Pathwise convergence of a numerical method for stochastic partial differential equations with correlated noise and local Lipschitz condition

Minoo Kamrani

[a]Department of Mathematics, Faculty of Sciences,
Razi University, Kermanshah, Iran

[b]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P. O. Box:19395-5746, Tehran, Iran.

Dirk Blömker

Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany

Abstract

In this paper we obtain a general statement concerning pathwise convergence of the full discretization of certain stochastic partial differential equations (SPDEs) with non-globally Lipschitz continuous drift coefficients. We focus on non-diagonal colored noise instead of the usual space-time white noise. By applying a spectral Galerkin method for spatial discretization and a numerical scheme in time introduced by Jentzen, Kloeden and Winkel we obtain the rate of path-wise convergence in the uniform topology. The main assumptions are either uniform bounds on the spectral Galerkin approximation or uniform bounds on the numerical data. Numerical examples illustrate the theoretically predicted convergence rate.

Keywords: stochastic partial differential equations; spectral Galerkin approximation; time discretization; colored noise; order of convergence; uniform bounds.

MSC2010: 60H35; 60H15; 60H10; 65M12; 65M60.

1 Introduction

Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and V be a Banach space. Suppose the space-time continuous stochastic process $X : [0, T] \times \Omega \to V$ is the unique solution of the following stochastic partial differential equation (SPDE)

$$\begin{aligned} dX_t &= [AX_t + F(X_t)]\, dt + dW_t \\ X_0 &= \xi, \end{aligned} \tag{1}$$

for $t \in [0, T]$, where the operator A denotes an unbounded operator, for example the Laplacian, with Dirichlet boundary conditions. The noise is given by an infinite dimensional Wiener process W_t, $t \in [0, T]$ defined later. We do not assume the F is global Lipschitz or that the solution exists for all times.

The main purpose of this article is to consider the numerical solution of (1) by Galerkin method where the noise is colored. A key point is the uniform bound on the numerical data that yields an error estimate similar to a-posteriori analysis. Alternatively, we can analytically bound the Galerkin approximation uniformly, which is for spectral methods in many cases straightforward to verify, by using energy-type a-priori estimates.

In [4] the Galerkin approximation was already considered for a stochastic Burgers equation with colored noise, but here we present this method in a more general setting, and not only for the Burgers equation. The main novelty, as in [4] or [3], is to bound the spatial and temporal discretization error in the uniform topology. The space of continuous or Hölder-continuous functions is a natural space for stochastic convolutions. For instance, for space-time white noise given by the generalized derivative of a standard cylindrical Wiener process, the stochastic convolution is L^2 in space, but then it is already space-time Hölder continuous. The idea of proof are bounds in space-time $W^{\alpha,p}$-spaces for large p, or the celebrated Kolmogorov test. In a recent publication [5] Cox & van Neerven established a time-discretization error in Hölder spaces, but the spatial error in UMD-spaces. We strongly believe, that working in fractional Sobolev-spaces $W^{\alpha,p}$ with small $\alpha > 0$ and large $p \gg 1$, should yield similar results than ours, but we present here a simple proof yielding uniform bounds in time only.

In [3, 12] the Galerkin approximation was considered for a simple case of SPDEs of the type of (1), either without time-discretization or in different spaces. Moreover, the Brownian motions in the Fourier expansion of the noise were independent. But in general the spatial covariance operator of the forcing does not necessarily commute with the linear operator A, thus we consider here the case where the Brownian motions are not independent.

Many authors have investigated the spectral Galerkin method for this kind of equation with space-time white noise. See for example [9, 10, 11, 12, 13, 14, 15]. There are also many articles about finite difference methods [1, 8, 9, 17, 18]. The existence and uniqueness of solutions of the stochastic equation was studied in [6, 7] for space-time white noise. In our proofs, as the nonlinearity might exhibit polynomial growth, we do not rely on the global existence of solutions, but assume that for example the numerical approximation remains uniformly bounded. In the limit of fine discretization, this will ensure global existence of the solutions and a global error bound for the numerical approximation.

Our aim here is to extend the results of [4] to the more general case of nonlinearities, with local Lipschitz conditions and polynomial growth. We obtain a general statement concerning pathwise convergence of the full discretization of certain SPDEs with non-globally Lipschitz continuous drift coefficients. For spatial discretization of equation (1) we apply a spectral Galerkin approximation as already discussed in [3] and for the time discretization we follow the method proposed in [12].

It should be mentioned that in this article we focus on the time discretization, while the spatial discretization error is obtained by the results of [3, 4], which we recall in Section 1. Not treated in [3] but already in [4], we consider here also the case of colored noise being not diagonal with respect to the eigenfunctions of the Laplacian.

As the final result we obtain an error estimate for the full space-time discretization for a wider class of SPDEs with colored noise in the uniform topology of continuous functions. The key assumption is a uniform bound on the numerical approximations, that allows for local Lipschitz-conditions only.

The paper is organized as follows. In Section 2 we give the setting and the assumptions and we recall the results on the spatial discretization error, while in Section 3 estimates for the temporal error are derived. Finally, in the last section two numerical examples are presented in order to illustrate the results.

2 Setting and assumptions

Let V, W be two $\mathbb{R}$-Banach spaces such that $V \subseteq W$. Suppose that the unbounded and invertible linear operator A generates an analytic semigroup S_t on V that extends to the larger space W, i.e., $S_t : W \to W$. Especially, $S_{t+s} = S_t S_s$ and $S_0 = Id$. Moreover let $P_N : V \to V, N \in \mathbb{N}$ be a sequence of bounded linear operators.

Consider the following assumptions already made in [3].

Assumption 1 (Semigroup)**.** *Suppose for the semigroup, that $S : (0,T] \to L(W,V)$ is a continuous mapping that commutes with P_N and satisfies for given constants $\alpha, \theta \in [0,1)$ and $\gamma \in (0,\infty)$*

$$\sup_{0<t\leq T} \left(t^\alpha \|S_t\|_{L(W,V)}\right) < \infty, \quad \sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \left(t^\alpha N^\gamma \|S_t - P_N S_t\|_{L(W,V)}\right) < \infty, \tag{2}$$

and

$$\|(-A)^\theta S_t\|_{L(V,V)} \leq C t^{-\theta} \quad \text{together with} \quad \|(-A)^{-\theta}(S_t - I)\|_{L(V,V)} \leq t^\theta. \tag{3}$$

The first assumption is crucial for the spatial discretization, while the second assumption (3) is mainly needed for the result on time-discretization, in order to bound differences of the semigroup. For example, for analytic semigroups generated by the Laplacian, (3) and the first property stated in (2) are usually straightforward to verify, see for example [16].

Assumption 2 (Nonlinearity)**.** *Let $F : V \to W$ be a continuous mapping, which satisfies the following local Lipschitz condition. There is a nonnegative integer $p \geq 0$ and a constant $L > 0$ such that for all $u, v \in V$*

$$\|F(u) - F(v)\|_W \leq L\|u - v\|_V (1 + \|u\|_V^p + \|v\|_V^p) \, . \tag{4}$$

Let us remark that it is not a major restriction that we assumed the operator A to be invertible, as we can always consider for some constant c the operator $\tilde{A} = A + cI$ and the nonlinearity $\tilde{F} = F - cI$.

