, which is white in time and possibly colored in space. Finally, the noise-strength ε^σ scales with the atomistic length scale, where σ is some parameter.

In applications, there are often different sources of the noise. If we split the noise $\varepsilon^\sigma \dot{\mathcal{W}} = \Delta \dot{\mathcal{V}} + \dot{W}$, then the term $\dot{\mathcal{V}}$ arises from fluctuations in the chemical potential, while $\dot{W}$ models fluctuations directly in the concentration. Here, $\dot{\mathcal{V}}$ needs to be a spatially smooth space–time noise since it is under the Laplacian, while the additive noise $\dot{W}$ might be rougher.

The stochastic Cahn–Hilliard equation is a model for the non-equilibrium dynamics of metastable states in phase transitions, [25,40,42]. The deterministic Cahn–Hilliard equation with $\dot{\mathcal{W}} = 0$ has been extended to a stochastic version by Cook, [25] (see also in [42]), incorporating thermal fluctuations in the form of an additive noise. Such a generalized Cahn–Hilliard model, [35], is based on the balance law for micro-forces. In this case, the additive term $\dot{\mathcal{V}}$ in the chemical potential is given by fluctuations in an external field. See [35,40]; cf. also [41], where the external gravity field is modeled. The noise $\dot{W}$, which is independent of the free energy, stands for the Gaussian noise in Model B of [40], in accordance with the original Cahn–Hilliard–Cook model.

We can model the noise either as the formal derivative of a Wiener process in the sense of Da Prato and Zabczyck [27] or as a derivative of a Brownian sheet in the sense of Walsh, [46]. Dalang and Quer-Sardanyons [28] showed that both approaches are actually equivalent. Thus, we focus on the Q-Wiener-process in the sense of [27], which may be defined by a series in the right orthonormal basis with independent coefficients given by a sequence of real valued Brownian motions.

Existence of stochastic solution for the problem (1.1) has been established under various assumptions on the noise-terms, see for example in [26,29]; we also refer to the results presented in [9,18,21,22].

1.1. *Phase transitions and noise*

Concerning the Cahn–Hilliard equation posed in one dimension, in [8] the authors analyzed the stochastic dynamics of the front motion of one dimensional interfaces. In the absence of noise we refer also to [14,15] for the dynamics of interfaces and the construction of a finite dimensional manifold parametrized by the interface positions. This is a key tool for studying the stochastic case, but it fails in dimension two and three, as interfaces are no longer points, but curves or surfaces respectively.

An interesting result is [17], where, on the unbounded domain, a single interface moves according to a fractional Brownian motion, which is in contrast to the usual Brownian motion in most of the other examples. Note that the one-dimensional case is significantly simpler, since the solutions can be fully parametrized by their finitely many zeros. Therefore, one needs only to consider the motion of solutions along a finite dimensional slow manifold, and the stability properties along such a manifold; see for example [13]. Here we try to follow similar ideas, despite the fact that the driving manifold is infinite dimensional, and parametrized by the curves or surfaces of the zero level set.

A slightly simpler model, due to the absence of mass-conservation, is the stochastic Allen Cahn-equation, the so called Model A of [40]. The interface motion of stochastic systems of Allen–Cahn type have been analyzed in [30]. In [19,33], the authors studied the stochastic one-dimensional Allen–Cahn equation with initial data close to one instanton or interface and proved that, under an appropriate scaling, the solution will stay close to the instanton shape, while the random perturbation will create a dynamic motion for this single interface. This is observed on a much faster time scale than in the deterministic case. This result has been also studied in [48] via an invariant measure approach.

If the initial data involves more than one interfaces, it is believed that these interfaces exhibit also a random movement, which is much quicker than in the deterministic case, while different interfaces should annihilate when they meet, [31]. We also refer to [45] or [36]. The limiting process should be related to a Brownian one (cf. in [32] for formal arguments). A full description of all the ideas for the analysis of the interface motion based on [24] and [8] is presented in [49]. In [44] the authors considered the stochastic Allen–Cahn equation driven by a multiplicative noise; they prove tightness of solutions for the sharp interface limit problem, and show convergence to phase-indicator functions; cf. also in [48] for the one-dimensional case with an additive space–time white noise for the proof of an exponential convergence towards a curve of minimizers of the energy.

The space–time white noise driven Allen–Cahn equation is known to be ill-posed in space dimensions greater than one, [27,46], and a renormalization is necessary to properly define the solutions. We refer to [37,39] for more details.

For a multi-dimensional stochastic Allen–Cahn equation driven by a mollified noise, in [38], it is shown that as the mollifier is removed, the solutions converge weakly to zero, independently of the initial condition. If the the noise strength converges to zero at a sufficiently fast rate, then the solutions converge to those of the deterministic equation, and the behavior is well described by the Freidlin–Wentzell theory. See also [16]. In [47], for a regularized noise, which is white in the limit, the author extending the classical result of Funaki [34] to spatial dimensions more than two, derived motion by mean curvature with an additional stochastic forcing for the sharp interface limit problem. Recently, in [1], for the case of an additive "mild" noise in the sense of [34,47], the first rigorous result on the generation of interface for the stochastic Allen–Cahn equation has been derived; the authors proved layer formation for general initial data, and established that the solution's profile near the interface remains close to that of a (squeezed) traveling wave, which means that a spatially uniform noise does not destroy this profile.

1.2. *Main results*

Due to phase separation, the solution u of the stochastic Cahn–Hilliard equation (1.1), which is related to the mass concentration, tends to split in regions where $u \approx \pm 1$ and with inner layers of order $\mathcal{O}(\varepsilon)$ between them. We shall study the motion of such layers in their sharp interface limit $\varepsilon \to 0$. *The rigorous complete description of the motion of interfaces in dimensions two and three stands for many years as a wide open question*. With this paper, we tried to contribute towards a full answer by providing first some formal asymptotics for the case of general noise strength. We were able to identify the intermediate regime $\sigma = 1$ when in the limit $\varepsilon \to 0$, the limiting solutions satisfy a stochastic Hele-Shaw problem (see (2.5)). While for smaller noise strength $\sigma > 1$ we conjecture that the limit solves a deterministic Hele-Shaw problem.

In addition, as a first rigorous step, we derive the sharp interface limit given by the deterministic Hele-Shaw problem in the case when the noise is sufficiently small (see Theorem 3.10). Unfortunately, we are not able to treat the case σ close to 1, we refer to Remark 3.11. A key problem is that spectral problems of the linearized operators are not yet understood in the general setting which is necessary for the proof (see Remark 3.13).

The result for the Cahn–Hilliard equation in the deterministic case (when $\dot{\mathcal{W}} = 0$) has been already studied in [2]. Given a solution of the deterministic Hele-Shaw problem, the authors constructed an approximation of (1.1) without noise admitting this solution, as the interface moves between ± 1. The analysis thereof will be the foundation of some of our results. The main technical problem of this strategy, is that the manifold of possible approximations is parametrized by an infinite dimensional space of closed curves or surfaces. Furthermore, the spectral information provided so far for the linearized problem, necessary for a qualitative study of the approximation is insufficient. This is due mainly to the fact that most of the larger eigenvalues actually are related to the fast motion of the interface itself, which do not obstruct the stability of the manifold. Thus, this approximation can only be valid on time-scales of order $\mathcal{O}(1)$. We refer to the main result in Theorem 3.10, where we approximate solutions of the stochastic Cahn–Hilliard equation by solutions of the deterministic Hele-Shaw problem only for very small noise and only on $\mathcal{O}(1)$-time-scales.

A simpler case is the motion of droplets for the two-dimensional Cahn–Hilliard or the mass conserving Allen–Cahn equation. Here, the solutions can be fully parametrized by finite dimensional data, namely the position and radius of the droplets. See [3,12], and [7] (stochastic problem), for droplets on the boundary, and [5,6] for droplets in the domain in the absence of noise. The approximation in these cases is valid for very long time-scales.

1.3. *Outline of the paper*

In Section 2, we present the formal derivation of the stochastic Hele-Shaw problem from (1.1) and identify the order of the noise strength that leads to a nontrivial stochastic modification of the deterministic limiting problem.

