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Convergence analysis of an adaptive interior penalty discontinuous Galerkin method for the biharmonic problem

Abstract: For the biharmonic problem, we study the convergence of adaptive C^0-Interior Penalty Discontinuous Galerkin (C^0-IPDG) methods of any polynomial order. We note that C^0-IPDG methods for fourth order elliptic boundary value problems have been suggested in [9, 17], whereas residual-type *a posteriori* error estimators for C^0-IPDG methods applied to the biharmonic equation have been developed and analyzed in [8, 18]. Following the convergence analysis of adaptive IPDG methods for second order elliptic problems [6], we prove a contraction property for a weighted sum of the C^0-IPDG energy norm of the global discretization error and the estimator. The proof of the contraction property is based on the reliability of the estimator, a quasi-orthogonality result, and an estimator reduction property. Numerical results are given that illustrate the performance of the adaptive C^0-IPDG approach.

Keywords: C^0-interior penalty discontinuous Galerkin method, biharmonic equation, residual type *a posteriori* error estimator, quasioptimal convergence.

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For second order elliptic boundary value problems, adaptive finite element methods (AFEM) are well established numerical tools that have been intensively studied in the literature (cf., e.g., [1, 3, 4, 16, 25, 28] and the references therein). The convergence analysis of AFEM for conforming discretizations has been initiated in [14] (cf. also [24]) with the most far reaching result so far given in [13]. Nonconforming discretizations based on the lowest order Crouzeix–Raviart elements have been addressed in [11], whereas for Interior Penalty Discontinuous Galerkin (IPDG) methods we refer to [6]. However, considerably less work has been devoted to AFEM for nonconforming discretizations of fourth order elliptic boundary value problems. As far as IPDG approaches are concerned, C^0-IPDG methods have been suggested in [15] (cf. also [30]) and subsequently analyzed in [9] focusing on an *a priori* error analysis. An *a posteriori* error analysis of quadratic C^0-IPDG methods based on residual-type *a posteriori* error estimators has been performed in [8], however, without addressing the issue of convergence.

The purpose of this contribution is to provide a convergence analysis of C^0-IPDG methods of any polynomial order for the biharmonic problem. The residual *a posteriori* error estimator consists of element and edge residuals and is a generalization to arbitrary polynomial degree $k \geqslant 2$ of the one considered in [8] for the case $k = 2$. The reliability of the estimator can be shown by similar techniques as in [8]. Together with the standard estimator reduction for Dörfler marking (Lemma 4.1) and a quasi-orthogonality result (Theorem 5.3) this results in a contraction property for a weighted sum of the C^0-IPDG energy norm of the global discretization error and the estimator (Theorem 6.1). We note that in case of IPDG approximations of second order elliptic boundary value problems a contraction property for the IPDG energy norm of the error has been established in [19, 22] based on the reliability of the estimator, its local efficiency up to data oscillations, as well as a quasi-orthogonality property. The proof of the local efficiency relies on the interior node property. In contrast to these results, the contraction property which will be established here does neither require the

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interior node property nor does it involve marking by data oscillations. Although the basic ingredients for the proof (reliability, estimator reduction, and quasi-orthogonality) are the same as in [6], their realizations are not straightforward and require to take into account the particular structure of the estimator. To our best knowledge this is the first contribution containing numerical results for high order IPDG approximations of the biharmonic problem that confirm the theoretically achievable quasi-optimal convergence rates.

1 C^0-interior penalty Discontinuous Galerkin method

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. For a given function $f \in L^2(\Omega)$ we consider the biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega \tag{1.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{1.1b}$$

We use standard notation from Lebesgue and Sobolev space theory [27]. In particular, $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$ stand for the inner product on $L^2(\Omega)$ and the associated norm. Moreover, $H^k(\Omega)$, $k \in \mathbb{N}$, refers to the Sobolev space with norm $\|\cdot\|_{k,\Omega}$ and seminorm $|\cdot|_{k,\Omega}$, whereas $H_0^k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the topology induced by $\|\cdot\|_{k,\Omega}$. The Sobolev spaces with broken index $s \in \mathbb{R}_+$ can be defined by interpolation and are referred to as $H^s(\Omega)$.

A weak formulation of (1.1) requires the computation of $u \in V := H_0^2(\Omega)$ such that

$$a(u,v) = (f,v)_{0,\Omega}, \quad v \in V \tag{1.2}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(v,w) = (D^2 v, D^2 w)_{0,\Omega} := \sum_{|\beta|=2} (D^\beta v, D^\beta w)_{0,\Omega}, \quad v,w \in V. \tag{1.3}$$

Let $\mathcal{T}_h(\Omega)$ be a geometrically conforming simplicial triangulation of Ω. For $D \subseteq \overline{\Omega}$, we denote by $\mathcal{E}_h(D)$ the set of edges of $\mathcal{T}_h(\Omega)$ in D. For $T \in \mathcal{T}_h(\Omega)$ and $E \in \mathcal{E}_h(\overline{\Omega})$ we denote by h_T and h_E the diameter of T and the length of E, and we set $h := \max\{h - T \mid T \in \mathcal{T}_h(\Omega)\}$. For two quantities A and B we write $A \lesssim B$, if there exists a constant $C > 0$ independent of h such that $A \leqslant CB$.

Denoting by $P_k(T)$, $k \in \mathbb{N}$, the linear space of polynomials of degree $\leqslant k$ on T, for $k \geqslant 2$ we refer to

$$V_h := \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h(\Omega)\} \tag{1.4}$$

as the finite element space of Lagrangian finite elements of type k (cf., e.g., [7]). For $D \subseteq \overline{\Omega}$, we denote by $\mathcal{N}_h(D)$ as the set of nodal points in D such that any $v_h \in V_h$ is uniquely determined by its degrees of freedom $v_h(a)$, $a \in \mathcal{N}_h(\Omega)$.

We note that $V_h \not\subset V$ and hence, V_h is a nonconforming finite element space for the approximation of the biharmonic problem (1.2). In particular, for $v_h \in V_h$ the normal derivative $\partial v_h/\partial n$ exhibits jumps across interior edges $E \in \mathcal{E}_h^\Omega$. After numbering of the elements $T \in \mathcal{T}_h(\Omega)$, for $E \in \mathcal{E}_h(\Omega)$, $E = T_i \cap T_j$, $i > j$, we set $T_E^+ := T_i$, $T_E^- := T_j$, and for $E \in \mathcal{E}_h^\Gamma$, $E = T_\ell \cap \Gamma$, we set $T_E := T_\ell$. Then, for $1 \leqslant \nu \leqslant 2$ we define averages and jumps according to

$$\left\{\frac{\partial^\nu v_h}{\partial n^\nu}\right\}_E := \begin{cases} \frac{1}{2}\left(\frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E^+} + \frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E^-}\right), & E \in \mathcal{E}_h(\Omega) \\ \frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E}, & E \in \mathcal{E}_h(\Gamma) \end{cases} \tag{1.5a}$$

$$\left[\frac{\partial^\nu v_h}{\partial n^\nu}\right]_E := \begin{cases} \frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E^+} - \frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E^-}, & E \in \mathcal{E}_h(\Omega) \\ \frac{\partial^\nu v_h}{\partial n^\nu}\Big|_{E\cap T_E}, & E \in \mathcal{E}_h(\Gamma) \end{cases} \tag{1.5b}$$

where n is the unit normal vector on E pointing in the direction from T_E^- to T_E^+ for $E \in \mathcal{E}_h(\Omega)$ and the exterior normal vector for $E \in \mathcal{E}_h(\Gamma)$.