2.1 The Ornstein-Uhlenbeck process

In this section we discuss properties of the Ornstein-Uhlenbeck process (or stochastic convolution) for a given Q-Wiener process W_t.

Assumption 3 (Ornstein-Uhlenbeck)**.** *Let $O : [0,T] \times \Omega \to V$ be a stochastic process with continuous sample paths and assume there exists some $\gamma \in (0,\infty)$ such that*

$$\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} N^\gamma \|O_t(\omega) - P_N(O_t(\omega))\|_V < \infty, \tag{5}$$

for every $\omega \in \Omega$.
Moreover, assume

$$\sup_{0 \le t_1 \le t_2 \le T} \frac{\left\| O_{t_2}(\omega) - O_{t_1}(\omega) \right\|_V}{(t_2 - t_1)^\theta} < \infty, \tag{6}$$

for some $\theta \in (0, \frac{1}{2})$.

In order to give an example for this assumption, we focus for the remainder of this subsection on $L^2[0,1]$ with basis functions e_k, which are given by the following standard Dirichlet basis,

$$e_k : [0,1] \to \mathbb{R}, \quad e_k(x) = \sqrt{2}\sin(k\pi x), \quad x \in [0,1], \quad k \in \mathbb{N}.$$

For every $k \in N$ define the real numbers $\lambda_k = (\pi k)^2 \in \mathbb{R}$.

Furthermore, let Q be a symmetric non-negative operator, given by the convolution with a translation invariant positive definite kernel q. This means

$$< Qe_k, e_l >= \int_0^1 \int_0^1 e_k(x) e_l(y) q(x-y) dy dx, \tag{7}$$

for $k, l \in \mathbb{N}$. Note that Q is diagonal with respect to the standard Fourier basis, but in general it is not diagonal with respect to the Dirichlet basis.

We think of Q being the covariance operator of a Wiener process W in $L^2(0,1)$ and q being the spatial correlation function of the noise process $\partial_t W_t$. See for example [2] for a detailed discussion.

Let $\beta^i : [0,T] \times \Omega \to \mathbb{R}, i \in \mathbb{N}$, be a family of Brownian motions that are not necessarily independent. If W is a Q-Wiener process in L^2, we can define $\beta^i(t) = \langle W_t, e_i \rangle$. See the discussion at end of this subsection.

Note that the variance of the Brownian motion β^i is $\sigma_i^2 = \langle Qe_i, e_i \rangle$, which means that for $\sigma_i \neq 0$ the process $\sigma_i^{-1}\beta^i(t)$ is a standard Brownian motion. Moreover, the β^i's are correlated with

$$\mathbb{E}\left[\beta^k(t)\beta^l(t)\right] = \langle Qe_k, e_l \rangle \cdot t, \quad \text{for all } k, l \in \mathbb{N}.$$

For the regularity of the noise we assume that for some $\rho > 0$

$$\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} i^{\rho-1} j^{\rho-1} |\langle Qe_i, e_j \rangle| < \infty. \tag{8}$$

This would mean that in case Q would be a diagonal operator the trace of $\Delta^{\rho-1}Q$ is finite. Moreover, we assume that $\xi : \Omega \to V$ is a measurable mapping with

$$\sup_{N \in \mathbb{N}} (N^{-\rho} \|\xi(\omega) - P_N(\xi(\omega))\|_V) < \infty$$

for every $\omega \in \Omega$.

From (8) and Lemma 3.7 in [4], it follows that there exists a stochastic process $O : [0,T] \times \Omega \to V$, which is the Ornstein-Uhlenbeck process (or stochastic convolution). It is defined by the semigroup generated by the Dirichlet Laplacian and the Wiener process $W_t = \sum_{k \in \mathbb{N}} \beta^k(t) e_k$, i.e, $O_t = S_t \xi + \int_0^t S_{t-s} dW_s$.

Furthermore, Lemma 4 in [4] assures that O satisfies Assumption 3, for all $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$ and $\gamma \in (0, \rho)$ with

$$\mathbb{P}\Big[\lim_{N\to\infty} \sup_{0\le t\le T} \Big\| O_t - S_t\xi - \sum_{i=1}^{N} \Big(-\lambda_i \int_0^t e^{-\lambda_i(t-s)}\beta^i(s)ds + \beta^i(t)\Big)e_i \Big\|_V = 0\Big] = 1. \tag{9}$$

Let us comment a little bit more on the Q-Wiener process. For example in this case when Q is a symmetric trace-class operator, there exists an orthonormal basis f_k given by eigenfunctions of Q with $\alpha_k^2 f_k = Qf_k$. Using standard theory of [6], there is a family of i.i.d. Brownian motions $\{B_k\}_{k\in\mathbb{N}}$ such that $W_t = \sum_{k\in\mathbb{N}} \alpha_k B_k(t) f_k \in L^2([0,1])$. We can then define

$$\beta_k(t) = \langle W_t, e_k\rangle_{L^2} = \sum_{\ell\in\mathbb{N}} \alpha_\ell B_\ell(t)\langle f_\ell, e_k\rangle_{L^2}.$$

2.2 Bounds and solutions

Let us first assume boundedness of the spectral Galerkin approximation. This will assure the existence of mild solutions later on. We will discuss later how to relax this condition to boundedness of the numerical data alone.

Assumption 4. *Let $X^N : [0,T]\times\Omega \to V,\quad N \in \mathbb{N}$, be a sequence of stochastic processes with continuous sample paths such that*

$$\sup_{N\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^N(\omega)\|_V < \infty \tag{10}$$

and

$$X_t^N(\omega) = \int_0^t P_N S_{t-s} F(X_s^N(\omega))ds + P_N(O_t(\omega)), \tag{11}$$

for every $t \in [0,T], \omega \in \Omega$ and every $N \in \mathbb{N}$.

From [3] we have the following theorem about the existence of solutions.

Theorem 1. *Let Assumptions 1-4 be fulfilled. Then, there exists a unique stochastic process $X : [0,T]\times\Omega \to V$ with continuous sample paths, which satisfies*

$$X_t(\omega) = \int_0^t S_{t-s}F(X_s(\omega))ds + O_t(\omega), \tag{12}$$

for every $t \in [0,T]$ and every $\omega \in \Omega$. Moreover, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : [0,\infty) \to \Omega$ such that

$$\sup_{0\le t\le T} \| X_t(\omega) - X_t^N(\omega) \|_V \le C(\omega)\cdot N^{-\gamma}, \tag{13}$$

holds for every $N \in \mathbb{N}$ and every $\omega \in \Omega$, where $\gamma \in (0,\infty)$ is given in Assumption 1 and Assumption 3.