In Section 3, we provide a rigorous definition of the setting and state the main results, which we then prove in Section 4. We concentrate on small noise strength and show in that case, that solutions of (1.1) in the sharp interface limit of $\varepsilon \to 0$ are well approximated by a Hele-Shaw problem. We will see that the main limitation towards a better approximation result is the lack of good bounds on the linearized operator.

2. Formal asymptotics

In this section we present some formal matching asymptotics applied to (1.1) that establish a first intuition towards a rigorous proof for the stochastic sharp interface limit. We remark first that the stochastic C–H equation (1.1) can be written as a system, where v is the chemical potential.

$$\begin{cases} \partial_t u = -\Delta v + \varepsilon^{\sigma}\dot{\mathcal{W}}, \\ v = -\frac{f'(u)}{\varepsilon} + \varepsilon\Delta u. \end{cases} \tag{2.1}$$

Formally in [10], and later more rigorously in [11] using a Hilbert expansions method, the asymptotic behavior for $\varepsilon \to 0$ of the following deterministic system has been analyzed:

$$\begin{cases} \partial_t u_\varepsilon = -\Delta v_\varepsilon + G_1, \\ v_\varepsilon = -\frac{f'(u_\varepsilon)}{\varepsilon} + \varepsilon\Delta u_\varepsilon + G_2, \end{cases}$$

where now $G_1(x,t;\varepsilon)$ and $G_2(x,t;\varepsilon)$ are deterministic forcing terms. The sharp interface limit problem in the multidimensional case demonstrated a local influence in phase transitions of forcing terms that stem from the chemical potential, while free energy independent terms act on the rest of the domain. In addition, the forcing may slow down the equilibrium.

Given an initial smooth closed $n-1$ dimensional hypersurface Γ_0 in $\mathcal{D}$ (this definition covers also the union of closed interfaces) then the limiting chemical potential

$$v := \lim_{\varepsilon\to 0^+}\big(\varepsilon\Delta u_\varepsilon - \varepsilon^{-1}f'(u_\varepsilon) + G_2\big), \tag{2.2}$$

satisfies the following Hele-Shaw free boundary problem (assuming that the limits exist)

$$\begin{cases} \Delta v = \lim_{\varepsilon\to 0^+} G_1 \quad \text{in } \mathcal{D}\backslash\Gamma(t), t>0, \\ \partial_n v = 0 \quad \text{on } \partial\mathcal{D}, \\ v = \lambda H + \lim_{\varepsilon\to 0^+} G_2 \quad \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \quad \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{2.3}$$

where $\Gamma(t)$ is the zero level surface of the limiting $u(t)$, which is for fixed time t a closed $n-1$ dimensional hypersurface of mean curvature $H = H(t)$ and of velocity $V = V(t)$ that divides the domain $\mathcal{D}$ in two open sets $\mathcal{D}^+(t)$ and $\mathcal{D}^-(t)$. The constant λ is positive, and n is the unit outward normal vector at the inner and outer boundaries.

According to the aforementioned arguments, each perturbation G_i has a different physical meaning and appears in a different equation when C–H is presented as a system. We will use some of the ideas of the deterministic asymptotic analysis, but we will see in the following that when the terms G_i are noise terms and small in ε, they still have an impact on the limiting behavior.

2.1. *Formal derivation of the stochastic Hele-Shaw problem*

In order to observe the limit behavior of (1.1) at larger noise strengths, we fix $\sigma = 1$ as we expect the noise strength ε to be the critical one. In order to avoid calculations with formal noise terms, where the order of magnitude and the definition of products is not always obvious, we use the following change of variables

$$\hat{u}_\varepsilon := u_\varepsilon - \varepsilon\mathcal{W},$$

where we assume that the Wiener process is spatially sufficiently smooth. Taking differentials, it follows that $\hat{u}_\varepsilon$ and v_ε solve the system

$$\begin{cases} \partial_t \hat{u}_\varepsilon = -\Delta v_\varepsilon, \\ v_\varepsilon = -\frac{1}{\varepsilon}f'(\hat{u}_\varepsilon + \varepsilon\mathcal{W}) + \varepsilon\Delta\hat{u}_\varepsilon + \varepsilon^2\Delta\mathcal{W}. \end{cases} \tag{2.4}$$

Observe that for spatially smooth noise on time-scales of order 1

$$\hat{u}_\varepsilon = u_\varepsilon + \mathcal{O}(\varepsilon),$$

is an approximate solution of (1.1) for small ε. Furthermore, the main advantage of the above representation as a system is based on the fact that (2.4) is now a random PDE without stochastic differentials and all terms appearing are spatially smooth and in time at least Hölder-continuous. Thus, we can treat all appearing quantities as functions and analyze the equation path-wisely, i.e., for every fixed realization of the underlying Wiener process $\mathcal{W}$.

Therefore, we are able to follow the ideas of the formal derivation presented in [10] and derive in the limit the following stochastic Hele-Shaw problem

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathcal{D}\backslash\Gamma(t), t > 0, \\ \partial_n v = 0 \quad \text{on } \partial\mathcal{D}, \\ v = \lambda H + \mathcal{W} \quad \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \quad \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{2.5}$$

where again H and V are the mean curvature and velocity respectively of the zero level surface $\Gamma(t)$. For positive $\varepsilon > 0$ the domain $\mathcal{D}$ admits the following disjoint decomposition

$$\mathcal{D} = \mathcal{D}_\varepsilon^+(t) \cup \mathcal{D}_\varepsilon^-(t) \cup \mathcal{D}_\varepsilon^I(t),$$

where

$$u_\varepsilon \approx 1 \quad \text{for } x \in \mathcal{D}_\varepsilon^+(t) \quad \text{and} \quad u_\varepsilon \approx -1 \quad \text{for } x \in \mathcal{D}_\varepsilon^-(t).$$

Moreover, $\mathcal{D}_\varepsilon^I(t)$ is a narrow interfacial region around $\Gamma(t)$ with thickness of order ε where u_ε is neither close to $+1$ nor -1.

In particular, we construct an inner solution close to the interface, and an outer solution away from it. Using the appropriate matching in orders of ε, we formally pass to the limit and derive the corresponding free boundary problem. To avoid additional technicalities, we also assume that the interface Γ does not intersect the boundary. In terms of simplicity of notation, we drop the subscript ε in all the calculations that follow.

2.2. *Outer expansion*

We consider that the inner interface is known, and seek the outer expansion far from it, i.e., an expansion in the form

$$\hat{u} = \hat{u}_0 + \varepsilon \hat{u}_1 + \cdots,$$
$$v = v_0 + \varepsilon v_1 + \cdots,$$

where "$+\cdots$" denote higher order terms and $u_0, u_1, \ldots, v_0, v_1, \ldots$ are smooth functions. We insert the outer expansion into the second equation of the stochastic system $(2.4)_2$ and obtain

$$\begin{aligned} v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2) = &-\frac{1}{\varepsilon}\big(f'(\hat{u}_0) + \varepsilon f''(\hat{u}_0)(\hat{u}_1 + \mathcal{W}) + \mathcal{O}(\varepsilon^2)\big) \\ &+ \varepsilon\Delta\big(\hat{u}_0 + \varepsilon\hat{u}_1 + \mathcal{O}(\varepsilon^2)\big) + \varepsilon^2\Delta\mathcal{W} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.6}$$

First collecting the terms of order $\mathcal{O}(\frac{1}{\varepsilon})$ in (2.6), we arrive at

$$f'(\hat{u}_0) = 0.$$

Thus, we get as in Remark 4.1, (1) of [2]

$$\hat{u}_0 = \pm 1.$$

In the second step, we collect the $\mathcal{O}(1)$-terms in (2.6) and derive

$$v_0 = -f''(\hat{u}_0)(\hat{u}_1 + \mathcal{W}) + \mathcal{O}(\varepsilon^2).$$

We plug now the outer expansion into the first equation of $(2.4)_1$ and obtain

$$\partial_t\big(\hat{u}_0 + \varepsilon\hat{u}_1 + \mathcal{O}(\varepsilon^2)\big) = -\Delta\big(v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2)\big).$$