We further refer to $M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$ as the set of matrix-valued functions on $\mathcal{T}_h(\Omega)$ such that for $W_h \in M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$ the restriction $W_h|_T$, $T \in \mathcal{T}_h(\Omega)$, is a 2×2 matrix with entries that are polynomials of order k.

Given a penalty parameter $\alpha > 1$, the C^0-IPDG method for the approximation of (1.2) requires the computation of $u_h \in V_h$ such that

$$a_h^{IP}(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{1.6}$$

Here, the mesh-dependent bilinear form $a_h^{IP}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is given according to

$$\begin{aligned} a_h^{IP}(v_h, w_h) := & \sum_{T\in\mathcal{T}_h(\Omega)} (D^2 v_h, D^2 w_h)_{0,T} + \sum_{E\in\mathcal{E}_h(\Omega)} \left(\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}_E, \left[\frac{\partial w_h}{\partial n} \right]_E \right)_{0,E} \\ & + \sum_{E\in\mathcal{E}_h(\Omega)} \left(\left[\frac{\partial v_h}{\partial n} \right]_E, \left\{ \frac{\partial^2 w_h}{\partial n^2} \right\}_E \right)_{0,E} + \alpha \sum_{E\in\mathcal{E}_h(\overline{\Omega})} h_E^{-1} \left(\left[\frac{\partial v_h}{\partial n} \right]_E, \left[\frac{\partial w_h}{\partial n} \right]_E \right)_{0,E}. \end{aligned} \tag{1.7}$$

We note that $a_h^{IP}(\cdot,\cdot)$ is not well defined for $v, w \in V$ which can be cured in terms of a lifting operator $L : L^2(\mathcal{E}_h(\overline{\Omega}), \mathbb{R}^2) \to M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$ given by

$$(L(q), W_h)_{0,\Omega} := \sum_{E\in\mathcal{E}_h(\overline{\Omega})} ([n \cdot q]_E, \{n \cdot W_h n\}_E)_{0,E} \tag{1.8}$$

for $W_h \in M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$. We refer to [20, 21] for lifting operators in case of DG approximations of second order problems and to [18] for a lifting operator associated with IPDG approximations of the biharmonic problem. The bilinear form $a_h^{IP}(\cdot,\cdot)$ can be extended to $V + V_h$ by means of

$$\begin{aligned} a_h^{IP}(v, w) := & \sum_{T\in\mathcal{T}_h(\Omega)} (D^2 v, D^2 w)_{0,T} + \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla w), D^2 v)_{0,T} \\ & + \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla v), D^2 w)_{0,T} + \alpha \sum_{E\in\mathcal{E}_h(\overline{\Omega})} h_E^{-1} \left(\left[\frac{\partial v}{\partial n} \right]_E, \left[\frac{\partial w}{\partial n} \right]_E \right)_{0,E} \end{aligned} \tag{1.9}$$

where with a slight abuse of notation we have also used $a_h^{IP}(\cdot,\cdot)$ for that extension.

The lifting operator satisfies the following stability estimate.

Theorem 1.1. *Let $L : L^2(\mathcal{E}_h(\overline{\Omega}), \mathbb{R}^2) \to M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$ be the lifting operator as given by* (1.8). *Then, there exists a positive constant C_L, depending only on the local geometry of the triangulation and on the polynomial order k, such that there holds*

$$\|L(q)\|_{0,\Omega}^2 \le C_L \sum_{E\in\mathcal{E}_h(\overline{\Omega})} h_E^{-1} \, \|[n \cdot q]_E\|_{0,E}^2, \quad q \in L^2(\mathcal{E}_h(\overline{\Omega}), \mathbb{R}^2). \tag{1.10}$$

Proof. For $q \in L^2(\mathcal{E}_h(\overline{\Omega}), \mathbb{R}^2)$ and $W_h \in M_h(\mathcal{T}_h(\Omega); \mathbb{R}^{2\times 2})$ we have

$$\|L(q)\|_{0,\Omega} = \sup_{\|W_h\|_{0,\Omega}\le 1} |(L(q), W_h)_{0,\Omega}|.$$

In view of (1.8) we find

$$|(L(q), W_h)_{0,\Omega}| \le \left(\sum_{E\in\mathcal{E}_h(\overline{\Omega})} \|[n \cdot q]_E\|_{0,E}^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}_h(\Omega)} \|n_{\partial T} \cdot W_h n_{\partial T}\|_{0,\partial T}^2 \right)^{1/2}$$

where $n_{\partial T}$ is the exterior unit normal on ∂T. Then, the trace inequality (cf., e.g., [29]):

$$\|n_{\partial T} \cdot W_h n_{\partial T}\|_{0,\partial T} \lesssim k \, h_T^{-1/2} \, \|W_h\|_{0,T}, \quad T \in \mathcal{T}_h(\Omega)$$

gives the assertion. □

On $V + V_h$ we introduce the mesh-dependent C^0-IPDG norm

$$\|v\|^2_{2,h,\Omega} := \sum_{T\in\mathcal{T}_h(\Omega)} |v|^2_{2,T} + \sum_{E\in\mathcal{E}_h(\overline{\Omega})} \frac{\alpha}{h_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|^2_{0,E}, \quad v \in V + V_h \tag{1.11}$$

where $|\cdot|^2_{2,T}$ stands for

$$|\cdot|^2_{2,T} := \sum_{|\beta|=2} \|D^\beta \cdot\|^2_{0,T}, \quad T \in \mathcal{T}_h. \tag{1.12}$$

It has been shown in [9] that for sufficiently large penalty parameter α there exists a positive constant $\gamma < 1$ such that

$$a_h^{IP}(v,v) \geqslant \gamma \|v\|^2_{2,h,\Omega}, \quad v \in V + V_h \tag{1.13}$$

whereas there exists a constant $C_1 > 1$ such that for any $\alpha \geqslant 1$

$$a_h^{IP}(v,w) \leqslant C_1 \|v\|_{2,h,\Omega} \|w\|_{2,h,\Omega}, \quad v,w \in V + V_h. \tag{1.14}$$

In particular, it follows from (1.13) and (1.14) that (1.6) admits a unique solution $u_h \in V_h$.