3 Time discretization

For the discretization in time of the finite dimensional SDE (11) we follow the method proposed in [12], which was also used in [4].

Definition 2. *Fix a small time-step $\Delta t = \frac{T}{M}$, $M \in \mathbb{N}$ and define discrete values via the random variables $Y_m^{N,M} : \Omega \to V$ for $m \in \{0, ..., M-1\}$ by*

$$Y_{m+1}^{N,M}(\omega) = S_{\Delta t}\Big(Y_m^{N,M}(\omega) + \Delta t (P_N F)(Y_m^{N,M}(\omega))\Big) + P_N\Big(O_{(m+1)\Delta t}(\omega) - S_{\Delta t} O_{m\Delta t}(\omega)\Big), \tag{14}$$

where $Y_0^{N,M} = P_N \xi$.

Thus $Y_m^{N,M}$, $m \in \{1, ..., M\}$ should be the approximation of the spectral Galerkin approximation X^N (see (18) below) at times $m \cdot \Delta t$. For simplicity of presentation of the main approximation result, we first assume in addition to (10) that our numerical data is uniformly bounded:

Assumption 5. *For the numerical scheme (14) we assume*

$$\sup_{0 \le m \le M} \sup_{N,M \in \mathbb{N}} \|Y_m^{N,M}\|_V < \infty. \tag{15}$$

Therefore, in all the examples we need to verify that both bounds (10) and (15) are true, which might be quite involved. We will comment later on the extension of the approximation result, in case either (10) or (15) is not verified. It should be mentioned that for simplicity of notation, during this section $C(\omega, \alpha, \theta) > 0$ is a random constant which changes from line to line, but could be explicitly calculated.

Remark 3. *There is also the a-posteriori perspective on the uniform bound on the data. If one fixes a constant $K(\omega)$ bounding the numerical data, then all the constants $C(\omega)$ in the proofs can be calculated in terms of K. Even if the bound on the numerical data by K is then only true for one N and M, then the error estimate holds for this pair (N, M). We comment on that in more detail in Section 3.1.*

Lemma 4. *Suppose Assumptions 1-4 are true. Let $X^N : [0,T] \times \Omega \to V$ be the unique adapted stochastic process with continuous sample paths in (11) and $O^N : [0,T] \times \Omega \to V$ is the stochastic process defined in Assumption 3 in (9). Then for all $\vartheta \in (0, 1-\alpha)$, there exists a random variable $C : \Omega \to [0, \infty)$ such that*

$$\Big\|(X_{t_2}^N(\omega) - O_{t_2}^N(\omega)) - (X_{t_1}^N(\omega) - O_{t_1}^N(\omega))\Big\|_V \le C(\omega)(t_2 - t_1)^{\vartheta},$$

for every $N \in \mathbb{N}$, $\omega \in \Omega$ and all $t_1, t_2 \in [0,T]$, with $t_1 < t_2$.

Note that α was introduced in Assumption 1.

Proof. The strategy of proof is somewhat similar to Lemma 4.1 in [4]. Here we recall the main arguments.

$$
\begin{aligned}
&\Big\|(X^N_{t_2}(\omega)-O^N_{t_2}(\omega))-(X^N_{t_1}(\omega)-O^N_{t_1}(\omega))\Big\|_V\\
&=\Big\|\int_{t_1}^{t_2}P_N S_{t_2-s}F(X^N_s(\omega))ds+\int_0^{t_1}P_N(S_{t_2-s}-S_{t_1-s})F(X^N_s(\omega))ds\Big\|_V\\
&\le C(\omega)\Big((t_2-t_1)^{1-\alpha}\\
&\qquad+\int_0^{t_1}\|P_N S_{t_1-s}(-A)^{\vartheta}\|_{L(W,V)}\|(-A)^{-\vartheta}(S_{t_2-t_1}-I)\|_{L(V,V)}ds\Big)\\
&\le C(\omega)\Big((t_2-t_1)^{1-\alpha}+(t_2-t_1)^{\vartheta}\int_0^{t_1}(t_1-s)^{-\alpha-\vartheta}ds\Big)\\
&\le C(\omega)(t_2-t_1)^{\vartheta},
\end{aligned}
\tag{16}
$$

where we have used (2) and (3). □

Now our aim is to obtain the discretization error in time

$$
\|X^N_{m\Delta t}(\omega)-Y^{N,M}_m(\omega)\|_V, \tag{17}
$$

where

$$
X^N_{m\Delta t}(\omega)=\int_0^{m\Delta t}P_N S_{m\Delta t-s}F(X^N_s(\omega))ds+O^N_{m\Delta t}(\omega) \tag{18}
$$

is the solution of the spatial discretization, which is evaluated at the grid points.

For this aim as an intermediate discretization, we consider the mapping $Y^N_m:\Omega\to V,\ m=1,2,...,M$ by

$$
Y^N_m(\omega)=\sum_{k=0}^{m-1}\int_{k\Delta t}^{(k+1)\Delta t}P_N S_{m\Delta t-k\Delta t}F(X^N_{k\Delta t}(\omega))ds+O^N_{m\Delta t}(\omega), \tag{19}
$$

and at first we obtain the error between the spectral Galerkin approximation and Y^N_m. Define for $\vartheta\in(0,\min\{\theta,1-\alpha\})$

$$
\begin{aligned}
R(\omega)&=\sup_{N\in\mathbb{N}}\sup_{0\le s\le T}\|F(X^N_s(\omega))\|_W+\sup_{N\in\mathbb{N}}\sup_{0\le s\le T}\|X^N_s(\omega)\|_V\\
&\quad+\sup_{0\le t_1,t_2\le T}\|O_{t_2}(\omega)-O_{t_1}(\omega)\|_V|t_2-t_1|^{-\vartheta}\\
&\quad+\sup_{N\in\mathbb{N}}\sup_{0\le t_1,t_2\le T}\|X^N_{t_2}(\omega)-O^N_{t_2}(\omega)-(X^N_{t_1}(\omega)-O^N_{t_1}(\omega))\|_V|t_2-t_1|^{-\vartheta}.
\end{aligned}
\tag{20}
$$

From Assumption 2 and 4 together with (6) and Lemma 4, the random variable $R:\Omega\to\mathbb{R}$ is finite.

Lemma 5. *Let Assumptions 1-4 be fulfilled and suppose $\vartheta\in(0,\min\{\theta,1-\alpha\})$. Then there exists a finite random variable $C:\Omega\to[0,\infty)$ such that for all $m\in\{0,1,...,M\}$ and every $M,N\in\mathbb{N}$*

$$
\|X^N_{m\Delta t}(\omega)-Y^N_m(\omega)\|_V\le C(\omega)(\Delta t)^{\vartheta}, \tag{21}
$$

for every $\omega\in\Omega$.

