Reflection of water waves by a submerged horizontal porous plate

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Abstract

On the basis of linear theory, an explicit expression is presented for the reflection coefficient R^{∞} when a plane wave is obliquely incident upon a semi-infinite porous plate in water of finite depth. The expression does not rely on knowledge of any of the complex-valued eigenvalues or corresponding vertical eigenfunctions in the region occupied by the plate. Also presented is a fast convergent expression for the reflection coefficient R when the plate is of finite length a and is backed by a rigid vertical wall. The special case of normal incidence is relevant to the design of submerged wave absorbers in a narrow wave tank.

1 Introduction

At last year's Workshop, a paper by I. H. Cho and M. H. Kim considered the effectiveness as a wave absorber of a single submerged horizontal porous plate, installed at the end of a narrow wave tank terminated by a vertical wall, as an alternative to using a number of vertical porous plates. The problem of determining the potential everywhere, and in particular the reflection coefficient R, was solved using matched eigenfunction expansions. This technique involves splitting the fluid domain into separate regions and constructing series expansions of eigenfunctions appropriate to the boundary conditions in each region. These expansions are then matched for continuity of potential (i.e. pressure) and horizontal velocity across the common interface between the regions. The orthogonality of the eigenfunctions yields an infinite system of equations for the unknowns in the eigenfunction series expansions, which is solved numerically by truncation.

There is a fundamental difficulty with using matched eigenfunction expansion methods here. This is that the boundary condition to be satisfied on the plate gives rise to a complex dispersion equation so that the eigenvalue problem is no longer self-adjoint and standard Sturm–Liouville theory does not apply. McIver (1998) shows how this can be overcome in a simpler related problem but it requires knowledge of which of the eigenvalues are multiple rather than simple zeroes of the dispersion relation since then the eigenfunctions are more complicated. Even if it is assumed that all the eigenvalues are simple, which is not true in general, it is not easy to track these complex zeroes over a range of parameters. The same problem arises in the paper presented by W. Bao and T. Kinoshita last year when considering the wave forces on a submerged fish cage, where the dispersion relation (their equation (7)) is even more complicated.

Here, we revisit the problem of the submerged plate and proceed differently. First, we assume that the plate is of infinite extent which allows us to use the Wiener–Hopf technique to derive an explicit solution for the reflection coefficient R^{∞} in a form which does not require knowledge of any of the eigenvalues in the porous-plate region. In addition, imposing the square-root singularity in the velocity of the fluid near the edge of the plate is crucial in obtaining the solution. This condition was not explicitly considered in either of the papers mentioned above or in the formulation of Wu *et al.* (1998) on which the Cho & Kim paper was based. We next consider the finite-plate problem and show how the eigenfunction method can be improved by using a so-called residue calculus technique (see, for example, Linton & McIver, 2001, pp. 148–166), which provides a powerful rapidly convergent solution which again builds in the necessary singularity. Because of space restrictions we are only able to present the governing equations, an outline of each method and a summary of the main results.

2 Governing equations

It is convenient to generalise the problem to include obliquely incident waves on a submerged porous half-plane. Then, on the basis of linear water-wave theory, there exists a harmonic velocity potential whose dependence on y and t is assumed to be proportional to $\mathrm{e}^{\mathrm{i}(\beta y-\omega t)}$ to allow for obliquely incident waves of frequency $\omega/2\pi$, where β is the component of the wavenumber in the y-direction. Then, the reduced (complex) potential is $\phi(x,z)$ and the full potential is $\mathbb{R}\mathrm{e}\,[\phi(x,z)\mathrm{e}^{\mathrm{i}(\beta y-\omega t)}]$. The problem is now two-dimensional and the free water surface is chosen to be $\{(x,z)\,|\,z=0\}$ and the fixed stiff porous plate occupies $\Gamma^{\mathrm{p}}=\{(x,z)\,|\,x>0,\ z=-d\}$ while the water domain is $\Omega=\{(x,z)\,|-h<z<0\}\backslash\bar{\Gamma}^{\mathrm{p}}$.