2 Residual-type *a posteriori* error estimator and its reliability

For adaptive mesh refinement we consider the residual-type *a posteriori* error estimator

$$\eta_h^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{E\in\mathcal{E}_h(\Omega)} \eta_E^2 + \sum_{E\in\mathcal{E}_h(\overline{\Omega})} \eta_{E,c}^2 \tag{2.1}$$

where the element residuals η_T, $T \in \mathcal{T}_h(\Omega)$, and the edge residuals η_E, $E \in \mathcal{E}_h(\Omega)$, as well as $\eta_{E,c}$, $E \in \mathcal{E}_h(\overline{\Omega})$, are given by

$$\eta_T^2 := h_T^4 \|f - \Delta^2 u_h\|^2_{0,T}, \quad T \in \mathcal{T}_h(\Omega) \tag{2.2a}$$

$$\eta_E^2 := h_E \left\| \left[\frac{\partial^2 u_h}{\partial n^2} \right]_E \right\|^2_{0,E} + h_E^3 \left\| \left[\frac{\partial}{\partial n} \Delta u_h \right]_E \right\|^2_{0,E}, \quad E \in \mathcal{E}_h(\Omega) \tag{2.2b}$$

$$\eta_{E,c}^2 := \alpha\, \hat{\eta}_{E,c}^2, \quad \hat{\eta}_{E,c}^2 := h_E^{-1} \left\| \left[\frac{\partial u_h}{\partial n} \right]_E \right\|^2_{0,E}, \quad E \in \mathcal{E}_h(\overline{\Omega}). \tag{2.2c}$$

For notational convenience we set

$$\eta_{h,c}^2 := \alpha\, \hat{\eta}_{h,c}^2, \quad \hat{\eta}_{h,c}^2 := \sum_{E\in\mathcal{E}_h(\overline{\Omega})} \eta_{E,c}^2. \tag{2.3}$$

The term $\hat{\eta}_{h,c}$ represents an upper bound for the consistency error

$$\inf_{v_h\in V_h^c} a_h^{IP}(u_h - v_h, u_h - v_h)$$

where $V_h^c \subset H_0^2(\Omega)$ stands for the C^1 conforming finite element space generated by the Argyris elements of the so-called TUBA family [2]. We use the enrichment operator (or recovery operator) $E_h : V_h \to V_h^c$ from [9] which is defined by averaging according to

$$N(E_h v_h) = |\omega_h^p|^{-1} \sum_{T\in\omega_h^p} (N v_h|_T), \quad v_h \in V_h$$

where p is any nodal point for V_h^c, N is any nodal variable at p, and $\omega_h^p := \bigcup\{T \in \mathcal{T}_h(\Omega) \mid \{p\} \cap \mathcal{N}_h(T) \neq \varnothing\}$. It follows from the mapping properties of E_h established in [9] that there exists a constant $C_{nc} > 0$, depending only on the local geometry of $\mathcal{T}_h(\Omega)$, such that

$$\inf_{v_h \in V_h^c} a_h^{IP}(u_h - v_h, u_h - v_h) \leqslant a_h^{IP}(u_h - E_h(u_h), u_h - E_h(u_h)) \leqslant C_{nc}\, \hat{\eta}_{h,c}^2. \tag{2.4}$$

The following result shows that η_h^2 provides an upper bound for the IPDG energy norm of the discretization error $u - u_h$. It can be shown by using similar techniques as in [8].

Theorem 2.1. *Let $u \in V$ and $u_h \in V_h$ be the unique solution of* (1.2) *and* (1.6), *and let η_h be given by* (2.1) *and* (2.2). *Then, there exists a constant $C_R > 0$, depending only on the local geometry of $\mathcal{T}_h$ and on k, such that*

$$a_h^{IP}(u - u_h, u - u_h) \leqslant C_R\, \eta_h^2. \tag{2.5}$$

3 Refinement strategy and estimator reduction

As a marking strategy for adaptive refinement we use Dörfler marking [14]. To this end, we reformulate the estimator η_h (cf. (2.1)) according to

$$\begin{aligned}
\eta_h &= \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \hat{\eta}_T^2 \Big)^{1/2} \\
\hat{\eta}_T^2 &:= h_T^4\, \|f - \Delta^2 u_h\|_{0,T}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_h(\partial T \cap \Omega)} \Big(\alpha h_E^{-1} \Big\| \Big[\frac{\partial u_h}{\partial n} \Big]_E \Big\|_{0,E}^2 \\
&\quad + h_E \Big\| \Big[\frac{\partial^2 u_h}{\partial n^2} \Big]_E \Big\|_{0,E}^2 + h_E^3 \Big\| \Big[\frac{\partial}{\partial n} \Delta u_h \Big]_E \Big\|_{0,E}^2 \Big) + \sum_{E \in \mathcal{E}_h(\partial T \cap \Gamma)} \alpha\, h_E^{-1} \Big\| \Big[\frac{\partial u_h}{\partial n} \Big]_E \Big\|_{0,E}^2 .
\end{aligned}$$

Then, given a constant $0 < \Theta < 1$, we compute a set $\mathcal{M}$ of elements $T \in \mathcal{T}_h(\Omega)$ such that

$$\Theta\, \eta_h^2 \leqslant \sum_{T \in \mathcal{M}} \hat{\eta}_T^2. \tag{3.1}$$

After having determined the set $\mathcal{M}$, a refined triangulation is generated by a recursive application of newest vertex bisection. Assuming a conforming initial triangulation that satisfies a certain labeling condition, this leads to quasi-optimal cardinality (cf. Section 4 in [13] and Subsection 3.4 in [6]). In particular, there exist constants $0 < \beta_1 < \beta_2$, depending only on the initial triangulation, such that for each triangle T of refinement level ℓ it holds $\beta_1 2^{-\ell/2} \leqslant h_T \leqslant \beta_2 2^{-\ell/2}$. Hence, if $\mathcal{T}_h(\Omega)$ is obtained from $\mathcal{T}_H(\Omega)$ by newest vertex bisection, for $T \in \mathcal{T}_H(\Omega)$ and $T \supset T' \in \mathcal{T}_h(\Omega)$ we have

$$\varkappa_1 h_{T'} \leqslant H_T \leqslant \varkappa_2 h_{T'} \tag{3.2}$$

where $\varkappa_1 := 2^{1/2}\beta_1/\beta_2$ and $\varkappa_2 := 2^{1/2}\beta_2/\beta_1$.

As in [13] (cf. also [6]), we can prove the following estimator reduction property.

Lemma 3.1. *Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, let $u_h \in V_h$, $u_H \in V_H$, and η_h, η_H be the associated C^0-IPDG solutions and error estimators, respectively, and let $\Theta > 0$ be the universal constant from* (3.1). *Then, for any $\tau > 0$ there exists a constant $C_\tau > 1$, depending only on the local geometry of the triangulations and on k, such that for $\varkappa(\Theta) := (1 + \tau)(1 - 2^{-1/2})\Theta$ there holds*

$$\eta_h^2 \leqslant \varkappa(\Theta)\, \eta_H^2 + C_\tau\, \|u_h - u_H\|_{2,h,\Omega}^2. \tag{3.3}$$

Proof. By definition of η_h and taking into account the inverse estimates

$$\|\Delta^2(u_h - u_H)\|_{0,T} \leqslant C_{\mathrm{inv}}^{(1)}\, k^4\, h_T^{-2}\, \|D^2(u_h - u_H)\|_{0,T}, \quad T \in \mathcal{T}_h(\Omega)$$
$$\left\|\frac{\partial^2(u_h - u_H)}{\partial n_{E\cap\partial T}^2}\right\|_{0,E} \leqslant C_{\mathrm{inv}}^{(2)}\, k^2\, h_E^{-1}\, \left\|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\right\|_{0,E}, \quad E \in \mathcal{E}_h(T)$$
$$\left\|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta(u_h - u_H)\right\|_{0,E} \leqslant C_{\mathrm{inv}}^{(3)}\, k^4\, h_E^{-2}\, \left\|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\right\|_{0,E}, \quad E \in \mathcal{E}_h(T)$$

where $C_{\mathrm{inv}}^{(i)}$, $i = 1, 2, 3$, are positive constants, depending only on the local geometry of the triangulations, we have

$$h_T^2\|f - \Delta^2 u_h\|_{0,T} \leqslant h_T^2\,\left(\|f - \Delta^2 u_H\|_{0,T} + C_{\mathrm{inv}}^{(1)} k^4 h_T^{-2}\|D^2(u_h - u_H)\|_{0,T}\right) \tag{3.4a}$$
$$h_E^{1/2}\left\|\frac{\partial^2 u_h}{\partial n_{E\cap\partial T}^2}\right\|_{0,E} \leqslant h_E^{1/2}\left\|\frac{\partial^2 u_H}{\partial n_{E\cap\partial T}^2}\right\|_{0,E} + C_{\mathrm{inv}}^{(2)} k^2 h_E^{-1/2}\left\|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\right\|_{0,E} \tag{3.4b}$$
$$h_E^{3/2}\left\|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta u_h\right\|_{0,E} \leqslant h_E^{3/2}\left\|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta u_H\right\|_{0,E} + C_{\mathrm{inv}}^{(3)} k^4 h_E^{-1/2}\left\|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\right\|_{0,E}. \tag{3.4c}$$