Proof. Here we follow the ideas from Lemma 4.2 of [4] with some modifications. From (18) we have

$$X^N_{m\Delta t}(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds + O^N_{m\Delta t}(\omega), \tag{22}$$

for every $m \in \{0, 1, ..., M\}$, and every $M \in \mathbb{N}$. Therefore

$$\begin{aligned} X^N_{m\Delta t}(\omega) - Y^N_m(\omega) \quad = \quad & \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \\ & - \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X^N_{k\Delta t}(\omega)) ds \\ & + \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \qquad (23) \\ & - \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} F(X^N_{k\Delta t}(\omega)) ds. \qquad (24) \end{aligned}$$

At first we obtain the bound for the last two integrals above. First

$$\|(23)\|_V \leq \int_{(m-1)\Delta t}^{m\Delta t} (m\Delta t - s)^{-\alpha} ds \sup_{0\leq s\leq t} \|F(X^N_s(\omega))\|_W \leq CR(\omega)(\Delta t)^{1-\alpha}.$$

Similarly

$$\|(24)\|_V \leq R(\omega)(\Delta t)^{1-\alpha}.$$

Now we insert the OU-process. Define $Z^N_{s,k\Delta t}(\omega) = O^N_s(\omega) - O^N_{k\Delta t}(\omega)$. Thus for every $m \in \{0, 1, ..., M\}$ we have,

$$\begin{aligned} & \left\| X^N_{m\Delta t}(\omega) - Y^N_m(\omega) \right\|_V \leq I_1 + I_2 + I_3 + R(\omega)(\Delta t)^{1-\alpha} \\ := & \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \left[F(X^N_s(\omega)) - F\big(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)\big) \right] ds \Big\|_V \\ & + \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \left[F(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)) - F(X^N_{k\Delta t}(\omega)) \right] ds \Big\|_V \\ & + \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \left[P_N S_{m\Delta t - s} - P_N S_{m\Delta t - k\Delta t} \right] F(X^N_{k\Delta t}(\omega)) ds \Big\|_V \\ & + R(\omega)(\Delta t)^{1-\alpha} \end{aligned}$$

For I_1 by using (4) and Lemma 4 we conclude with a probably different random constant $C(\omega)$ that

$$
\begin{aligned}
I_1 &\leq L \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - s)^{-\alpha} \left\| X_s^N(\omega) - (X_{k\Delta t}^N(\omega) + Z_{s,k\Delta t}^N(\omega)) \right\|_V \\
&\quad \cdot \Big(1 + \|X_s^N(\omega)\|_V^p + \|X_{k\Delta t}^N(\omega) + Z_{s,k\Delta t}^N(\omega)\|_V^p\Big) ds \\
&\leq C(\omega) \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - s)^{-\alpha} (s - k\Delta t)^{\vartheta} \Big(1 + 2R^p(\omega) + (s - k\Delta t)^{p\vartheta}\Big) ds \\
&\leq C(\omega)(\Delta t)^{\vartheta}.
\end{aligned}
\tag{25}
$$

For the second term I_2 by (4) we derive very similarly

$$
\begin{aligned}
I_2 &\leq L \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \left\| P_N S_{m\Delta t - s} \right\|_{L(W,V)} \left\| X_{k\Delta t}^N(\omega) + Z_{s,k\Delta t}^N(\omega) - X_{k\Delta t}^N(\omega) \right\|_V \\
&\quad \cdot \Big(1 + \left\| X_{k\Delta t}^N(\omega) + Z_{s,k\Delta t}^N(\omega) \right\|_V^p + \|X_{k\Delta t}^N(\omega)\|_V^p\Big) \\
&\leq C(\omega)(\Delta t)^{\vartheta}.
\end{aligned}
$$

Finally, for the third term I_3 we drive

$$
\begin{aligned}
I_3 &\leq \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \left\| P_N S_{m\Delta t - k\Delta t}(S_{k\Delta t - s} - I) F(X_{k\Delta t}^N(\omega)) \right\|_V ds \\
&\leq C(\omega) \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - k\Delta t)^{-\alpha-\theta} (k\Delta t - s)^{\theta} \|F(X_{k\Delta t}^N(\omega))\|_W ds \\
&\leq C(\omega)(\Delta t)^{\vartheta},
\end{aligned}
\tag{26}
$$

where we used (3) for the semigroup. □

The first main result of this section is stated below.

Theorem 6. *Let Assumptions 1-5 be fulfilled and suppose $\vartheta \in (0, \min\{\theta, 1-\alpha\})$. Then there exists a finite random variable $C : \Omega \to [0, \infty)$ such that for all $m \in \{0, 1, ..., M\}$ and every $M, N \in \mathbb{N}$*

$$
\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V \leq C(\omega)(\Delta t)^{\vartheta},
$$

for all $\omega \in \Omega$, where $X^N : [0,T] \times \Omega \to V$ is the unique adapted stochastic process with continuous sample paths, defined in Assumption 4 and $Y_m^{N,M} : \Omega \to V$, for $m \in \{0, 1, ..., M\}$ and $N, M \in \mathbb{N}$ is given in (14).

Proof. Recall $\Delta t = \frac{T}{M}$, with $M \in \mathbb{N}$. Note that $Y_m^{N,M} : \Omega \to V$ satisfies

$$
Y_m^{N,M}(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(Y_k^{N,M}(\omega)) ds + P_N O_{m\Delta t}(\omega). \tag{27}
$$

Therefore (neglecting the ω in the notation)

$$\begin{aligned}
\|Y_m^N(\omega) - Y_m^{N,M}\|_V &= \left\| \Delta t \sum_{k=0}^{m-1} P_N S_{m\Delta t - k\Delta t}(F(X_{k\Delta t}^N) - F(Y_k^{N,M})) \right\|_V \\
&\le L\Delta t \sum_{k=0}^{m-1} (m\Delta t - k\Delta t)^{-\alpha} \left\| X_{k\Delta t}^N - Y_k^{N,M} \right\|_V \left(1 + \|X_{k\Delta t}^N\|_V^p + \left\| Y_k^{N,M} \right\|_V^p \right) \\
&\le C \sum_{k=0}^{m-1} \Delta t (m\Delta t - k\Delta t)^{-\alpha} \left\| X_{k\Delta t}^N - Y_k^{N,M} \right\|_V .
\end{aligned} \tag{28}$$

Now from Lemma 5 and (28) we derive

$$\begin{aligned}
\left\| X_{m\Delta t}^N(\omega) - Y_m^{M,N}(\omega) \right\|_V &\le \left\| X_{m\Delta t}^N(\omega) - Y_m^N(\omega) \right\|_V + \left\| Y_m^N(\omega) - Y_m^{M,N}(\omega) \right\|_V \\
&\le C(\omega) \Big((\Delta t)^\vartheta + \sum_{k=0}^{m-1} \Delta t (m\Delta t - k\Delta t)^{-\alpha} \left\| X_{k\Delta t}^N(\omega) - Y_k^{N,M}(\omega) \right\|_V \Big).
\end{aligned}$$