As $\hat{u}_0$ is a constant in the outer expansion we have $\partial_t\hat{u}_0 = 0$, and thus, collecting the $\mathcal{O}(1)$ terms yields

$$-\Delta v_0 = 0.$$

Collecting finally in the third step all $\mathcal{O}(\varepsilon)$-terms we arrive at

$$\partial_t\hat{u}_1 = -\Delta v_1.$$

2.3. *Inner expansion*

Let x be a point in $\mathcal{D}$ that at time t is near the interface $\Gamma(t)$. Let us introduce the stretched normal distance to the interface, $z := \frac{d}{\varepsilon}$, where $d(x,t)$ is the signed distance from the point x in $\mathcal{D}$ to the interface $\Gamma(t)$, such that $d(x,t) > 0$ in $\mathcal{D}_\varepsilon^+$ and $d(x,t) < 0$ in $\mathcal{D}_\varepsilon^-$. Obviously Γ has the representation

$$\Gamma(t) = \big\{x \in \mathcal{D} : d(x,t) = 0\big\}.$$

If Γ is smooth, then d is well defined and smooth near Γ, and $|\nabla d| = 1$ in a neighborhood of Γ. Following [2] and [43], we seek for an inner expansion valid for x near Γ of the form

$$\hat{u} = q\left(\frac{d(x,t)}{\varepsilon}, x, t\right) + \varepsilon Q\left(\frac{d(x,t)}{\varepsilon}, x, t\right) + \cdots,$$
$$v = \tilde{q}\left(\frac{d(x,t)}{\varepsilon}, x, t\right) + \varepsilon \tilde{Q}\left(\frac{d(x,t)}{\varepsilon}, x, t\right) + \cdots,$$

where again "$+\cdots$" denote higher order terms that we neglect and $q, Q, \ldots, \tilde{q}, \tilde{Q}, \ldots$ are sufficiently smooth. It will be convenient to require that the quantities depending on z, x, t are defined for x in a full neighborhood of Γ but do not change when x varies normal to Γ with z held fixed, [43]. We insert the inner expansion into $(2.4)_2$, utilize that $|\nabla d|^2 = 1$, and thus obtain the following expression

$$\tilde{q} + \varepsilon\tilde{Q} + \mathcal{O}(\varepsilon^2) = -\frac{1}{\varepsilon}\big(f'(q) + \varepsilon f''(q)(Q + \mathcal{W})\big) + \varepsilon\left(\frac{\partial_z q}{\varepsilon}\Delta d + \frac{\partial_{zz} q}{\varepsilon^2} + \partial_z Q\Delta d + \frac{\partial_{zz} Q}{\varepsilon}\right) + \varepsilon^2\Delta W. \tag{2.7}$$

We collect the terms of order $\mathcal{O}(\frac{1}{\varepsilon})$ and derive

$$\partial_{zz} q - f'(q) = 0.$$

By matching now the terms of order $\mathcal{O}(1)$ in (2.7), we obtain

$$\tilde{q} = -f''(q)Q + \partial_{zz} Q - f''(q)\mathcal{W} + \partial_z q\Delta d,$$

or equivalently

$$\tilde{q} - \partial_z q\Delta d = \partial_{zz} Q - f''(q)Q - f''(q)\mathcal{W}. \tag{2.8}$$

We define the linearized Allen–Cahn operator

$$\mathcal{L}Q = \partial_{zz} Q - f''(q)Q.$$

Then (2.8) is written as

$$\tilde{q} - \partial_z q \Delta d = \mathcal{L}Q - f''(q)\mathcal{W}. \tag{2.9}$$

This equation is solvable if for any $\chi \in \mathrm{Ker}(\mathcal{L}^*)$ it holds that $\chi \perp (\tilde{q} - \partial_z q \Delta d + f''(q)\mathcal{W})$, or equivalently if

$$\int_{-\infty}^{\infty} \chi \cdot \big(\tilde{q} - \partial_z q \Delta d + f''(q)\mathcal{W}\big)\, dz = 0. \tag{2.10}$$

Obviously, for any x on Γ it holds that $d(x,t) = 0$ and $\Delta d(x,t) = H(x,t)$. Replacing in (2.10) we obtain the following sufficient condition on the interface Γ:

$$\tilde{q} = \lambda H - f''(q)\mathcal{W}. \tag{2.11}$$

Plugging the inner expansion into $(2.4)_1$ we obtain

$$\frac{\partial_z q}{\varepsilon} d_t + \partial_z Q d_t + \mathcal{O}(\varepsilon) = -\bigg(\frac{\partial_z \tilde{q}}{\varepsilon}\Delta d + \frac{\partial_{zz}\tilde{q}}{\varepsilon^2} + \partial_z \tilde{Q}\Delta d + \frac{\partial_{zz}\tilde{Q}}{\varepsilon}\bigg). \tag{2.12}$$

We collect the terms of order $\mathcal{O}(\varepsilon^{-2})$ and arrive at

$$\partial_{zz}\tilde{q} = 0,$$

which implies that for some functions a and b

$$\tilde{q} = a(x,t)z + b(x,t).$$

To proceed further, the matching condition for the inner and outer expansions must be developed. In general, these are obtained by the following procedure (see [20]). Fixing $x \in \Gamma$, we seek to match the expansions by requiring formally for $z \to \infty$

$$\tilde{q} + \varepsilon\tilde{Q} + \mathcal{O}\big(\varepsilon^2\big) = v_0 + \varepsilon v_1 + \mathcal{O}\big(\varepsilon^2\big),$$

and thus in order $\mathcal{O}(1)$

$$v_0 = \lim_{z\to\infty} \tilde{q} = \lim_{z\to\infty}\big(a(x,t)z + b(x,t)\big).$$

We obtain $a = 0$ and thus, $\tilde{q} = b$. Hence, utilizing (2.11) we have that on the interface

$$v_0 = \lambda H - f''(q)\mathcal{W} = \lambda H + \mathcal{W},$$

where we used that q solves the Euler–Lagrange equation

$$-q''(z) + f'\big(q(z)\big) = 0, \quad z \in \mathbb{R},$$
$$\lim_{z\to\pm\infty} q(z) = \pm 1, \quad q(0) = 0,$$

while on the inner interface with $z = d/\varepsilon = 0$ we have $f''(q) = 3q^2 - 1 = -1$ since $q(0) = 0$.

What is still missing is the evolution law, which should come from the inner expansion. From (2.12), we collect the terms of order $\mathcal{O}(1/\varepsilon)$ and obtain

$$\partial_z q d_t = -\partial_z \tilde{q}\Delta d - \partial_{zz}\tilde{Q}.$$

Recall that $-d_t = V$, while $\Delta d = H$ (see for example [2]) and integrate over z from $-\infty$ to ∞ to derive

$$-\int_{-\infty}^{\infty} \partial_z q V\, dz = -\int_{-\infty}^{\infty} \partial_{zz}\tilde{Q}\, dz.$$

From the matching conditions we get

$$q(+\infty) = 1 \quad \text{and} \quad q(-\infty) = -1.$$

Hence, we have

$$V = \frac{1}{2}\big[\partial_z \tilde{Q}(+\infty) - \partial_z \tilde{Q}(-\infty)\big].$$

Thus the stochastic Hele-Shaw problem (2.5) is established formally as the sharp interface limit, in the case $\sigma = 1$.

Remark 2.1. In the case $\sigma > 1$, we follow the same construction of inner and outer solutions as above and obtain in the limit, the deterministic Hele-Shaw problem ([2])

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{D}\backslash\Gamma(t), t > 0, \\ \partial_n v = 0 & \text{on } \partial\mathcal{D}, \\ v = \lambda H & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where H and V are the mean curvature and velocity respectively of the zero level surface $\Gamma(t)$ contained in the interfacial region $\mathcal{D}^I_\varepsilon(t)$.

Remark 2.2. Note that the change of variables

$$\hat{u}_\varepsilon := u_\varepsilon - \varepsilon^\sigma \mathcal{W},$$

implies that

$$\hat{u}_\varepsilon := u_\varepsilon + \mathcal{O}\big(\varepsilon^\sigma\big).$$

Thus, only if $\sigma \geq 1$, $\hat{u}$ is permitted to be expanded as in the presented inner expansion using q. The key difference to the deterministic analysis is that in the nonlinearity we have $\frac{1}{\varepsilon} f(\hat{u}_\varepsilon + \varepsilon^\sigma \mathcal{W})$. In case $\sigma > 1$ there is no contribution of $\mathcal{W}$ in an asymptotic expansion in terms of order $\mathcal{O}(1)$ and $\mathcal{O}(1/\varepsilon)$, while for $\sigma = 1$ there is an impact of $\mathcal{W}$ on terms of order $\mathcal{O}(1)$.