By an application of Young's inequality, in view of (1.11), (3.2) and observing $\alpha \geqslant 1$, from (3.4) and the marking and refinement strategy we deduce the existence of $C_{\mathrm{ER}} > 1$, depending only on the local geometry of the triangulations and on k, such that for $\tau > 0$ there holds

$$\eta_h^2 \leqslant (1+\tau)(1 - 2^{-1/2})\Theta\left(\sum_{T\in\mathcal{T}_H(\Omega)} H_T^4\|f - \Delta^2 u_H\|_{0,T}^2 + \sum_{E\in\mathcal{E}_H(\Omega)}\left(H_E\left\|\left[\frac{\partial^2 u_H}{\partial n}\right]_E\right\|_{0,E}^2 + H_E^3\left\|\left[\frac{\partial}{\partial n_E}\Delta u_H\right]_E\right\|_{0,E}^2\right)\right)$$
$$+ \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \alpha\, H_E^{-1}\,\left\|\left[\frac{\partial u_H}{\partial n}\right]_E\right\|_{0,E}^2 + (1+\tau^{-1})\, C_{\mathrm{ER}}\,\|u_h - u_H\|_{2,h,\Omega}^2 \tag{3.5}$$

which gives the assertion with $C_\tau := (1+\tau^{-1})\, C_{\mathrm{ER}}$. □

Remark 3.1. If we choose $\tau = 2^{-1/2}$ and observe $0 < \Theta \leqslant 1$, we have

$$\varkappa(\Theta) = \frac{1}{2}\,\Theta \leqslant \frac{1}{2}. \tag{3.6}$$

4 Quasi-orthogonality

As a further significant ingredient of the convergence analysis, in this section we prove quasi-orthogonality of the C^0-IPDG approach. We first provide a mesh perturbation result in Subsection 4.1 and then establish quasi-orthogonality in Subsection 4.2.

4.1 Mesh perturbation result

In the convergence analysis of IPDG methods for second order elliptic boundary value problems, mesh perturbation results estimating the coarse mesh error in the fine mesh energy norm from above by its coarse mesh energy norm have played a central role in the convergence analysis as a prerequisite for establishing a quasi-orthogonality result (cf., e.g., [6, 19, 22]). Here, we provide the following mesh perturbation result.

Lemma 4.1. *Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$. Then, there exists a constant $C_P > 0$, depending only on the local geometry of the triangulations and on k, such that for any $\varepsilon > 0$ and $v \in V + V_H$ there holds*

$$a_h^{IP}(v,v) \leqslant (1+\varepsilon)\, a_H^{IP}(v,v) + \frac{C_P}{\varepsilon}\left(\sum_{E\in\mathcal{E}_h(\bar\Omega)} h_E^{-1}\,\left\|\left[\frac{\partial v}{\partial n}\right]_E\right\|_{0,E}^2 + \sum_{E\in\mathcal{E}_H(\bar\Omega)} H_E^{-1}\,\left\|\left[\frac{\partial v}{\partial n}\right]_E\right\|_{0,E}^2\right). \tag{4.1}$$

Proof. For $v \in V + V_H$ we have

$$a_h^{IP}(v,v) = \sum_{T\in\mathcal{T}_h(\Omega)} \|D^2 v\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h(\overline{\Omega})} \frac{\alpha}{h_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E}^2 + 2 \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla v), D^2 v)_{0,T}. \tag{4.2}$$

Obviously, there holds

$$\sum_{T\in\mathcal{T}_h(\Omega)} \|D^2 v\|_{0,T}^2 = \sum_{T\in\mathcal{T}_H(\Omega)} \|D^2 v\|_{0,T}^2. \tag{4.3}$$

Moreover, in view of (3.2) we have

$$\sum_{E\in\mathcal{E}_h(\overline{\Omega})} \frac{1}{h_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E,h}^2 \leqslant \varkappa_2^{-1} \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \frac{1}{H_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E,H}^2. \tag{4.4}$$

Using (4.3) and (4.4) in (4.2) and observing $\varkappa_2^{-1} < 1$, we find

$$\begin{aligned} a_h^{IP}(v,v) &\leqslant a_H^{IP}(v,v) + (\varkappa_2^{-1} - 1) \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \frac{\alpha}{H_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E}^2 + 2 \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla v), D^2 v)_{0,T} - 2 \sum_{T\in\mathcal{T}_H(\Omega)} (L(\nabla v), D^2 v)_{0,T} \\ &\leqslant a_H^{IP}(v,v) + 2 \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla v), D^2 v)_{0,T} - 2 \sum_{T\in\mathcal{T}_H(\Omega)} (L(\nabla v), D^2 v)_{0,T}. \end{aligned} \tag{4.5}$$

Using Young's inequality, (1.10), and (1.13), we find

$$\begin{aligned} 2 \left| \sum_{T\in\mathcal{T}_h(\Omega)} (L(\nabla v), D^2 v)_{0,T} \right| &\leqslant 2 \sum_{T\in\mathcal{T}_h(\Omega)} \|L(\nabla v)\|_{0,T} \|D^2 v\|_{0,T} \leqslant \frac{\gamma}{\varepsilon} \|L(\nabla v)\|_{0,\Omega}^2 + \frac{\varepsilon}{2\gamma} \sum_{T\in\mathcal{T}_h(\Omega)} \|D^2 v\|_{0,T}^2 \\ &\leqslant 2\,\frac{C_L \gamma}{\varepsilon} \sum_{E\in\mathcal{E}_h(\overline{\Omega})} \frac{1}{h_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E,h}^2 + \frac{\varepsilon}{2}\, a_h^{IP}(v,v). \end{aligned} \tag{4.6}$$

Further, taking Young's inequality and (1.10) as well as (4.3) into account, it follows that

$$2 \left| \sum_{T\in\mathcal{T}_H(\Omega)} (L(\nabla v), D^2 v)_{0,T} \right| \leqslant 2\,\frac{C_L \gamma}{\varepsilon} \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \frac{1}{H_E} \left\| \left[\frac{\partial v}{\partial n} \right]_E \right\|_{0,E,h}^2 + \frac{\varepsilon}{2}\, a_h^{IP}(v,v). \tag{4.7}$$

Finally, using (4.6) and (4.7) in (4.5), we deduce (4.1) with $C_P := 2C_L\gamma$. □