Hölder and Young's inequality yields

$$\left\| X_{m\Delta t}^N(\omega) - Y_m^{M,N}(\omega) \right\|_V^2 \le C(\omega) \Big((\Delta t)^{2\vartheta} + \sum_{k=0}^{m-1} \Delta t \left\| X_{k\Delta t}^N(\omega) - Y_k^{M,N}(\omega) \right\|_V^2 \Big).$$

Hence from Gronwall's lemma we conclude

$$\left\| X_{m\Delta t}^N(\omega) - Y_m^{M,N}(\omega) \right\|_V^2 \le C(\omega)(\Delta t)^{2\vartheta}.$$

□

3.1 Main results – Full Discretization

Combining Theorem 6 for the time discretization and Theorem 1 for the spatial discretization, yields the following result on the full discretization

Theorem 7. *Let Assumptions 1-5 be true. Let $X : [0,T] \times \Omega \to V$ be the solution of the SPDE (12) and $Y_m^{N,M} : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ the numerical solution given by (14). Fix $\vartheta \in (0, \min\{\theta, 1-\alpha\})$, then there exists a finite random variable $C : \Omega \to [0, \infty)$ such that*

$$\|X_{m\Delta t}(\omega) - Y_m^{N,M}(\omega)\|_V \le C(\omega) \left(N^{-\gamma} + (\Delta t)^\vartheta \right), \tag{29}$$

for all $m \in \{0, 1, ..., M\}$ and every $M, N \in \mathbb{N}$.

For simplicity of presentation we supposed in Theorem 7 both the full discretization (15) and the Galerkin approximation (10) to be uniformly bounded.

Following the proofs, it is easy to verify that it is sufficient to assume only one of those assumptions. Let us comment in more detail on the extension of the approximation result in that case. Let us focus on the case where the uniform bound (15) for the full discretization is not satisfied.

First it is easy to verify that the following minor modification of our main result is true. Its proof follows directly, by observing, that the proof of the main

Theorem 7 never uses the dependence on M or N. If the numerical data is bounded only for a given fixed M and N (see (30)), then all the estimates of the previous section go through, and the constant $C(\omega)$ in Theorem 7 does not depend on M and N but only on the bound on the numerical data.

Theorem 8. *Let Assumptions 1-4 be true. Fix $\vartheta \in (0, \min\{\theta, 1-\alpha\})$ and fix a non-negative random constant $K(\omega)$. Let $X : [0,T] \times \Omega \to V$ be the solution of the SPDE (12) and $Y_m^{N,M} : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ be the numerical solution given by (14).*

Then there exists a finite random variable $C : \Omega \to [0, \infty)$, depending on K, but independent of M and N, such that the following is true:

If for one choice of $N, M \in \mathbb{N}$ we have

$$\sup_{0 \le m \le M} \|Y_m^{N,M}\|_V \le K(\omega), \tag{30}$$

then for all $m \in \{0, 1, ..., M\}$

$$\|X_{m\Delta t}(\omega) - Y_m^{N,M}(\omega)\|_V \le C(\omega)\left(N^{-\gamma} + (\Delta t)^{\vartheta}\right). \tag{31}$$

Remark 9. *This is a somewhat a-posteriori view of the error estimate. We first calculate for fixed N and M the numerical approximation, and then decide whether it is smaller than an a-priori determined constant, which then implies the error estimate.*

One could also determine with some effort the dependence of $C(\omega)$ on $K(\omega)$ and then use the numerical data itself in the error estimate.

To proceed, note that in the proofs we can always bound every occurrence of $\|Y_m^{N,M}\|_V$ by the bounded $\|X_t^N\|_V$ and the error

$$e_t^{N,M} = \sup_{m\Delta t \le t} \|Y_m^{N,M} - X_{m\Delta t}^N\|_V \ .$$

I.e., we use for $m \le t/\Delta t$

$$\|Y_m^{N,M}\|_V^p \le C_p \|X_{m\Delta t}^N\|_V^p + C_p (e_t^{N,M})^p \ .$$

If we now assume a-priori that $e_t^{N,M} \le 1$, which is easily true, for sufficiently small $t > 0$, then we can proceed completely analogous as in the proofs of Theorem 7.

By Theorem 8 this implies now that for the error, probably with a different $C(\omega)$ that

$$e_t^{N,M} \le C(\omega)(\Delta t)^{\vartheta} \ .$$

As the right hand-side is independent of M and N, we can a-posteriori conclude, that as long as $C(\omega)(\Delta t)^{\vartheta} \le 1$ our initial guess on $e^{N,M}$ was true, and we finally derive the following theorem:

Theorem 10. *Let Assumptions 1-4 be true. Let $X : [0,T] \times \Omega \to V$ be the solution of the SPDE (12) and $Y_m^{N,M} : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ be the numerical solution given by (14).*

Then there exists a finite random variable $C : \Omega \to [0, \infty)$ such that the error estimate (29) holds provided $0 < \Delta t < C(\omega)^{-\vartheta}$.

The case when the uniform bound (10) on the spectral Galerkin approximation fails, is verified in a similar way, by bounding in the whole proof $\|X_t^N\|_V^p$ by $\|Y_m^{N,M}\|_V^p$ and $e_t^{N,M}$.

4 Numerical results

In this section we consider two examples for the numerical solution of stochastic equations by the method given in (14). Let $V = W = C([0,\pi],\mathbb{R})$ be the $\mathbb{R}$-Banach space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the norm $\|v\|_V = \|v\|_W := \sup_{x\in[0,\pi]} |v(x)|$ for every $v \in V = W$, where $|.|$ is the absolute value of a real number. Moreover, consider as orthonormal L^2-basis the smooth eigenfunctions

$$e_k : [0,\pi] \to \mathbb{R}, \quad e_k(x) = \sqrt{2/\pi}\sin(kx), \qquad \text{for every } x \in (0,\pi).$$

Denote the Laplacian with Dirichlet boundary conditions on $[0,\pi]$ by A, such that $Ae_k = -k^2 e_k$. Let the operators $P_N, N \in \mathbb{N}$ be the spectral Galerkin operators given by the orthogonal projection $P_N v = \sum_{i=1}^{N} \int_0^{\pi} e_i(s)v(s)ds \cdot e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ are an orthonormal basis of eigenfunctions of A.

We define the mapping $S : [0,T] \to L(V)$ by

$$(S_t)v(x) = \sum_{i\in\mathbb{N}} e^{-\lambda_i t} \int e_i(s)v(s)ds \cdot e_i(x), \tag{32}$$

where $\lambda_i = -i^2$. It is well known that A generates the analytic semigroup $(S_t)_{t\geq 0}$ on V, see [16]. From Lemma 4.1 in [3] and Lemma 1 in [4] we recall that (2) is satisfied for $\gamma \in (0, \frac{3}{2})$ and $\alpha \in (\frac{1}{4} + \frac{\gamma}{2}, 1)$. Moreover, from [16] we know that (3) is satisfied for any $\theta \in (0,1)$.