Remark 2.3. When $0 \leq \sigma < 1$ the strategy presented in this section fails. For this case, we might think of avoiding the change of variables and apply the formal asymptotics of [10] directly instead. If we split the noise term $\dot{\mathcal{W}}$ in (1.1), we obtain the following stochastic system with two noise sources (recall that $\varepsilon^\sigma \dot{\mathcal{W}} = \Delta \dot{V} + \dot{W}$)

$$\begin{cases} \partial_t u = -\Delta v + \dot{W}, \\ v = -\frac{f'(u)}{\varepsilon} + \varepsilon \Delta u + \dot{V}. \end{cases}$$

The sharp interface limit should coincide to (2.3), but for $\lim_{\varepsilon\to 0^+} G_1$, $\lim_{\varepsilon\to 0^+} G_2$ replaced by

$$\lim_{\varepsilon\to 0^+} \dot{W}(\cdot,\varepsilon), \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \dot{V}(\cdot,\varepsilon),$$

respectively. When $0 < \sigma < 1$ we would obtain that both these limits are 0 and thus the limiting problem is a deterministic Hele-Shaw problem without the contribution of the noise. But this is a very dangerous reasoning, as the noise terms $\dot{V}$ and $\dot{W}$, even if they would be ε-independent are not of order $\mathcal{O}(1)$, and we would still expect an impact of the noise terms on the limiting problem.

3. The sharp interface limit

The main result of this paper is that (1.1), as ε tends to zero, may have a deterministic or a stochastic profile depending on the strength of the additive noise in terms of ε. Only large noise perturbations with $\sigma = 1$ generate a stochastic limit problem. Here, we discuss the limit for smaller noise strength.

Let us first precisely state our problem. We assume that noise is induced by the formal derivative of a Q-Wiener process $\mathcal{W}$ in a Fourier series representation (see [27]); for simplicity, the only ε-dependence will appear in the noise strength, and thus, for the rest of this paper we shall use the notation $\varepsilon^\sigma d\mathcal{W}(x,t)$ for the additive noise, where $\sigma \in \mathbb{R}$ is a scaling parameter.

Assumption 3.1. Let $\mathcal{W}$ be a Q-Wiener process on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$\mathcal{W}(t) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k,$$

for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in $L^2(\mathcal{D})$, independent real-valued Brownian motions $(\beta_k)_{k\in\mathbb{N}}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, and real-valued coefficients α_k such that $Qe_k = \alpha_k^2 e_k$. Furthermore, we assume that the noise has some weak smoothness in space, i.e., Q satisfies

$$\operatorname{trace}\big(\Delta^{-1} Q\big) < \infty. \tag{3.1}$$

To deal with a mass-conserving stochastic problem, we impose the condition

$$\int_{\mathcal{D}} \mathcal{W}(t)\, dx = 0.$$

Note that (3.1) implies that the Wiener-process $\mathcal{W}(t)$ is $H^{-1}(\mathcal{D})$-valued. This is the minimal requirement for the approximation theorem presented in the sequel; we might need more regularity, in order to have the stochastic Hele-Shaw limit problem well defined, or while performing the formal asymptotics.

Recall (1.1) in Itô-formulation

$$du_\varepsilon = \Delta\big(-\varepsilon\Delta u_\varepsilon + \varepsilon^{-1} f'(u_\varepsilon)\big)\, dt + \varepsilon^\sigma d\mathcal{W}(x,t), \tag{3.2}$$

associated to Neumann conditions on the boundary of $\mathcal{D}$, so that the equation is still mass conservative.

The following theorem for the existence of solutions is well known. See for example [26].

Theorem 3.2. *Let $\mathcal{D}$ be a rectangle in dimensions $1, 2, 3$. If $Q = I$ or $\operatorname{trace}(\Delta^{-1+\delta} Q) < \infty$ for $\delta > 0$, then the following holds true*:

(1) *if u_0 is in $H^{-1}(\mathcal{D})$, there exists a unique solution for the problem (1.1) in $C([0,T]; H^{-1}(\mathcal{D}))$*,
(2) *if u_0 is in $L^2(\mathcal{D})$, then the solution for the problem (1.1) is in $L^\infty(0,T; L^2(\mathcal{D}))$.*

Note that the previous theorem could be extended for general Lipschitz domains in dimensions 2 and 3 under some additional assumptions of minimum eigenfunctions growth, cf. the arguments in [9]. For the analysis underlying our results and to avoid technicalities, we will for the remainder of the paper always assume:

Assumption 3.3. For any initial condition in $H^{-1}(\mathcal{D})$ there exists a unique solution for the problem (1.1) in $C([0,T]; H^{-1}(\mathcal{D}))$, which is sufficiently regular such that we can apply Itô-formula to the H^{-1}-norm.

The regularity required for this assumption is usually straightforward to verify. For example [26] applies the Itô-formula to a finite dimensional spectral Galerkin-approximation and then passes to the limit. Introducing the chemical potential v_ε, the equation is as in the formal derivation rewritten as a stochastic system. Indeed,

$$\text{for } T > 0 \quad \text{let } \mathcal{D}_T := \mathcal{D} \times (0,T),$$

then (3.2) is written as

$$\begin{aligned} du_\varepsilon &= -\Delta v_\varepsilon \, dt + \varepsilon^\sigma d\mathcal{W} \quad \text{in } \mathcal{D}_T, \\ v_\varepsilon &= -\frac{1}{\varepsilon} f'(u_\varepsilon) + \varepsilon \Delta u_\varepsilon \quad \text{in } \mathcal{D}_T, \end{aligned} \tag{3.3}$$

subject to Neumann boundary conditions

$$\frac{\partial u_\varepsilon}{\partial n} = \frac{\partial \Delta u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial\mathcal{D} \times (0, T).$$

Our main analytic theorem considers *a sufficiently small noise resulting to a deterministic sharp interface* limiting behavior. In particular, we analyze the case

$$\sigma \gg \sigma_0 = 1,$$

where σ_0 is the conjectured borderline case, where according to our formal calculation the noise has an impact on the limiting model. Under some assumptions on the initial condition $u_\varepsilon(0)$, the limit of u_ε and v_ε as $\varepsilon \to 0$ solves the deterministic Hele-Shaw problem on a given time interval $[0, T]$. We will state the precise formulation of this argument in Theorem 3.10, and then present the rigorous proof.

While our rigorous result can only treat very small noise, the formal derivation of Section 2.1 motivates the following conjecture implying a *stochastic* sharp interface limit:

Conjecture 3.4. *For $\sigma = 1$ the limit of u_ε and v_ε solves the stochastic Hele-Shaw problem*

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathcal{D}\backslash\Gamma(t), t > 0, \\ \partial_n v = 0 \quad \text{on } \partial\mathcal{D}, \\ v = \lambda H + \mathcal{W} \quad \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \quad \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{3.4}$$

Remark 3.5. In Section 2 by formal asymptotics, we only presented an indication for the correctness of the conjecture. A rigorous proof of this conjecture remains open at the moment. We hope to attack the problem to its full generality in the near future.

Remark 3.6. Note that $\mathcal{W}$ is a Wiener process, and the equation $v = \lambda H + \mathcal{W}$ on $\Gamma(t)$ appearing in (3.4), has a rigorous mathematical meaning in terms of functions. In fact, no noise is present, while a random equation appears on $\mathcal{D}\backslash\Gamma(t)$ in the following sense. For any given t, $\Gamma(t)$ is defined by its velocity V and thus is known, and the unknown function v on $\Gamma(t)$ is a stochastic process. Thus, the problem for fixed t is posed in between the inner boundary $\Gamma = \Gamma(t)$ and the outer boundary $\partial\mathcal{D}$ as follows

$$\Delta v = 0 \quad \text{in } \mathcal{D}\backslash\Gamma, \qquad \partial_n v = 0 \quad \text{on } \partial\mathcal{D}, \qquad v = \lambda H + \mathcal{W} \quad \text{on } \Gamma, \tag{3.5}$$

the inner boundary condition $v|_\Gamma$ being a realization of a t-dependent stochastic process.