The mathematical description of the time-harmonic problem is as follows. It is assumed that a plane wave, making an angle θ_{inc} with the x-axis, is incident on the submerged plate from $x=-\infty$. Then, the reduced complex potential ϕ satisfies

$$
\begin{aligned}
-\Delta\phi+\beta^2\phi&=0 && \text{in } \Omega, && \text{(1a)}\\
\partial_z\phi&=0 && \text{on } z=-h, && \text{(1b)}\\
\partial_z\phi&=\alpha\phi && \text{on } z=0, && \text{(1c)}\\
\partial_z\phi|_{z=-d_-}=\partial_z\phi|_{z=-d_+}&=\mathrm{i}\mu[\phi] && \text{on } \Gamma^{\mathrm{p}}, && \text{(1d)}\\
|\nabla\phi|&=\mathcal{O}(r^{-1/2}) && && \\
&\text{as } r=\left(x^2+(z+d)^2\right)^{1/2}\to 0. && && \text{(1e)}
\end{aligned}
$$

Equation (1d) is derived from the assumption that the normal velocity of the fluid passing through the porous plate is proportional to the pressure difference across it, see Chwang (1983) for a fuller discussion. Here $\alpha=\omega^2/g$, $\mu=\sigma\omega$, where σ has positive real part and is related to the properties of the porous plate, and $[\phi]$ denotes the jump of ϕ across Γ^{p}: $[\phi]=\phi|_{z=-d_-}-\phi|_{z=-d_+}$. Equation (1e) reflects the infinite speed of the fluid around the sharp edge of the plate. Finally, radiation conditions are needed appropriate to the scattering of the incident wave. In what follows, we write $c=h-d$ for brevity.

Consider in $x<0$ the expression $\mathrm{e}^{\pm\kappa_n x}\psi_n(z)$, where $\psi_n(z)=\frac{\cos k_n(z+h)}{\cos k_n h}$, $n=0,1,2,\ldots$, $\kappa_n=(k_n^2+\beta^2)^{1/2}$ and where the numbers k_n, $n\geq 1$, are the positive real roots of the relation $\alpha+k_n\tan k_n h=0$ and $k_0=-\mathrm{i}k$ is its sole negative imaginary root. The positive real wavenumber k is thus related to α by $\alpha=k\tanh kh$. Then, $\mathrm{e}^{\pm\kappa_n x}\psi_n(z)$ satisfies (1a), (1b) and (1c). Thus, a wave of unit potential amplitude, obliquely incident from $x=-\infty$ on the submerged plate, has the form $\mathrm{e}^{\mathrm{i}\kappa x}\psi_0(z)$, where $\kappa\equiv\mathrm{i}\kappa_0=k\cos\theta_{\mathrm{inc}}$ and $\beta=k\sin\theta_{\mathrm{inc}}$. We can now complete the conditions on ϕ by demanding $\phi\sim(\mathrm{e}^{\mathrm{i}\kappa x}+R^\infty\mathrm{e}^{-\mathrm{i}\kappa x})\psi_0(z)$ as $x\to-\infty$ and for the plate region $\phi\to 0$ as $x\to\infty$ so that the effect of the plate is to progressively reduce the wave amplitude along its length.

3 Wiener–Hopf solution

The solution is straightforward but long and relies crucially on being able to write $K(s)=K^{\mathrm{n}}(s)/K^{\mathrm{d}}(s)$, where

$$
\begin{aligned}
K^{\mathrm{n}}(s)&=\gamma\sinh\gamma h-\alpha\cosh\gamma h, && \text{(2a)}\\
K^{\mathrm{d}}(s)&=\gamma\sinh\gamma c(\gamma\sinh\gamma d-\alpha\cosh\gamma d)\\
&\quad-\mathrm{i}\mu(\gamma\sinh\gamma h-\alpha\cosh\gamma h), && \text{(2b)}
\end{aligned}
$$

in the form $K(s)=K_+(s)K_-(s)$, where $K_\pm(s)$ is non-zero and regular in the upper/lower half of the complex s-plane, and $\gamma=(s^2+\beta^2)^{1/2}$. Once this is done, the reflection coefficient is

$$
R^\infty=\frac{k^3\sinh^2 kcK_+(\kappa)K_-(-\kappa)}{\kappa^2(2kh+\sinh 2kh)}. \qquad (3)
$$