4.2 Quasi-orthogonality

The quasi-orthogonality result can be derived using the conforming approximations $u_H^c \in V_H^c$, $u_h^c \in V_h^c$ of (1.2) which are given as the unique solutions of

$$\begin{aligned} a(u_H^c, v_H^c) &= (f, v_H^c), \quad v_H^c \in V_H^c \\ a(u_h^c, v_h^c) &= (f, v_h^c), \quad v_h^c \in V_h^c. \end{aligned} \tag{4.8}$$

In particular, we assume that the data of the problem, i.e., the domain Ω and the right-hand side f, are such that the solution u of (1.2) satisfies $u \in H^{2+\varkappa}(\Omega) \cap V$ for some $\varkappa > 1/2$. Then there exists a constant $C_{ap} > 0$, independent of H and h, such that the following *a priori* error estimates hold true

$$\|u - u_H^c\|_{2,\Omega} \leqslant C_{ap} H^{\varkappa} \|u\|_{2+\varkappa,\Omega}, \qquad \|u - u_h^c\|_{2,\Omega} \leqslant C_{ap} h^{\varkappa} \|u\|_{2+\varkappa,\Omega}. \tag{4.9}$$

Lemma 4.2. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the C^0-IPDG solutions of* (1.6) *and error estimators, respectively. Moreover, let $u_h^c \in V_h^c$ and $u_H^c \in V_H^c$ be the conforming approximations of* (1.2) *according to* (4.8)*. Then, for $u_h^{nc} := u_h - u_h^c$ and $u_H^{nc} := u_H - u_H^c$ it holds*

$$\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2 \leqslant \frac{2C_{nc}}{\gamma\alpha} \left(\eta_h^2 + \eta_H^2 \right) \tag{4.10}$$

where C_{nc} is the constant from (2.4)*.*

Proof. Due to (1.13) we have

$$\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2 \leqslant 2\left(\|u_h^{nc}\|_{2,h,\Omega}^2 + \|u_H^{nc}\|_{2,h,\Omega}^2\right) \leqslant \frac{2}{\gamma}\left(a_h^{IP}(u_h^{nc}, u_h^{nc}) + a_h^{IP}(u_H^{nc}, u_H^{nc})\right). \tag{4.11}$$

On the other hand, in view of (2.4) it holds

$$\frac{2}{\gamma}\, a_h^{IP}(u_h^{nc}, u_h^{nc}) \leqslant \frac{2C_{nc}}{\gamma}\, \hat{\eta}_{h,c}^2 \leqslant \frac{2C_{nc}}{\gamma\alpha}\, \eta_{h,c}^2 \leqslant \frac{2C_{nc}}{\gamma\alpha}\, \eta_h^2. \tag{4.12}$$

Likewise, taking

$$\sum_{E\in\mathcal{E}_h(\bar{\Omega}} \frac{1}{h_E} \left\| \left[\frac{\partial u_H^{nc}}{\partial n} \right]_E \right\|_{0,E}^2 \leqslant \varkappa_2^{-1} \sum_{E\in\mathcal{E}_H(\bar{\Omega})} \frac{1}{H_E} \left\| \left[\frac{\partial u_H^{nc}}{\partial n} \right]_E \right\|_{0,E}^2$$

into account, we find

$$\frac{2}{\gamma}\, a_h^{IP}(u_H^{nc}, u_H^{nc}) \leqslant \frac{2\varkappa_2^{-1} C_{nc}}{\gamma\alpha}\, \eta_H^2. \tag{4.13}$$

Noting that $\varkappa_2^{-1} < 1$, we conclude by using (4.12) and (4.13) in (4.11). □

The quasi-orthogonality result reads as follows.

Theorem 4.1. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the associated C^0-IPDG solutions of* (1.6) *and error estimators, respectively, and let $e_h := u - u_h$ and $e_H := u - u_H$ be the fine and coarse mesh errors. Further, assume that* (4.9) *holds true. Then, for any* $0 < \varepsilon < 1$ *and sufficiently small mesh width H there exists a constant $C_Q > 0$, independent of H and h, such that it holds*

$$a_h^{IP}(e_h, e_h) \leqslant (1+\varepsilon)\, a_H^{IP}(e_H, e_H) - \frac{\gamma}{4}\|u_h - u_H\|_{2,h,\Omega}^2 + \frac{C_Q}{\alpha\varepsilon}\left(\eta_{h,c}^2 + \eta_{H,c}^2\right). \tag{4.14}$$

The proof of Theorem 4.1 will be provided by a series of lemmas.

Lemma 4.3. *Under the assumptions of Theorem* 4.1, *for any* $0 < \varepsilon < 1$ *it holds*

$$a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) \leqslant (1+\varepsilon)\, a_H^{IP}(e_H, e_H) + \left(1 + \frac{4}{\varepsilon}\right) \frac{C_P + 2C_1C_{nc}}{\alpha\gamma}\, (\eta_{h,c}^2 + \eta_{H,c}^2). \tag{4.15}$$

Proof. Using $u_h + u_h^c - u_H^c = u_H - u_H^{nc} + u_h^{nc}$, (1.14), and Young's inequality twice, it follows that for any $0 < \varepsilon_2 < 1$ it holds

$$\begin{aligned} a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) &= a_h^{IP}(e_H - (u_h^{nc} - u_H^{nc}), e_H - (u_h^{nc} - u_H^{nc})) \\ &= a_h^{IP}(e_H, e_H) - 2a_h^{IP}(e_H, u_h^{nc} - u_H^{nc}) + a_h^{IP}(u_h^{nc} - u_H^{nc}, u_h^{nc} - u_H^{nc}) \\ &\leqslant a_h^{IP}(e_H, e_H) + 2a_h^{IP}(e_H, e_h)^{1/2} a_h^{IP}(u_h^{nc} - u_H^{nc}, u_h^{nc} - u_H^{nc})^{1/2} + a_h^{IP}(u_h^{nc} - u_H^{nc}, u_h^{nc} - u_H^{nc}) \\ &\leqslant a_h^{IP}(e_H, e_H) + 2C_1^{1/2} a_h^{IP}(e_H, e_H)^{1/2} \|u_h^{nc} - u_H^{nc})\|_{2,h,\Omega} + C_1\|u_h^{nc} - u_H^{nc})\|_{2,h,\Omega}^2 \\ &\leqslant (1+\varepsilon_2) a_h^{IP}(e_H, e_H) + C_1\left(1 + \frac{1}{\varepsilon_2}\right)\|u_h^{nc} - u_H^{nc})\|_{2,h,\Omega}^2. \end{aligned} \tag{4.16}$$

An application of Lemma 4.1 (with $0 < \varepsilon_1 < 1$) and of Lemma 4.2 to the right-hand side of 4.16 yields

$$\begin{aligned} a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) &\leqslant (1+\varepsilon_1)(1+\varepsilon_2) a_H^{IP}(e_H, e_H) \\ &\quad + \frac{1}{\alpha\gamma}\left(\frac{1+\varepsilon_2}{\varepsilon_1}\, C_P + 2C_1C_{nc}\left(1 + \frac{1}{\varepsilon_2}\right)\right)(\eta_{h,c}^2 + \eta_{H,c}^2). \end{aligned} \tag{4.17}$$

Finally, choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon/4$, $0 < \varepsilon < 1$, in (4.17) gives the assertion. □