3.1. *Statement of the main theorem*

In this section, we shall state the main analytic theorem of this paper, concerning the sharp interface limiting profile for sufficiently small noise strength. To approximate the stochastic solution we use the same approximations u_ε^A and v_ε^A as in [2] proposed in the absence of noise. For a precise definition see further below. In our proof we follow the ideas of the proof of their Theorem 2.1, and need to adapt the analysis to the presence of noise.

The main difference concerns the noise in the equation for the residual

$$R := u_\varepsilon - u_\varepsilon^A. \tag{3.6}$$

We show bounds for this error in our main Theorem 3.10 below, while in Remark 3.11 we comment on the smallness of the noise necessary for the main result. Unfortunately, we are quite far away from treating the case $\sigma > 1$ close to 1. Another problem is the weak estimate on the spectral stability of the linearized operator stated in Proposition 3.9. See also Remark 3.13.

Assumption 3.7. Let the family $\{\Gamma(t)\}_{t\in[0,T]}$ of smooth closed hypersurfaces together with the functions $\{v(t)\}_{t\in[0,T]}$ be a solution of the deterministic Hele-Shaw problem (i.e., equation (3.4) with $\mathcal{W} = 0$) such that the interfaces do not intersect with the boundary $\partial\mathcal{D}$, i.e., $\Gamma(t) \subset \mathcal{D}$ for all $t \in [0, T]$.

With Γ from Assumption 3.7 the authors in [2] construct a pair of approximate solutions $(u_\varepsilon^A, v_\varepsilon^A)$, so that $\Gamma(t)$ is the zero level set of $u_\varepsilon^A(t)$, which satisfies

$$\begin{aligned} du_\varepsilon^A &= -\Delta v_\varepsilon^A \, dt \quad \text{in } \mathcal{D}_T, \\ v_\varepsilon^A &= -\frac{1}{\varepsilon} f'\big(u_\varepsilon^A\big) + \varepsilon \Delta u_\varepsilon^A + r_\varepsilon^A \quad \text{in } \mathcal{D}_T, \end{aligned} \tag{3.7}$$

for boundary conditions

$$\frac{\partial u_\varepsilon^A}{\partial n} = \frac{\partial \Delta u_\varepsilon^A}{\partial n} = 0 \quad \text{on } \partial\mathcal{D}.$$

We recall that u_ε^A approximates the deterministic version of equation (1.1) (i.e., for $\mathcal{W} = 0$). The error term r_ε^A is bounded in terms of ε, and depending on the smoothness of Γ and the number of approximation steps, the bound on r_ε^A can be arbitrarily small. For details see relation (4.30) and Theorem 4.12 in [2].

We will summarize the results of [2] that we need for our proof in the following Theorem.

Theorem 3.8. *Under the Assumption 3.7, for any $K > 0$ there exists a pair $(u_\varepsilon^A, v_\varepsilon^A)$ of solutions to (3.7), such that*

$$\|r_\varepsilon^A\|_{C^0(\mathcal{D}_T)} \le C\varepsilon^{K-2}.$$

Moreover, it holds that

$$\|v_\varepsilon^A - v\|_{C^0(\mathcal{D}_T)} \le C\varepsilon,$$

and finally for x away from $\Gamma(t)$ (i.e., $d(x, \Gamma(t)) \ge c\varepsilon$)

$$\big|u_\varepsilon^A(t,x) - 1\big| \le C\varepsilon \quad \text{or} \quad \big|u_\varepsilon^A(t,x) + 1\big| \le C\varepsilon.$$

We present now the following spectral estimate, useful in our proof; we refer to [23] for dimensions larger than two, and to [4] for dimension two. Unfortunately, this estimate is also the key problem to extend the approximation result beyond time-scales of order 1.

Proposition 3.9 (Proposition 3.1 of [2]). *Let u_ε^A be the approximation given in Theorem 3.8. Then for all $w \in H^1(\mathcal{D})$ satisfying Neumann boundary conditions such that $\int_{\mathcal{D}} w\,dx = 0$, the following estimate is valid*

$$\int_{\mathcal{D}} \left[\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon} f''\big(u_\varepsilon^A\big) w^2\right] dx \ge -C_0 \|\nabla w\|_{L^2}^2.$$

Our main theorem provides bounds for the residual R:

Theorem 3.10 (Main theorem). *Let Assumption 3.1 for the noise and 3.3 for the existence of solutions be true.*
Fix a time $T > 0$ and any $p \in (2, 3]$ such that $p \le (8 + 2d)/(2 + d)$, a radius ε^γ with

$$\gamma > \frac{1}{p-2}\left[1 + \frac{2p + d(p-2)}{2p - d(p-2)} \cdot \frac{p+2}{p}\right]$$

and a noise strength ε^σ with

$$\sigma > \gamma + \frac{2p + d(p-2)}{2p - d(p-2)} \cdot \frac{p+2}{p}.$$

Then for some small $\kappa > 0$ and a generic constant $C > 0$ there is for all large $\ell > 0$ a constant C_ℓ such that the following is true:

For all families $\{\Gamma(t)\}_{t\in[0,T]}$ of closed hypersurfaces satisfying Assumption 3.7, with corresponding approximation u_ε^A and v_ε^A defined above in Theorem 3.8 and u_ε solution of the stochastic Cahn–Hilliard equation (3.2) such that $u_\varepsilon(0) = u_\varepsilon^A(0)$, the following probability estimates hold:

$$\mathbb{P}\big(\|R\|_{L^p([0,T]\times\mathcal{D})} \le \varepsilon^\gamma\big) \ge 1 - C_\ell \varepsilon^\ell,$$
$$\mathbb{P}\big(\|R\|^2_{L^\infty(0,T,H^{-1})} \le C\big[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}\big]\big) \ge 1 - C_\ell \varepsilon^\ell,$$

and

$$\mathbb{P}\big(\|R\|^2_{L^2(0,T,H^1)} \le C\big[\varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa}\big]\big) \ge 1 - C_\ell \varepsilon^\ell,$$

where $R := u_\varepsilon - u_\varepsilon^A$ is the error defined in (3.6).

Remark 3.11. Let us remark that in dimension $d = 2$ one can easily check that we obtain the smallest possible value both for σ and γ for $p = 3$. In that case $\gamma > 6$ and $\sigma > 23/3$. This is in well agreement with the γ derived in [2], but unfortunately we can only consider very small noise strength. But it seems that using the H^{-1}-norm and spectral information available there is no improvement possible.

For dimension $d = 3$ again the noise strength is small, but the result is not that clear. While the smallest value for γ is still attained at $p = 3$ (with $\gamma > 6$ and $\sigma > 11$) we obtain the smallest value of σ for some $p < 3$. Note that due to the additional restriction $p \le (8 + 2d)/(2 + d) = 14/5$.

Remark 3.12. Let us remark on the fact that we take $R(0) = 0$. We could take more general initial conditions $u_\varepsilon(0)$ for the Cahn–Hilliard equation. Looking closely into the proof of Theorem 3.10 we could allow for an initial error $R(0)$ with $\|R(0)\|^2_{H^{-1}} = \mathcal{O}(\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa})$.

Remark 3.13. Let us state two main problems with the approach presented.

First, the spectral estimate in Theorem 3.9 yields an unstable eigenvalue of order $\mathcal{O}(1)$. This immediately restricts any approximation result to time scales of order $\mathcal{O}(1)$. But we strongly believe that this eigenvalue represents only a motion of the interfaces itself. One would need spectral information orthogonal to the space of all possible approximations u_ε^A, which are parametrized by the hypersurfaces Γ. But this does not seem to be available at the moment.

Moreover, later in the closure of the estimate we can only allow $\sigma > \sigma_0$ large enough, i.e., for sufficiently small noise strength. Here, an additional problem is that the H^{-1}-norm is not strong enough to control the nonlinearity, and from the Spectral Theorem 3.9, we do not get any higher order norms that would help in the estimate. Nevertheless, if we start with higher order norms like L^2, for instance, then there are no spectral estimates available at all.