It is easy to split the numerator of $K(s)$ using its infinite-product decomposition into its known factors such as $(1+\gamma^2/k_n^2)$ for $n=0,1,2,\ldots$ but the location of the complex, possibly multiple, zeroes of the denominator is not so easy. We avoid this by using a Cauchy integral method which results, after simplifying, in

$$
R^\infty=-\exp(-2\mathrm{i}\Theta(\kappa)). \qquad (4)
$$

Here,

$$
\begin{aligned}
\Theta(\kappa)&=I(\kappa)+2\arctan(\beta/\kappa)+\chi(\kappa)\\
&\quad+\sum_{n=1}^{\infty}\Big(\arctan(\kappa/\kappa_n)\\
&\quad-\arctan(\kappa/c_n')-\arctan(\kappa/d_n')\Big), \qquad (5)
\end{aligned}
$$

where $\kappa_n = (k_n^2 + \beta^2)^{1/2}$, $c'_n = (c_n^2 + \beta^2)^{1/2}$, $d'_n = (d_n^2 + \beta^2)^{1/2}$, $c_n = n\pi/c$, $d_n = n\pi/d$, and $\chi(s) = s\pi^{-1}(h\log h - c\log c - d\log d)$. Finally,

$$I(s) = \frac{1}{\pi}\int_0^\infty \frac{\log(F(st))}{t^2-1}\,\mathrm{d}t,$$

where $F(s) = \gamma^2 \sinh\gamma c \sinh\gamma d / K^{\mathrm{d}}(s)$, and which can be converted into a numerically favourable expression for real argument.

It is of interest to determine R^∞ on the assumption that the zeroes of the denominator of $K(s)$ are all simple and of the form $\gamma = \gamma_n = \pm\mathrm{i}l_n$, or $s = \pm\mathrm{i}\lambda_n$, $n = 0, 1, 2, \dots$, where the l_n are all complex. Care must be taken in the numbering of the roots, which we also denote as $\lambda_n^\pm$, $n = 1, 2, \dots$, as they arise as perturbations of the roots above (below) the plate in the limit case of an impermeable dock ($\sigma = 0$). Then, we can expand

$$F(s) = \frac{\gamma^2 \sinh\gamma c \sinh\gamma d}{(-\mathrm{i}\mu\alpha)\prod_{n=1}^\infty (1+\gamma^2/l_n^2)}$$
$$= \frac{(s^2+\beta^2)^2 cd \prod_{n=1}^\infty ((d_n'^2+s^2)/d_n^2)((c_n'^2+s^2)/c_n^2)}{(-\mathrm{i}\mu\alpha)\prod_{n=1}^\infty ((\lambda_n^{+2}+s^2)/l_n^{+2})((\lambda_n^{-2}+s^2)/l_n^{-2})}$$

and the infinite product in the denominator converges because of the behaviour of the l_n as $n \to \infty$. To evaluate $I(\kappa)$ we make use of the result $\frac{1}{\pi}\int_0^\infty \frac{\log(a^2+b^2t^2)}{t^2-1}\,\mathrm{d}t = \arctan(b/a)$ and we find that

$$I(\kappa) = 2\arctan(\kappa/\beta) + \sum_{n=1}^\infty \Big(\arctan(\kappa/d'_n) - \arctan(\kappa/\lambda_n^+) + \arctan(\kappa/c'_n) - \arctan(\kappa/\lambda_n^-)\Big),$$

so that

$$\Theta(\kappa) = \pi + \chi(\kappa) + \sum_{n=1}^\infty \Big(\arctan(\kappa/\kappa_n) - \arctan(\kappa/\lambda_n^+) - \arctan(\kappa/\lambda_n^-)\Big). \quad (6)$$

4 Finite plate: eigenfunction matching and residue calculus

The fluid region is now $\{(x,z)\,|\,-\infty < x < a,\ -h < z < 0\}$ with the submerged plate occupying $\{(x,z)\,|\,0 < x < a,\ z = -d\}$ with a rigid vertical wall at $x = a$, $-h < z < 0$. If the l_m are assumed known and simple, then vertical eigenfunctions of the form

$$\Psi_m(z) = \begin{cases} \overline{\Psi}_m(z), & -d < z < 0, \\ \underline{\Psi}_m(z), & -h < z < -d, \end{cases} \quad (7)$$