Lemma 4.4. *Under the assumptions of Theorem* 4.1 *it holds*

$$a_h^{IP}(e_h, u_h^c - u_H^c) \leqslant C_{ap} H^{1/2+\varkappa}\left(C_1\|u_h - u_H\|_{2,h,\Omega}^2 + (2 + C_1)\|u\|_{5/2+\varkappa,\Omega}^2 + \|f\|_{0,\Omega}^2\right). \tag{4.18}$$

Proof. We have

$$a_h^{IP}(e_h, u_h^c - u_H^c) = a_h^{IP}(u - E_H(u_H), u_h^c - u_H^c) - a_h^{IP}(u_H - E_H(u_H), u_h^c - u_H^c) + a_h^{IP}(u_H - u_h, u_h^c - u_H^c). \quad (4.19)$$

Since $E_H(u_H) \in V_H^c \subset V_h^c$ is an admissible test function in (4.8), it holds

$$a_h^{IP}(u_h^c - u_H^c, E_H(u_H)) = 0. \quad (4.20)$$

On the other hand, $u_h^c - u_H^c \in H_0^2(\Omega)$ is an admissible test function in (1.2) and hence, it holds

$$a_h^{IP}(u, u_h^c - u_H^c) = (f, u_h^c - u_H^c). \quad (4.21)$$

We set $e_h^c := u - u_h^c$, $e_H^c := u - u_H^c$. Using (4.20), (4.21), (4.9), $h \leqslant H$, as well as Young's inequality, for the first term on the right-hand side in (4.19) we obtain

$$\begin{aligned} a_h^{IP}(u - E_H(u_H), u_h^c - u_H^c) &= (f, u_h^c - u_H^c)_{0,\Omega} = (f, e_H^c - e_h^c) \leqslant \|f\|_{0,\Omega}(\|e_h^c\|_{0,\Omega} + \|e_H^c\|_{0,\Omega}) \\ &\leqslant 2C_{ap}H^{1/2+\varkappa}\|f\|_{0,\Omega}\|u\|_{5/2+\varkappa,\Omega} \leqslant C_{ap}H^{1/2+\varkappa}(\|f\|_{0,\Omega}^2 + \|u\|_{5/2+\varkappa,\Omega}^2). \end{aligned} \quad (4.22)$$

Further, in view of (1.14), (4.9), (2.4), and Young's inequality, for the second term on the right-hand side in (4.19) it follows that

$$\begin{aligned} a_h^{IP}(u_H - E_H(u_H), u_h^c - u_H^c) &= a_h^{IP}(u_H - E_H(u_H), e_H^c - e_h^c) \\ &\leqslant a_h^{IP}(u_H - E_H(u_H), u_H - E_H(u_H))^{1/2} a_h^{IP}(e_h^c - e_H^c, e_h^c - e_H^c) \\ &\leqslant C_1^{1/2} C_{nc}^{1/2} \eta_{H,c}(\|e_h^c\|_{2,h,\Omega} + \|e_H^c\|_{2,h,\Omega}) \leqslant 2C_1^{1/2} C_{nc}^{1/2} C_{ap} H^{1/2+\varkappa} \eta_{H,c}\|u\|_{5/2+\varkappa,\Omega} \\ &\leqslant C_{ap}H^{1/2+\varkappa}(C_1 C_{nc}\eta_{H,c}^2 + \|u\|_{5/2+\varkappa,\Omega}^2). \end{aligned} \quad (4.23)$$

Finally, applying (1.14), (4.9), and Young's inequality again, the third term on the right-hand side in (4.19) can be estimated from above according to

$$\begin{aligned} a_h^{IP}(u_h - u_H, u_h^c - u_H^c) &= a_h^{IP}(u_h - u_H, e_H^c - e_h^c) \leqslant C_1\|u_h - u_H\|_{2,h,\Omega}(\|e_h^c\|_{2,h,\Omega} + \|e_H^c\|_{2,h,\Omega}) \\ &\leqslant 2C_1 C_{ap} H^{1/2+\varkappa}\|u_h - u_H\|_{2,h,\Omega}\|u\|_{5/2+\varkappa,\Omega} \\ &\leqslant C_1 C_{ap} H^{1/2+\varkappa}(\|u_h - u_H\|_{2,h,\Omega}^2 + \|u\|_{5/2+\varkappa,\Omega}^2). \end{aligned} \quad (4.24)$$

The assertion follows from (4.22), (4.23), and (4.24). □

Lemma 4.5. *Under the assumptions of Theorem* 4.1 *it holds*

$$a_h^{IP}(u_h^c - u_H^c, u_h^c - u_H^c) \geqslant \frac{\gamma}{2}\|u_h^c - u_H^c\|_{2,h,\Omega}^2 - 2C_{nc}(\eta_{h,c}^2 + \eta_{H,c}^2). \quad (4.25)$$

Proof. Using (1.13), $u_h^c - u_H^c = u_h - u_H - (u_h^{nc} - u_h^{nc})$, the left-hand side of the triangle inequality, and Young's inequality, we get

$$\begin{aligned} a_h^{IP}(u_h^c - u_H^c, u_h^c - u_H^c) &\geqslant \gamma\|u_h^c - u_H^c\|_{2,h,\Omega}^2 \geqslant \gamma(\|u_h - u_H\|_{2,h,\Omega} - \|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega})^2 \\ &= \gamma(\|u_h - u_H\|_{2,h,\Omega}^2 - 2\|u_h - u_H\|_{2,h,\Omega}\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega} + \|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2) \\ &\geqslant \frac{\gamma}{2}\|u_h - u_H\|_{2,h,\Omega}^2 - \gamma\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2. \end{aligned} \quad (4.26)$$

The assertion follows from (4.26) and Lemma 4.2. □

Proof of Theorem 4.1. We have

$$a_h^{IP}(e_h, e_h) = a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) - 2a_h^{IP}(e_h, u_h^c - u_H^c) - a_h^{IP}(u_h^c - u_H^c, u_h^c - u_H^c).$$

Using Lemmas 4.3, 4.4, 4.5 and observing $\varepsilon < 1$, it follows that

$$\begin{aligned} a_h^{IP}(e_h, e_h) &\leqslant (1+\varepsilon)\, a_H^{IP}(e_H, e_H) - \Big(\frac{\gamma}{2} - 2C_1 C_{ap}\, H^{\varkappa}\Big)\|u_h - u_H\|_{2,h,\Omega}^2 \\ &\quad + \frac{1}{\alpha\varepsilon}\frac{5(C_p + 2C_{nc}(1 + C_1))}{\gamma}(\eta_{h,c}^2 + \eta_{H,c}^2) + C_{ap}\Big((2 + C_1)\|u\|_{2+\varkappa,\Omega}^2 + \|f\|_{0,\Omega}^2\Big) H^{\varkappa}. \end{aligned} \quad (4.27)$$

We may choose $H_0 > 0$ such that for $H \leqslant H_0$

$$2C_1 C_{ap}\, H^{\varkappa} \leqslant \frac{\gamma}{4} \tag{4.28a}$$

and with a constant $C_{as} > 0$, independent of H

$$H^{\varkappa} \leqslant C_{as}\, \hat{\eta}_{H,c}^2 = \frac{C_{as}}{\alpha}\, \eta_{H,c}^2 \leqslant \frac{C_{as}}{\alpha\varepsilon}\, \eta_{H,c}^2. \tag{4.28b}$$

The assertion now follows from (4.27) and (4.28a), (4.28b). □

5 Contraction property

We now use the error reduction property (3.3), the quasi-orthogonality (4.14), and the reliability (2.5) to prove the following contraction property.