We use the OU-process $O : [0,T] \times \Omega \to V$ as defined in the example in Section 2.1. Therefore O satisfies Assumption 3, for all $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$ and $\gamma \in (0,\rho)$. The covariance operator Q is given as a convolution operator

$$\langle Qe_k, e_l\rangle = \int_0^{\pi}\int_0^{\pi} e_k(x)e_l(y)q(x-y)dydx. \tag{33}$$

We obtain our numerical result with two kernels

$$q_1(x-y) = \frac{1}{h}\max\{0, 1 - \frac{1}{h^2}|x-y|\} \tag{34}$$

and

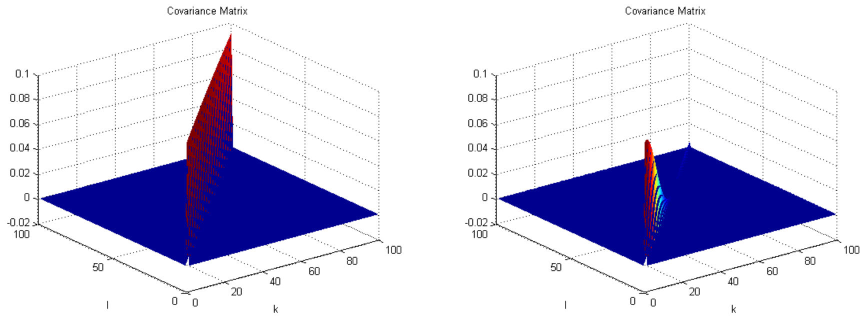
$$q_2(x-y) = \max\{0, 1 - \frac{1}{h}|x-y|\}. \tag{35}$$

In Figures 1 and 2 we plotted the Covariance Matrix for $h = 0.1$ and $h = 0.01$ for the kernels (34) and (35). By some numerical calculations we can show that the condition on Q from (8) is satisfied for any $\rho \in (0, \frac{1}{2})$, as we can calculate explicitly the Fourier-coefficients of $(x,y) \mapsto q(x-y)$ and check for summability.

For simplicity fix the following smooth deterministic initial condition

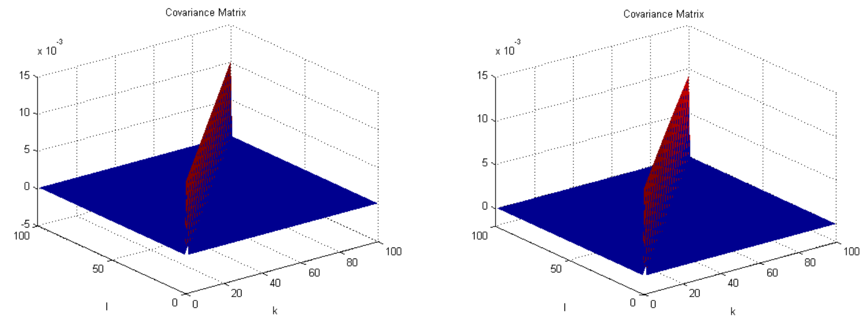
$$\xi(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \quad \text{for all } x \in [0,\pi].$$

Now we consider two types of nonlinearity, globally Lipschitz and locally Lipschitz, as given by the following examples.



(a) (b)

Figure 1: Covariance Matrix $\langle Qe_k, e_l \rangle_{k,l}$ for $k, l \in \{1, 2, \cdots, 100\}$, for $h = 0.1$ by (a) kernel (34) and (b) kernel (35)



(a) (b)

Figure 2: Covariance Matrix $\langle Qe_k, e_l \rangle_{k,l}$ for $k, l \in \{1, 2, \cdots, 100\}$, for $h = 0.01$ (a) kernel (34) and (b) kernel (35)

Example 1 Consider the Nemytskii operator $F : V \to V$ given by $(F(v))(x) = f(v(x))$ for every $x \in [0, \pi]$ and every $v \in V$, where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(y) = 5 \cdot \frac{1-y}{1+y^2}.$$

This generates a globally Lipschitz nonlinearity. Thus Assumption 2 is true. The stochastic equation (1) now reads as

$$dX_t = \Big[\frac{\partial^2}{\partial x^2} X_t + 5\frac{1-X_t}{1+X_t^2}\Big] dt + dW_t, \quad X_0(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \tag{36}$$

with Dirichlet boundary conditions $X_t(0) = X_t(\pi) = 0$ for $t \in [0, 1]$.

The finite dimensional SDE (11) reduces to

$$dX_t^N = \Big[\frac{\partial^2}{\partial x^2} X_t^N + 5P_N \frac{1-X_t^N}{1+(X_t^N)^2}\Big] dt + dP_N W_t, \quad X_0^N(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x),$$

with $X_t^N(0) = X_t^N(\pi) = 0$ for $t \in [0, 1]$ and $x \in [0, \pi]$, and all $N \in \mathbb{N}$.

Now in our simple example we can verify rigorously that the numerical data is uniformly bounded. We derive

$$\begin{aligned} \|X_t^N(\omega)\|_V &= \Big\| \int_0^t P_N S_{t-s} F(X_s^N(\omega)) ds + P_N(O_t(\omega)) \Big\|_V \\ &\le \int_0^t \|P_N S_{t-s}\|_{L(V,V)} \|F(X_s^N(\omega))\|_V ds + \|P_N(O_t(\omega))\|_V \\ &\le C \int_0^t \|P_N S_{t-s}\|_{L(V,V)} (1 + \|X_s^N(\omega)\|_V) ds + C, \end{aligned} \tag{37}$$

from Gronwall's lemma we conclude

$$\sup_{N\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^N(\omega)\|_V < \infty.$$

In a similar way we obtain

$$\sup_{N,M\in\mathbb{N}} \sup_{0\le m\le M} \|Y_m^{N,M}(\omega)\|_V < \infty.$$

Using that here $\theta \in (0, \min\{\frac{1}{2}, \frac{\varrho}{2}\})$, Theorem 7 yields the existence of a unique solution $X : [0, \pi] \times \Omega \to C^0([0, \pi])$ of the SPDE (36) such that

$$\sup_{0\le x\le\pi} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)| \le C(\omega) \left(N^{-\gamma} + (\Delta t)^{\vartheta}\right),$$

for $m = 1, ..., M,\ M = \frac{1}{\Delta t}$, such that $\gamma \in (0, \frac{1}{2})$, $\vartheta \in (0, \frac{1}{4})$.

Example 2 Consider for the nonlinearity the Nemytskii operator $F : V \to V$ given by $(F(v))(x) = f(v(x))$ for every $x \in [0, \pi]$ and every $v \in V$, where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(y) = -y^3.$$

This generates a locally Lipschitz nonlinearity which satisfies Assumption 2. The stochastic equation (1) now reads as

$$dX_t = \Big[\frac{\partial^2}{\partial x^2} X_t - X_t^3\Big] dt + dW_t, \quad X_0(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \tag{38}$$

with Dirichlet boundary conditions $X_t(0) = X_t(\pi) = 0$ for $t \in [0,1]$.