$m = 0, 1, 2, \dots$, where

$$\overline{\Psi}_m(z) = p_m(l_m \cos l_m z + \alpha \sin l_m z),$$
$$\underline{\Psi}_m(z) = q_m \cos l_m(z+h),$$

can be used to expand the potential for $0 < x < a$. Here, p_m and q_m are chosen to satisfy the first equality in (1d) and such that the Ψ_m have unit L^2-norm, also cf. Wu *et al.* (1998). The second equality in (1d) is automatically satisfied since $K^{\mathrm{d}}(\pm\mathrm{i}\lambda_n) = 0$ or, equivalently, if the l_m satisfy the dispersion relation

$$\frac{l_m}{\mathrm{i}\mu} + \frac{l_m \sin l_m h + \alpha \cos l_m h}{(l_m \sin l_m d + \alpha \cos l_m d)\sin l_m c} = 0. \quad (8)$$

Then, we can write

$$\phi^-(x,z) = \psi_0(z)\mathrm{e}^{-\kappa_0 x} + \sum_{m=0}^\infty a_m \psi_m(z)\mathrm{e}^{\kappa_m x}, \quad (9)$$

$$\phi^+(x,z) = \sum_{m=0}^\infty b_m \Psi_m(z) \frac{\cosh\lambda_m(a-x)}{\cosh\lambda_m a}, \quad (10)$$

for $x < 0$ and $0 < x < a$, respectively. We now match the potential and its horizontal derivative across $x = 0$, multiply by Ψ_n and integrate over $(-h, 0)$ to get an infinite system of linear algebraic equations for the a_m and b_m, which can be truncated and solved numerically. This is the usual eigenfunction matching method. But further analytical progress is possible. We eliminate b_m and re-arrange to obtain

$$\left(\frac{1}{\kappa_0+\lambda_n} + \frac{\mathrm{e}^{-2\lambda_n a}}{\kappa_0-\lambda_n}\right) = \sum_{m=0}^\infty A_m \left(\frac{1}{\kappa_m-\lambda_n} + \frac{\mathrm{e}^{-2\lambda_n a}}{\kappa_m+\lambda_n}\right), \quad (11)$$

$n = 0, 1, 2, \dots$, where $A_m = a_m \psi'_m(-d)/\psi'_0(-d)$, and, from (9), $R = A_0$.

Neglecting the terms involving the exponentials is equivalent to letting $a \to \infty$ and gives the infinite system for the semi-infinite plate which is simply $\sum_{m=0}^\infty A_m^\infty/(\kappa_m - \lambda_n) = 1/(\kappa_0 + \lambda_n)$, $n = 0, 1, 2, \dots$, where $R^\infty = A_0^\infty$ and $A_m^\infty \sim m^{-1/2}$ as $m \to \infty$ to ensure the correct singularity at the edge of the plate. This equation can be solved explicitly using residue calculus when it is found that R^∞ is the same as (4) together with (6). Returning to (11), a rapidly convergent expression for R can be developed, again using residue calculus, namely

$$R = R^\infty\Big(1 - \sum_{n=0}^\infty \frac{B_n}{\lambda_n - \kappa_0}\Big)\Big/\Big(1 - \sum_{n=0}^\infty \frac{B_n}{\lambda_n + \kappa_0}\Big),$$

where the B_m satisfy $B_m + \sum_{n=0}^{\infty} K_{mn} B_n = C_m$, $m = 0, 1, 2, \ldots$, where $K_{mn} = C_m/(\lambda_m + \lambda_n)$ and $C_m \sim \mathrm{e}^{-2(\lambda_m a + \chi(\lambda_m))}$ for $m \to \infty$. Thus, the equation for the B_m is rapidly convergent and a good approximation is given by the term $n = 0$ in R. It turns out that $B_0 \approx 2\lambda_0 \mathrm{e}^{-2(\lambda_0 a + \chi(\lambda_0) + \theta_0)}$, where $\theta_0 = \operatorname{arctanh}(\lambda_0/\kappa_1) - \operatorname{arctanh}(\lambda_0/\lambda_1^-) + \sum_{n=2}^{\infty} \big(\operatorname{arctanh}(\lambda_0/\kappa_n) - \operatorname{arctanh}(\lambda_0/\lambda_n^+) - \operatorname{arctanh}(\lambda_0/\lambda_n^-) \big)$.