4. The proof of the main Theorem 3.10

4.1. *Idea of proof*

For the proof we define for $p \in (2, 3]$ and $\sigma > \gamma > 0$ (both fixed later), the stopping time

$$T_\varepsilon := \inf\left\{t \in [0,T] : \left(\int_0^t \|R(s)\|^p_{L^p}\, ds\right)^{1/p} > \varepsilon^\gamma\right\}, \tag{4.1}$$

where the convention is that $T_\varepsilon = T$ if the condition is never true.

The general strategy for the proof of the main theorem is the following:

(1) Use Itô-formula for $d\|R\|^2_{H^{-1}}$.
(2) Consider all estimates up to T_ε only.
(3) Bound the stochastic integrals (at least on a set with high probability).
(4) Show that $T_\varepsilon = T$ with high probability using the bound derived for $\int_0^t \|R\|_{L^p}\, dt$ up to T_ε.

4.2. *A differential equation for the error*

Let us first derive an SPDE for R from (3.6), using (3.7) and (3.3), as follows

$$\begin{aligned} dR &= du_\varepsilon - du_\varepsilon^A = \Delta v_\varepsilon^A\, dt - \Delta v_\varepsilon\, dt + \varepsilon^\sigma d\mathcal{W} \\ &= \left[-\frac{1}{\varepsilon}\Delta f'\big(u_\varepsilon^A\big) + \varepsilon\Delta^2 u_\varepsilon^A + \Delta r_\varepsilon^A + \frac{1}{\varepsilon}\Delta f'(u_\varepsilon) - \varepsilon\Delta^2 u_\varepsilon\right] dt + \varepsilon^\sigma d\mathcal{W} \\ &= \frac{1}{\varepsilon}\big[\Delta f'\big(u_\varepsilon^A + R\big) - \Delta f'\big(u_\varepsilon^A\big)\big]\, dt + \big[-\varepsilon\Delta^2 R + \Delta r_\varepsilon^A\big]\, dt + \varepsilon^\sigma d\mathcal{W}. \end{aligned} \tag{4.2}$$

4.3. *The H^{-1}-norm of R*

The approximate solutions u_ε^A and v_ε^A are, by their construction, functions in $C^2(\overline{\mathcal{D}_T})$, while u_ε^A satisfies for all $t \in [0, T]$

$$\int_{\mathcal{D}} u_\varepsilon^A(t)\, dx = 0.$$

Since (3.2) is mass conservative, we can conclude that mass conservation also holds for R, i.e., for all $t \in [0, T]$

$$\int_{\mathcal{D}} R(t)\, dx = 0.$$

Observe that the operator $-\Delta$ is a symmetric positive operator on the space

$$\mathcal{H}^2 := \left\{ w \in C^2(\overline{\mathcal{D}}) : \int_{\mathcal{D}} w\, dx = 0 \text{ and } \partial_n w = 0 \text{ on } \partial\mathcal{D} \right\}.$$

Therefore, by elliptic regularity, the operator $-\Delta : \mathcal{H}^2 \to L^2$ is bijective. So, we can invert it and for any $t \in [0, T]$ there exists a unique $\psi(t) \in \mathcal{H}^2$ such that

$$-\Delta\psi(t) = R(t), \quad \text{or equivalently} \quad (-\Delta)^{-1} R(t) = \psi(t). \tag{4.3}$$

With the scalar product $\langle \cdot, \cdot \rangle$ in L^2 the H^{-1}-norm of R is given by

$$\|R\|^2_{H^{-1}} = \big\|(-\Delta)^{-1/2} R\big\|^2_{L^2} = \big\|(-\Delta)^{1/2}\psi\big\|^2_{L^2} = \|\nabla\psi\|^2_{L^2} = \langle \psi, R \rangle.$$

Since

$$\langle d\psi, R \rangle = \langle -\Delta\, dR, R \rangle = \langle dR, -\Delta R \rangle,$$

considering the Itô-differential, we obtain

$$\begin{aligned} \frac{1}{2} d\|R\|^2_{H^{-1}} &= \langle \psi, dR \rangle + \frac{1}{2}\langle d\psi, dR \rangle = \langle \psi, dR \rangle + \frac{1}{2}\varepsilon^{2\sigma}\big\langle (-\Delta)^{-1} d\mathcal{W}, d\mathcal{W} \big\rangle \\ &= \langle \psi, dR \rangle + \frac{1}{2}\varepsilon^{2\sigma} \operatorname{tr}\big(Q^{1/2}(-\Delta)^{-1} Q^{1/2}\big). \end{aligned} \tag{4.4}$$

Here, by Assumption 3.1 the trace in the previous estimate is bounded. So, using (3.3) and (3.7), we arrive at

$$\begin{aligned}\langle \psi, dR\rangle &= \big\langle \psi, d\big(u_\varepsilon - u_\varepsilon^A\big)\big\rangle = \big\langle \psi, (-\Delta)\big(v_\varepsilon - v_\varepsilon^A\big)\,dt + \varepsilon^\sigma\, d\mathcal{W}\big\rangle \\ &= \big\langle R, \big(v_\varepsilon - v_\varepsilon^A\big)\big\rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W}\rangle. \end{aligned} \tag{4.5}$$

Using again (3.3) and (3.7) in order to replace the v's, yields the following equality

$$\begin{aligned}\langle \psi, dR\rangle &= -\varepsilon^{-1}\big\langle R, f'(u_\varepsilon) - f'\big(u_\varepsilon^A\big)\big\rangle dt + \varepsilon\big\langle R, \Delta\big(u_\varepsilon - u_\varepsilon^A\big)\big\rangle dt - \big\langle R, r_\varepsilon^A\big\rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W}\rangle \\ &= -\varepsilon^{-1}\big\langle R, \big(f'(u_\varepsilon) - f'\big(u_\varepsilon^A\big)\big)\big\rangle dt - \varepsilon\|\nabla R\|^2\,dt - \big\langle R, r_\varepsilon^A\big\rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W}\rangle. \end{aligned} \tag{4.6}$$

Definition 4.1. For any positive integer p, we define the L^p-norms

$$\|f\|_{p,\mathcal{D}} := \left(\int_{\mathcal{D}} |f|^p\,dx\right)^{1/p} \quad \text{and} \quad \|f\|_{p,\mathcal{D}_t} := \left(\int_0^t \int_{\mathcal{D}} |f|^p\,dx\,ds\right)^{1/p}.$$

Also, we denote by $\|\cdot\|$ the usual $L^2(\mathcal{D})$-norm and by $\|\cdot\|_{L^p}$ the $L^p(\mathcal{D})$-norm.

Applying Taylor's formula to expand $f'(u_\varepsilon)$ around u_ε^A, with residual $\mathcal{N}(u_\varepsilon^A, R)$, we have

$$f'(u_\varepsilon) - f'\big(u_\varepsilon^A\big) = f''\big(u_\varepsilon^A\big)R + \mathcal{N}\big(u_\varepsilon^A, R\big).$$

The crucial bound for the nonlinearity in the residual is the following result from Lemma 2.2 of [2]. It is based on a direct representation of the remainder $\mathcal{N}$ in the Taylor expansion together with the fact that u_ε^A is uniformly bounded.