The finite dimensional SDE (11) reduces to

$$dX_t^N = \Big[\frac{\partial^2}{\partial x^2} X_t^N - P_N (X_t^N)^3\Big] dt + dP_N W_t, \quad X_0^N(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x),$$

with $X_t^N(0) = X_t^N(\pi) = 0$ for $t \in [0,1]$ and $x \in [0,\pi]$, and all $N \in \mathbb{N}$.

Using Theorem 10 it remains to verify (10) from Assumption 4. This is straightforward by using first the estimate in L^2. We sketch only the main ideas here.

Define the standard transformation $y_t^N = X_t^N - P_N O(t)$. Thus

$$\partial_t y_t^N = \frac{\partial^2}{\partial x^2} y_t^N - (y_t^N + P_N O_t)^3 \ .$$

Taking the L^2-scalar product with y_t^N and an estimate on the polynomial yield

$$\tfrac{1}{2}\partial_t \|y_t^N\|_{L^2}^2 = -\|\partial_x y_t^N\|_{L^2}^2 - \tfrac{1}{2}\|y_t^N\|_{L^4}^4 + C\|P_N O_t\|_{L^4}^4 \ ,$$

where C is a constant independent of N. This gives a random bound in the space $L^2([0,T],H^1) \cap L^\infty([0,T],L^2) \cap L^4([0,T],L^4)$. Now by integration

$$\sup_{[0,T]} \|y_t^N\|_{L^2}^2 + \int_0^T \|\partial_x y_s^N\|_{L^2}^2 ds \le C \int_0^T \|P_N O_s\|_{L^4}^4 ds. \tag{39}$$

Using Agmon inequality yields

$$\begin{aligned} \|y_t^N\|_{L^4([0,T],V)}^2 &\le C\|y_t^N\|_{L^2([0,T],H^1)} \cdot \|y_t^N\|_{L^\infty([0,T],L^2)} \\ &\le 2C(\|y_t^N\|_{L^2([0,T],H^1)}^2 + \|y_t^N\|_{L^\infty([0,T],L^2)}^2). \end{aligned} \tag{40}$$

By (39) and (40) we get

$$\|y_t^N\|_{L^4([0,T],V)}^2 \le C \int_0^T \|P_N O_s\|_{L^4}^4 ds \le C(\omega) < \infty. \tag{41}$$

where we used that $\sup_{[0,T]} \|P_N O_t\|_V$ is, uniformly in N, bounded (see Lemma 3.8 in [4]). This is sufficient now to verify the bound in V. From the mild formulation and (41) we have

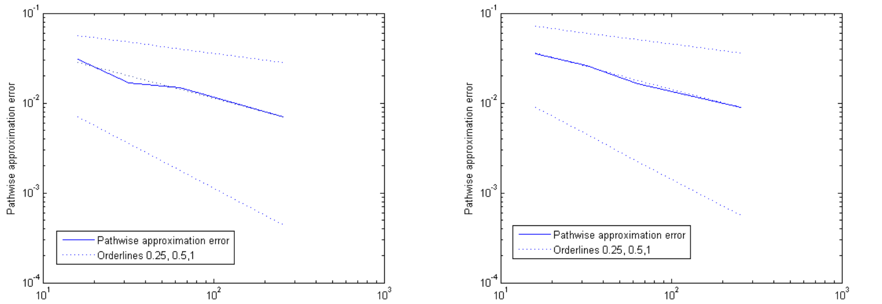
$$\begin{aligned} \|y_t^N\|_V &\le C \int_0^t \|P_N S_{t-s}\|_{L(V,V)} \cdot \|y_s^N + P_N O_s\|_V^3 ds \\ &\le C\big(\|y^N\|_{L^3(0,T,V)}^3 + \|P_N O\|_{L^3(0,T,V)}^3\big) \\ &\le C\|P_N O\|_{L^4(0,T,V)}^3, \end{aligned}$$

with a constant depending on T. This shows that $\|y_t^N\|_V$ is bounded and thus $\|X_t^N\|_V$ is bounded. Now our main result for the unique solution $X : [0,\pi]\times\Omega \to C^0([0,\pi])$ of the SPDE (38) implies for sufficiently small Δt

$$\sup_{0\le x\le\pi} |X_{m\Delta t}(\omega,x) - Y_m^{N,M}(\omega,x)| \le C(\omega)\left(N^{-\gamma} + (\Delta t)^{\vartheta}\right), \tag{42}$$

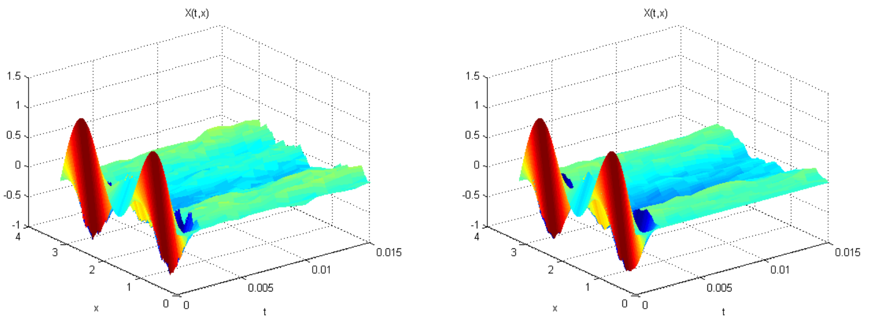
for $m = 1,...,M$, $\;M = \frac{1}{\Delta t}$, such that $\gamma \in (0,\frac{1}{2})$, $\vartheta \in (0,\frac{1}{4})$.

Let us now explain briefly how we implement our numerical results. The main



(a) (b)

Figure 3: Pathwise approximation error (43) against N for $N \in \{16, 32, ..., 256\}$ with convolution operator with kernel (34) for (a) $h = 0.1$ and (b) $h = 0.01$, for one random $\omega \in \Omega$.