Theorem 5.1. *Let $u \in H_0^2(\Omega)$ be the unique solution of* (1.2). *Further, let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the C^0-IPDG solutions of* (1.6) *and error estimators, respectively. Then, there exist constants $0 < \delta < 1$ and $\rho > 0$, depending only on the local geometry of the triangulations, the parameter Θ from the Dörfler marking, and on k, such that for sufficiently large penalty parameter α the fine mesh and coarse mesh discretization errors $e_h := u - u_h$ and $e_H = u - u_H$ satisfy*

$$a_h^{IP}(e_h, e_h) + \rho\, \eta_h^2 \leqslant \delta \left(a_H^{IP}(e_H, e_H) + \rho\, \eta_H^2 \right). \tag{5.1}$$

Proof. Multiplying the estimator reduction property (3.3) by $\gamma/(4C_\tau)$ and substituting the result into the quasi-orthogonality (4.14), we obtain

$$a_h^{IP}(e_h, e_h) + \rho\, \eta_h^2 \leqslant (1+\varepsilon)\, a_H^{IP}(e_H, e_H) + \left(\frac{C_Q}{\alpha\varepsilon} - \frac{\gamma}{4C_\tau} + \rho \right) \eta_h^2 + \left(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\varkappa(\Theta)}{4C_\tau} \right) \eta_H^2. \tag{5.2}$$

If we choose $\alpha > (4C_Q C_\tau)/(\gamma\varepsilon)$, we have $\rho := \gamma/(4C_\tau) - C_Q/(\alpha\varepsilon) > 0$, and it follows from (5.2) that

$$a_h^{IP}(e_h, e_h) + \rho\, \eta_h^2 \leqslant (1+\varepsilon)\, a_H^{IP}(e_H, e_H) + \left(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\varkappa(\Theta)}{4C_\tau} \right) \eta_H^2. \tag{5.3}$$

Now, taking advantage of the reliability result

$$a_H^{IP}(e_H, e_H) \leqslant C_R\, \eta_H^2$$

(cf. (2.5)), for $0 < \delta < 1$ we obtain

$$a_h^{IP}(e_h, e_h) + \rho\, \eta_h^2 \leqslant \delta\, a_H^{IP}(e_H, e_H) + \left(C_R(1+\varepsilon-\delta) + \left(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\varkappa(\Theta)}{4C_\tau} \right) \right) \eta_H^2. \tag{5.4}$$

We choose δ such that

$$\rho = \frac{\gamma}{4C_\tau} - \frac{C_Q}{\alpha\varepsilon} = \delta^{-1} \left(C_R(1+\varepsilon-\delta) + \left(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\varkappa(\Theta)}{4C_\tau} \right) \right). \tag{5.5}$$

Solving for δ, we obtain

$$\delta = \frac{C_R\left(1+\varepsilon\right) + \dfrac{C_Q}{\alpha\varepsilon} + \dfrac{\gamma\varkappa(\Theta)}{4C_\tau}}{C_R + \dfrac{\gamma}{4C_\tau} - \dfrac{C_Q}{\alpha\varepsilon}}. \tag{5.6}$$

For instance, if we choose $\tau = \tau^* := 2^{-1/2}$ and $\varepsilon := \gamma/(16 C_R C_{\tau^*})$, we have $\varepsilon < 1$ (due to $\gamma < 1$, $C_R > 1$, $C_{\tau^*} > 1$), and, observing (3.6), it follows that

$$\delta = \frac{C_R + \frac{\gamma}{16 C_{\tau^*}} + \frac{16 C_Q C_R C_{\tau^*}}{\alpha\gamma} + \frac{\gamma\Theta}{8 C_{\tau^*}}}{C_R + \frac{\gamma}{4 C_{\tau*}} - \frac{16 C_Q C_R C_{\tau^*}}{\alpha\gamma}}. \tag{5.7}$$

Looking for α such that

$$\frac{\gamma}{16 C_{\tau^*}} + \frac{16 C_Q C_R C_{\tau^*}}{\alpha\gamma} + \frac{\gamma\Theta}{8 C_{\tau^*}} < \frac{\gamma}{4 C_{\tau*}} - \frac{16 C_Q C_R C_{\tau^*}}{\alpha\gamma}$$

we find that $0 < \delta < 1$ for

$$\alpha > \frac{512 C_Q C_R C_{\tau^*}^2}{(3 - 2\Theta)\gamma^2}. \tag{5.8}$$

This concludes the proof of the contraction property. □

The contraction property (5.1) is an essential ingredient to prove quasi-optimality of the adaptive approach with respect to a certain approximation class of functions depending on the regularity of the solution (cf. [6] for IPDG approximations of second order elliptic boundary value problems).

6 Numerical results

We provide a detailed documentation of the performance of the adaptive C^0-IPDG method for an illustrative example taken from [8].

Example 6.1. We choose Ω as the L-shaped domain $\Omega := (-1, +1)^2 \setminus ([0, 1) \times (-1, 0])$ and choose f in (1.1a) such that

$$u(r, \varphi) = (r^2 \cos^2\varphi - 1)^2 (r^2 \sin^2\varphi - 1)^2\, r^{1+z}\, g(\varphi) \tag{6.1}$$

is the exact solution of (1.1a), (1.1b), where

$$\begin{aligned} g(\varphi) := {}& \Big(\frac{1}{z-1} \sin\Big(\frac{3(z-1)\pi}{2}\Big) - \frac{1}{z+1} \sin\Big(\frac{3(z+1)\pi}{2}\Big)\Big)\Big(\cos((z-1)\varphi) - \cos((z+1)\varphi)\Big) \\ & - \Big(\frac{1}{z-1} \sin((z-1)\varphi) - \frac{1}{z+1} \sin((z+1)\varphi)\Big)\Big(\cos\Big(\frac{3(z-1)\pi}{2}\Big) - \cos\Big(\frac{3(z-1)\pi}{2}\Big)\Big) \end{aligned}$$

and $z \approx 0.54448$ is a non-characteristic root of $\sin^2(3z\pi/2) = z^2 \sin^2(3\pi/2)$.

For the documentation of the performance of the adaptive C^0-IPDG scheme, we have run simulations for polynomial degrees $2 \leqslant k \leqslant 6$. Since each of the constants C_Q, C_R, and $C_{\tau*}$ in (5.8) depends on k^4, (5.8) leads to the requirement $k^{16} \lesssim \alpha$. The numerical simulations revealed that this requirement is far too restrictive. In fact, the choice $\alpha = 2.5(k+1)^2$ turned out to be sufficient to achieve stability and to yield optimal convergence rates. The numerical evaluation of the element residuals has been taken care of by the collapsed Gauss–Jacobian-type quadrature formulas from [23] which worked fine even for triangles containing the origin as a vertex.

For $k = 2, 4, 6$, Figures 1–6 show the adaptively refined meshes after 10 adaptive cycles (top left), the convergence histories in terms of the broken C^0-IPDG energy norm of the error $a_h^{IP}(u - u_h, u - u_h)^{1/2}$ as a function of the total number of degrees of freedom (DOF) on a logarithmic scale (top right), the decrease of the estimator as a function of the DOF (bottom left), as well as the computed effectivity indices $\eta_h / a_h^{IP}(u - u_h, u - u_h)^{1/2}$ (bottom right) for uniform refinement and adaptive refinement with $\Theta = 0.7$ and $\Theta = 0.3$ in the Dörfler marking.