Lemma 4.2. *Let $p \in (2,3]$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, then it holds that*

$$-\int \varepsilon^{-1}\mathcal{N}\big(u_\varepsilon^A, R\big)R \le c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p. \tag{4.7}$$

Thus, we obtain

$$\begin{aligned}-\frac{1}{\varepsilon}\big\langle R, \big(f'(u_\varepsilon) - f'\big(u_\varepsilon^A\big)\big)\big\rangle &= -\frac{1}{\varepsilon}\big\langle R, f''\big(u_\varepsilon^A\big)R\big\rangle - \frac{1}{\varepsilon}\big\langle R, \mathcal{N}\big(u_\varepsilon^A, R\big)\big\rangle \\ &\le -\frac{1}{\varepsilon}\big\langle R, f''\big(u_\varepsilon^A\big)R\big\rangle + c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p. \end{aligned} \tag{4.8}$$

Relations (4.4), (4.6) and (4.8) yield the following first key estimate

$$\begin{aligned}&\frac{1}{2}d\|\nabla\psi\|^2 + \varepsilon\|\nabla R\|^2\,dt + \frac{1}{\varepsilon}\big\langle R, f''\big(u_\varepsilon^A\big)R\big\rangle dt \\ &\quad \le c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p\,dt + \|R\|_{p,\mathcal{D}}\big\|r_\varepsilon^A\big\|_{q,\mathcal{D}}\,dt + \varepsilon^\sigma\langle\psi, d\mathcal{W}\rangle + C_{\mathrm{tr}}\varepsilon^{2\sigma}\,dt. \end{aligned} \tag{4.9}$$

From this *a-priori* estimate, we now can derive a uniform bound for $\|\nabla\psi\|$ and later a mean square bound on $\|\nabla R\|$. Both estimates still involve the L^p-norm of R on the right hand side, and we use the stopping time T_ε to control this.

4.4. *Technical lemmas*

We first need the following lemma of Burkholder–Davis–Gundy type for stochastic integrals. Recall the stopping time T_ε from (4.1).

Lemma 4.3. *Let f be a continuous real valued function, and $\Delta\psi = R$ as before. Then for all $\kappa > 0$, $\ell > 1$ there exists a constant $C = C(\ell, T, \kappa)$ such that*

$$\mathbb{P}\left(\sup_{[0,T_\varepsilon]}\left|\int_0^t f\langle\psi, d\mathcal{W}\rangle\right| \ge \varepsilon^{\gamma-\kappa}\right) \le C\varepsilon^{\ell\kappa}\|f\|_{L^{2p/(p-2)}}^\ell.$$

Proof. We shall use the Chebychev's inequality. Thus, we need to bound the moments first. Applying Burkholder–Davis–Gundy inequality (using that $\Delta^{-1}Q\Delta^{-1}$ is a bounded operator by assumption), we obtain

$$\begin{aligned}\mathbb{E}\sup_{[0,T_\varepsilon]}\left|\int_0^t f(s)\big\langle\psi(s), d\mathcal{W}(s)\big\rangle\right|^\ell &\le C_\ell\mathbb{E}\left|\int_0^{T_\varepsilon} f^2(s)\big\langle\psi(s), Q\psi(s)\big\rangle ds\right|^{\ell/2}\\ &\le C\mathbb{E}\left|\int_0^{T_\varepsilon} f^2(s)\|R(s)\|_{L^2}^2\, ds\right|^{\ell/2}\\ &\le C\mathbb{E}\left|\left(\int_0^{T_\varepsilon}\|R(s)\|_{L^p}^p\, ds\right)^{2/p}\right|^{\ell/2}\left|\int_0^{T_\varepsilon} f^{2p/(p-2)}\right|^{(p-2)\ell/(2p)}\\ &\le C\|f\|_{L^{2p/(p-2)}}^\ell\varepsilon^{\gamma\ell}.\end{aligned}$$

Here, we applied Hölder's inequality and the definition of $T_\varepsilon \le T$.

Furthermore, using Chebychev's inequality, we obtain the result as follows

$$\mathbb{P}\left(\sup_{[0,T_\varepsilon]}\left|\int_0^t f\langle\psi, d\mathcal{W}\rangle\right| \ge \varepsilon^{\gamma-\kappa}\right) \le \varepsilon^{-\ell(\gamma-\kappa)}\mathbb{E}\sup_{[0,T_\varepsilon]}\left|\int_0^t f\langle\psi, d\mathcal{W}\rangle\right|^\ell \le C\varepsilon^{\ell\kappa}\|f\|_{L^{2p/(p-2)}}^\ell. \qquad \square$$

Now we present the following stochastic version of Gronwall's lemma.

Lemma 4.4. *Let X, $\mathcal{F}_i$, λ be real valued processes, and $\mathcal{G}$ be a Hilbert-space valued one. Furthermore, assume that*

$$dX = [\lambda X + \mathcal{F}_1]\,dt + \langle\mathcal{G}, d\mathcal{W}\rangle,$$

and that

$$\mathcal{F}_1 \le \mathcal{F}_2.$$

Then the following inequality holds true

$$X(t) \le e^{\Lambda(t)}X(0) + \int_0^t e^{\Lambda(t)-\Lambda(s)}\mathcal{F}_2(s)\,ds + \int_0^t e^{\Lambda(t)-\Lambda(s)}\big\langle\mathcal{G}(s), d\mathcal{W}(s)\big\rangle,$$

for

$$\Lambda(t) := \int_0^t \lambda(s)\,ds.$$

Proof. We define

$$Y(t) := X(t)e^{-\Lambda(t)}.$$

By the definition of the process Y, we obtain easily

$$dY = e^{-\Lambda}\,dX - \lambda Y\,dt = e^{-\Lambda}\mathcal{F}_1\,dt + e^{-\Lambda}\langle\mathcal{G}, d\mathcal{W}\rangle,$$

and

$$\begin{aligned}Y(t) &= Y(0) + \int_0^t e^{-\Lambda(s)}\mathcal{F}_1(s)\,ds + \int_0^t e^{-\Lambda(s)}\big\langle\mathcal{G}(s), d\mathcal{W}(s)\big\rangle\\ &\le X(0) + \int_0^t e^{-\Lambda(s)}\mathcal{F}_2(s)\,ds + \int_0^t e^{-\Lambda(s)}\big\langle\mathcal{G}(s), d\mathcal{W}(s)\big\rangle.\end{aligned}$$

Multiplying the inequality with $e^{\Lambda(t)}$, and using the definition of Y, we derive the stated stochastic version of Gronwall's inequality. $\square$

4.5. *Uniform bound on $\nabla\psi$*

Using the spectral estimate of Proposition 3.9, we get from (4.9)

$$d\|\nabla\psi\|^2 \le \big[C\|\nabla\psi\|^2 + c\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}} + 2\|R\|_{p,\mathcal{D}}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}} + C\varepsilon^{2\sigma}\big]\,dt + 2\varepsilon^\sigma\langle\psi, d\mathcal{W}\rangle. \tag{4.10}$$

We apply Lemma 4.4. Since $R(0)=0$ implies $\nabla\psi(0)=0$, this yields

$$\begin{aligned}\big\|\nabla\psi(t)\big\|^2 &\le \int_0^t e^{C(t-s)}\big[c\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}} + \|R\|_{p,\mathcal{D}}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}} + C_{\mathrm{tr}}\varepsilon^{2\sigma}\big]\,ds + \int_0^t e^{C(t-s)}\varepsilon^\sigma\big\langle\psi, d\mathcal{W}(s)\big\rangle \\ &\le e^{CT}\int_0^t\big[c\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}} + \|R\|_{p,\mathcal{D}}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}} + C_{\mathrm{tr}}\varepsilon^{2\sigma}\big]\,ds + \varepsilon^\sigma e^{CT}\bigg|\int_0^t e^{-Cs}\big\langle\psi, d\mathcal{W}(s)\big\rangle\bigg|.\end{aligned}$$

Furthermore, from Lemma 4.3 we obtain on a subset with high probability

$$\sup_{t\in[0,T_\varepsilon]}\bigg|\int_0^t e^{-Cs}\big\langle\psi(s), d\mathcal{W}(s)\big\rangle\bigg| \le C\varepsilon^{\gamma-\kappa}.$$

Thus, we arrive at

$$\begin{aligned}\big\|\nabla\psi(t)\big\|^2 &\le C\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}_t} + C\|R\|_{p,\mathcal{D}_t}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}_t} + C\varepsilon^{2\sigma}t + C\varepsilon^{\sigma+\gamma-\kappa} \\ &\le C\big[\varepsilon^{p\gamma-1} + \varepsilon^\gamma\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}_T} + \varepsilon^{2\sigma} + \varepsilon^{\sigma+\gamma-\kappa}\big] \\ &\le C\big[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}\big],\end{aligned} \tag{4.11}$$

where we used that $\gamma<\sigma$ and that κ is sufficiently small, together with Theorem 3.8. Finally, we verified the following lemma:

Lemma 4.5. *For all $p\in[2,3)$, $\sigma>1$, $\kappa>0$, and $\gamma<\sigma$ we have*

$$\big\|R(t)\big\|^2_{L^\infty(0,T_\varepsilon,H^{-1})} \le C\big[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}\big], \tag{4.12}$$

with probability larger that $1-C_\ell\varepsilon^\ell$ for all $\ell>0$.