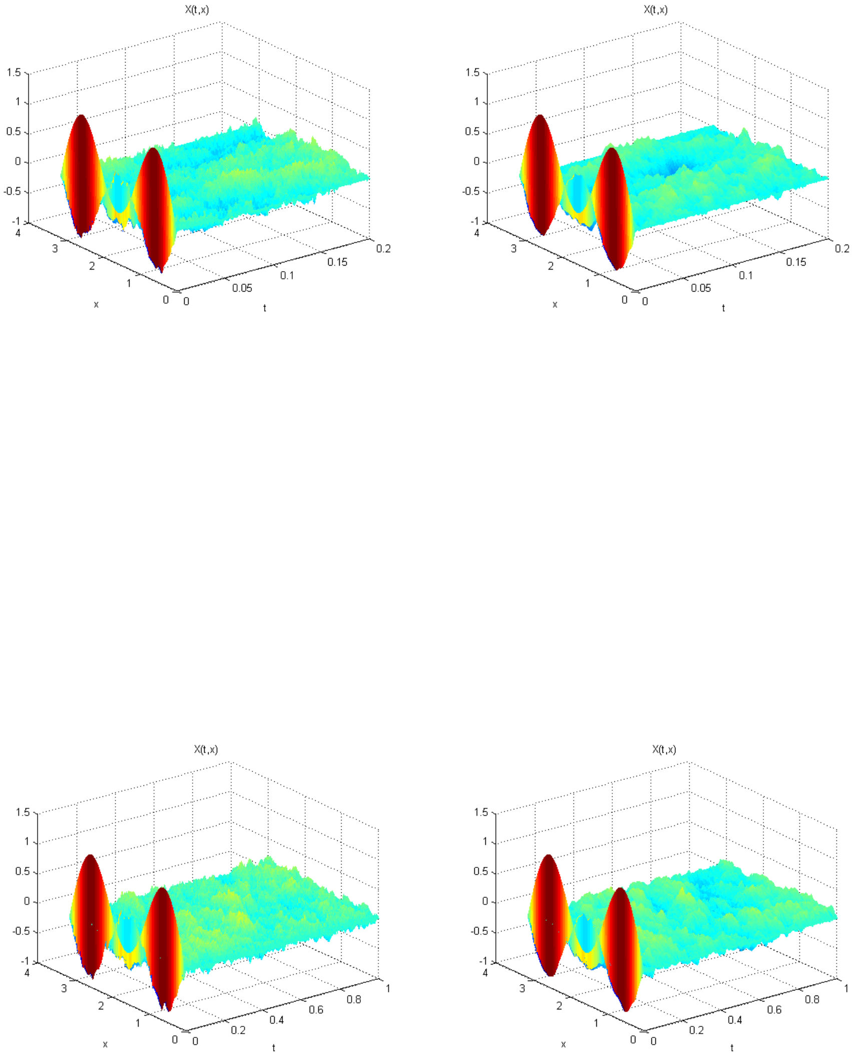
part is generating the Brownian motions $X = (X_1, X_2, \cdots, X_N)$ that are correlated such that $X \sim N(0, \Sigma)$, which $Cov(X_i, X_j) = \Sigma_{ij}$.

For this assume C is a $n \times m$ Matrix and let $Z = (Z_1, \cdots, Z_N)^T$, with $Z_i \sim N(0, 1)$, for $i = 1, \cdots, N$. Then obviously $C^T Z \sim N(0, C^T C)$. Therefore our aim clearly reduces to finding C such that $C^T C = \Sigma$, which can for instance be achieved by Cholesky. By using $\Delta t = \frac{T}{N^2}$, the solutions $X_t^N(\omega, x)$ of the finite dimensional SODEs converge uniformly in $t \in [0, 1]$ and $x \in [0, \pi]$ to the solution $X_t(\omega, x)$ of the stochastic evolution equation (38) with the rate $\frac{1}{2}$, as N goes to infinity for all $\omega \in \Omega$. In Figure 3 the path-wise approximation error

$$\sup_{0 \le x \le \pi} \sup_{0 \le m \le M} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)|, \tag{43}$$

is plotted against N, for $N \in \{16, 32, \cdots, 256\}$. As a replacement for the true unknown solution, we use a numerical approximation for N sufficiently large.

Figure 3 confirms that the order of convergence is $\frac{1}{2}$, as we expected from Theorem 10. Obviously, these are only two examples, but all out of a few hundred calculated examples behave similarly. Even their mean seem to behave with the same order of the error. Nevertheless, we did not calculate sufficiently many realizations to estimate the mean satisfactory, nor did we proof in the general setting, that the mean converges.



(a) (b)

Figure 4: $X_t(\omega, x)$, $x \in [0, \pi]$, $t \in (0, T)$ for $T \in \{3/200, 0.2, 1\}$, given by (36) for $h = 0.1$ with the covariance operator by (a) kernel (34) and (b) kernel (35), for one random $\omega \in \Omega$.

Finally, as an example in Figures 4, $X_t(\omega)$, is plotted for $t \in [0, T]$ for $T \in \{\frac{3}{200}, 0.2, 1\}$, for $h = 0.1$ and with convolution operator (33) given by kernel (34) and (35).

Acknowledgements M. Kamrani's work was in part supported by a grant from IPM (No. 93600053).

References

[1] A. Alabert, I. Gyöngy, *On numerical approximation of stochastic Burgers equation*, From stochastic calculus to mathematical finance, Springer, Berlin, 2006, 1–15.

[2] D. Blömker, *Nonhomogeneous noise and Q-Wiener processes on bounded domains.* Stochastic Anal. Appl. 23-2 (2005), 255–273.

[3] D. Blömker, A. Jentzen, *Galerkin Approximations for the Stochastic Burgers Equation*, SIAM J. Numer. Anal. 51-1 (2013), 694–715.

[4] D. Blömker, M. Kamrani, S.M. Hosseini, *Full discretization of Stochastic Burgers Equation with correlated noise*, IMA J. Numer. Anal. (2013) 33(3), 825-848.

[5] S. Cox, J.A. van Neerven, *Pathwise Hölder Convergence of the Implicit Euler Scheme for Semi-Linear SPDEs with Multiplicative Noise*, http://arxiv.org/pdf/1201.4465v1.pdf.

[6] G. Da Prato, D. Gatarek, *Stochastic Burgers equation with correlated noise*, Stochastics Rep. 52 (1995), 29–41.

[7] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, vol. 44 of Encyclopedia of of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

[8] I. Gyöngy, D. Nualart, *On the stochastic Burgers equation in the real line*, Ann. Probab. 27 (1999), 782–802.

[9] I. Gyöngy, *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise II*, Potential Anal. 11 (1999), 1–37.

[10] E. Hausenblas, *Numerical analysis of semilinear stochastic evolution equations in Banach spaces*, J. Comput. Appl. Math. 147(2) (2002), 485–516.

[11] A. Jentzen, *Pathwise Numerical Approximations of SPDEs with Additive Noise under Non-global Lipschitz Coefficients*, Potential Anal. 31 (2009), 375–404.

[12] A. Jentzen, P. Kloeden, G. Winkel, *Efficient simulation of nonlinear parabolic Spdes with additive noise*, Annals of Applied Probability. 21(3) (2011), 908–950.

[13] P.E. Kloeden, S. Shott, *Linear-implicit strong schemes for Itô-Galerkin approximations of stochastic PDEs*, J. Appl. Math. Stochastic Anal. 14(1), (2001), 47–53. Special issue: Advances in applied stochastics.

[14] Di Liu, *Convergence of the spectral method for stochastic Ginzburg equation driven by space-time white noise*, Comm. Math. Sci 1(2), (2003), 361–375.

[15] G. Lord, T. Shardlow, *Postprocessing for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal. 45(2) (2007), 870–889.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44. New York, Springer-Verlag (1983).

[17] T. Shardlow, *Numerical methods for stochastic parabolic PDEs*, Numer. Funct. Anal. Optim. 20 (1999), 121–145.

[18] H. Yoo, *Semi-discretization of stochastic partial differential equations on R by a finite difference method*, Mathematics of computation. 69 (1999), 653–666.