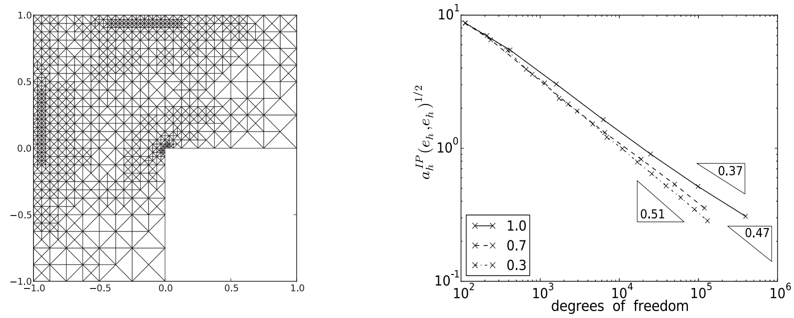

Figure 1. $k = 2$: Refined mesh after 10 adaptive cycles (left) and convergence history (right).

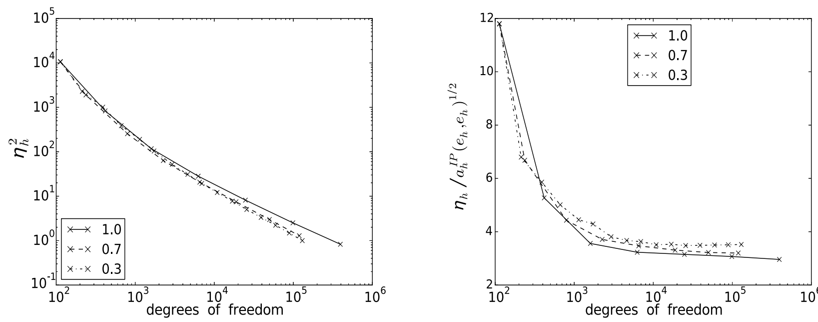

Figure 2. $k = 2$: Estimator reduction (left) and effectivity indices (right).

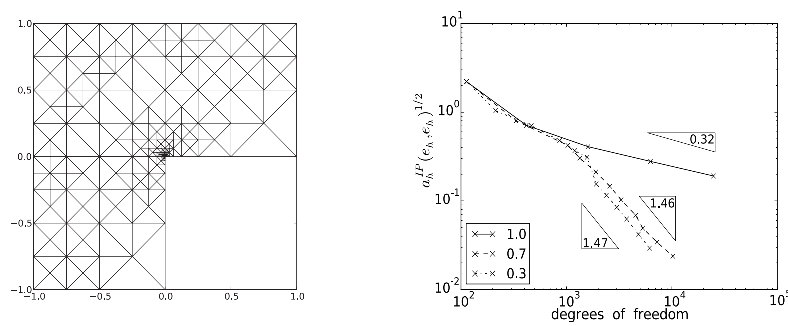

Figure 3. $k = 4$: Refined mesh after 10 adaptive cycles (left) and convergence history (right).

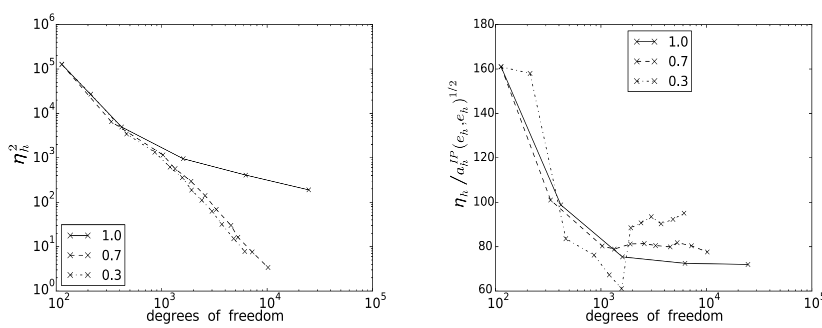

Figure 4. $k = 4$: Estimator reduction (left) and effectivity indices (right).

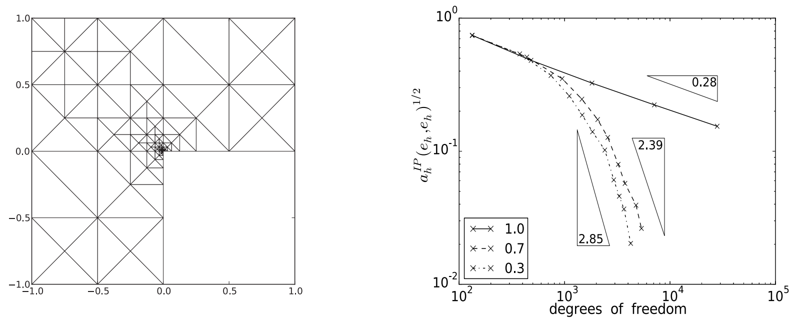

Figure 5. $k = 6$: Refined mesh after 10 adaptive cycles (left) and convergence history (right).

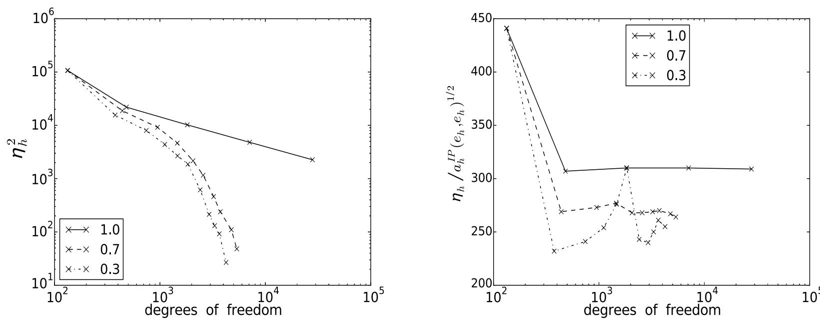

Figure 6. $k = 6$: Estimator reduction (left) and effectivity indices (right).

As has been shown in [9], we have

$$a_h^{IP}(u - u_h, u - u_h)^{1/2} \lesssim \left(\sum_{T \in \mathcal{T}_h(\Omega)} (\mathrm{diam}(T))^{2min(\alpha(T),k-1)} |u|^2_{2+\alpha(T),T} \right)^{1/2}$$

where $\alpha(T)$, $T \in \mathcal{T}_h(\Omega)$, is the local index of elliptic regularity. We note that $\min(\alpha(T), k-1) = z \approx 0.544$ for elements T having a vertex at the origin and $\min(\alpha(T), k-1) = k-1$ elsewhere. Consequently, the expected optimal convergence rates are slightly less than 0.5 for $k = 2$, 1.5 for $k = 4$, and 2.5 for $k = 6$. Figures 1 (right), 3 (right), and 5 (right) show that these optimal convergence rates are asymptotically achieved by the adaptive algorithm. Moreover, as in case of IPDG methods for second order elliptic boundary value problems [19] and H-IPDG methods for Maxwell's equations [12] we observe a different convergence behavior depending on the choice of Θ in the Dörfler marking. The effectivity indices show a clear dependence on the polynomial degree k.

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