4.6. *Mean square bound on ∇R*

We return to relation (4.9) and shall use the estimate

$$-\varepsilon^{-1}\int_0^t\int_{\mathcal{D}} f'\big(u^A_\varepsilon\big)R^2\,dx \le \varepsilon^{-2/p}\|R\|^2_{p,\mathcal{D}_t},$$

presented in [2] on p. 171. Its proof is based on Hölder inequality together with the fact that the set where the value of u^A_ε is not close to either $+1$ or -1, has a small measure. More precisely, the measure is controlled by

$$\mathrm{measure}\big\{(x,t)\in\mathcal{D}_T : f''\big(u^A_\varepsilon\big)<0\big\} \le C\varepsilon, \quad \varepsilon\in(0,1].$$

Therefore, integrating (4.9) and using $\nabla\psi(0)=0$ (since $R(0)=0$), we arrive at

$$\varepsilon\|\nabla R\|^2_{2,\mathcal{D}_t} \le \varepsilon^{-2/p}\|R\|^2_{p,\mathcal{D}_t} + c\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}_t} + \|R\|_{p,\mathcal{D}_t}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}_t} + \varepsilon^\sigma\int_0^t\langle\psi, d\mathcal{W}\rangle + C_{\mathrm{tr}}\varepsilon^{2\sigma}t.$$

Revoking again Lemma 4.3, we obtain on a set of high probability

$$\varepsilon\|\nabla R\|^2_{2,\mathcal{D}_{T_\varepsilon}} \le \varepsilon^{-2/p}\|R\|^2_{p,\mathcal{D}_{T_\varepsilon}} + c\varepsilon^{-1}\|R\|^p_{p,\mathcal{D}_{T_\varepsilon}} + \|R\|_{p,\mathcal{D}_{T_\varepsilon}}\big\|r^A_\varepsilon\big\|_{q,\mathcal{D}_T} + \varepsilon^{\sigma+\gamma-\kappa} + C_{\mathrm{tr}}\varepsilon^{2\sigma}T,$$

where we used that $T_\varepsilon \le T$. Moreover, the definition of T_ε implies for all $t \in [0, T_\varepsilon]$

$$\varepsilon \|\nabla R\|^2_{2,\mathcal{D}_{T_\varepsilon}} \le \varepsilon^{-2/p}\varepsilon^{2\gamma} + c\varepsilon^{-1}\varepsilon^{p\gamma} + \varepsilon^{\gamma}\left\|r^A_\varepsilon\right\|_{q,\mathcal{D}_T} + \varepsilon^{\sigma+\gamma-\kappa} + C\varepsilon^{2\sigma}.$$

Here, the constant depends on the final time T. Using again $\gamma < \sigma$ and κ sufficiently small, together with Theorem 3.8, we obtain

$$\varepsilon \|\nabla R\|^2_{2,\mathcal{D}_{T_\varepsilon}} \le C\big[\varepsilon^{-2/p}\varepsilon^{2\gamma} + \varepsilon^{-1}\varepsilon^{p\gamma} + \varepsilon^{\sigma+\gamma-\kappa}\big]. \tag{4.13}$$

Note that as $p > 2$, a short calculation shows that

$$\varepsilon^{-2/p}\varepsilon^{2\gamma} > \varepsilon^{-1}\varepsilon^{p\gamma} \quad \Longleftrightarrow \quad \frac{1}{p} < \gamma,$$

which we assume from now on, as we expect both γ and σ to be bigger that 1. We verified the following lemma:

Lemma 4.6. *For all $p \in [2, 3)$, $\kappa > 0$, $\sigma > 1$, and $\gamma \in (\frac{1}{p}, \sigma)$ we have*

$$\left\|R(t)\right\|^2_{L^2(0,T_\varepsilon,H^1)} \le C\big[\varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa}\big] \tag{4.14}$$

with probability larger that $1 - C_\ell\varepsilon^\ell$ for all $\ell > 0$.

4.7. *Final step*

In the final part of the proof it remains to show that $T_\varepsilon = T$ on our set of high probability. Thus, we shall use our estimates of the previous two Lemmas to show that $\|R\|_{p,\mathcal{D}}$ is not larger that ε^γ.

Observe first, that the following trivial interpolation inequality holds true

$$\|R\|^2_{2,\mathcal{D}} = -\int_{\mathcal{D}} R\Delta\psi\, dx = \int_{\mathcal{D}} \nabla R \nabla\psi\, dx \le \|\nabla R\|_{2,\mathcal{D}}\|\nabla\psi\|_{2,\mathcal{D}}. \tag{4.15}$$

We use the Sobolev's embedding of H^α into L^p with $\alpha := d(\frac{1}{2} - \frac{1}{p}) = \frac{d(p-2)}{2p}$, and then interpolate H^α between L^2 and H^1. We need $\alpha \in [0, 1]$, which is assured by $2 < p \le 3 < \frac{2d}{(d-2)}$. This gives,

$$\|R\|_{p,\mathcal{D}} \le C\|R\|_{H^\alpha} \le C\|R\|^{1-\alpha}_{2,\mathcal{D}}\|\nabla R\|^\alpha_{2,\mathcal{D}}.$$

Thus, using (4.15) we obtain

$$\begin{aligned}
\|R\|^p_{p,\mathcal{D}} &\le C\|R\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{2}}\|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)}{2}} \\
&\le C\|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}}\|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)}{2}+\frac{2p-d(p-2)}{4}} = C\|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}}\|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)+2p}{4}}.
\end{aligned}$$

Integration yields

$$\begin{aligned}
\|R\|^p_{p,\mathcal{D}_t} &\le C\sup_{[0,t]}\|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}} \cdot \int_0^t \|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)+2p}{4}}\, ds \\
&\le C\sup_{[0,t]}\|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}} \cdot \|\nabla R\|_{2,\mathcal{D}_t}^{\frac{d(p-2)+2p}{4}},
\end{aligned}$$

where we used the concavity of the root, as $\frac{2p-d(p-2)}{8} \le 1$ together with t being bounded by the constant T.

Now we use (4.12) and (4.14) and arrive at

$$\|R\|^p_{p,\mathcal{D}} \le C\big[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}\big]^{\frac{2p-d(p-2)}{8}}\big[\varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa}\big]^{\frac{d(p-2)+2p}{8}}. \tag{4.16}$$

Thus, pulling out $\varepsilon^{2\gamma}$ from both brackets, and as $\sigma - \gamma \geq 0$ and κ is small, we arrive at

$$\begin{aligned}\varepsilon^{-\gamma}\|R\|_{p,\mathcal{D}} &\leq C \cdot \left[\varepsilon^{(p-2)\gamma-1} + \varepsilon^{\sigma-\gamma-\kappa}\right]^{\frac{2p-d(p-2)}{8p}} \left[\varepsilon^{-1-2/p} + \varepsilon^{-1+\sigma-\gamma-\kappa}\right]^{\frac{d(p-2)+2p}{8p}} \\ &\leq C \cdot \left[\varepsilon^{(p-2)\gamma-1} + \varepsilon^{\sigma-\gamma-\kappa}\right]^{\frac{2p-d(p-2)}{8p}} \cdot \varepsilon^{-(1+2/p)\frac{d(p-2)+2p}{8p}}.\end{aligned}$$

By Lemmas 4.5 and 4.6 the previous bound holds with probability larger than $1 - C_\ell \varepsilon^\ell$. In order to show that $T_\varepsilon = T$ holds with high probability, we need to prove that the right hand side of the previous equation is smaller than 1.

As the second factor is larger than one, we need the first one to be smaller than 1 and to compensate the larger factor. Hence, we need

$$\gamma > \frac{1}{p-2}\left[1 + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p}\right] > \frac{1}{p}$$

and provided κ is sufficiently small

$$\sigma > \gamma + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p} > \gamma.$$

Hence, the proof of theorem is now complete.

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