5 Numerical comparisons

We first compare R^∞ calculated using the expression from the Wiener–Hopf approach using the Cauchy integral method (W–H) to the result from simple eigenfunction matching (EM). The roots of (8) are tedious to calculate and are found by using their asymptotics to obtain an initial guess, which feeds into a numerical solver based on Meylan & Gross (2003). For $kh = 1.22$, $\sigma = 0.32$, $c/h = 0.71$, $\theta_{\text{inc}} = 0$, we obtain:

W–H	0.0145 − 0.0567i
EM with 50 roots	0.0140 − 0.0569i
EM with 200 roots	0.0144 − 0.0568i

while, for $kh = 0.78$, $\sigma = 0.00032$, $c/h = 0.92$, $\theta_{\text{inc}} = 0$, we obtain:

W–H	0.4969 − 0.1095i
EM with 50 roots	0.4963 − 0.1121i
EM with 200 roots	0.4967 − 0.1102i

It can be seen that the results using EM agree with those from the W–H approach but the convergence of EM is very slow (relative errors of 0.24% and 0.14% for 200 roots in the examples above). The behaviour of $|R^\infty|$ for different parameters is shown in figures 1 and 2.

The fourth decimal place accuracy can be obtained by the semi-infinite-plate residue calculus in the examples above by truncating the series (6) at only $n = 9$ and $n = 26$, resp. This agreement validates the applicability of the residue calculus method (RC). For the finite plate with $a/h = 7.36$ and the other data as in the first example above, we obtain:

RC	0.0205 − 0.0523i
RC 0th-term approx.	0.0206 − 0.0617i
EM with 50 roots	0.0196 − 0.0615i
EM with 200 roots	0.0199 − 0.0614i

while for the second example with $\sigma = 0.032$ and $a/h = 10.78$, we find:

RC	0.2916 − 0.1926i
RC 0th-term approx.	0.2527 − 0.1933i
EM with 50 roots	0.2549 − 0.1966i
EM with 200 roots	0.2556 − 0.1956i

It can be seen that the convergence of the EM is even much slower for the finite plate and the RC 0th-term approximation already yields more accurate results than the EM with 200 roots!

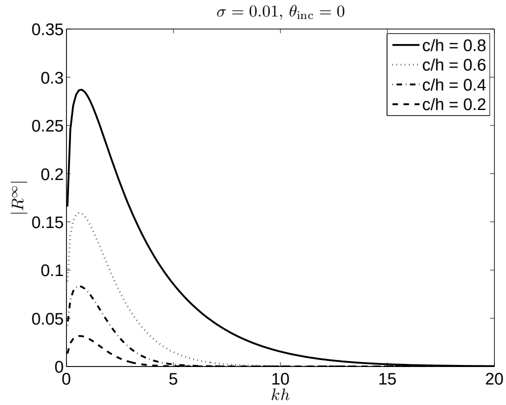

Fig. 1: $|R^\infty|$ versus kh for $\sigma = 0.01$, $\theta_{\text{inc}} = 0$ and different c/h.

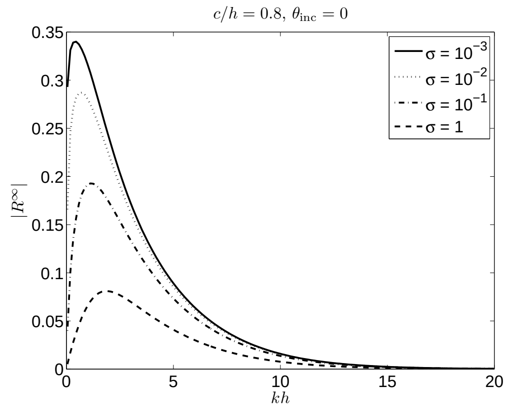

Fig. 2: $|R^\infty|$ versus kh for $c/h = 0.8$, $\theta_{\text{inc}} = 0$ and different σ.

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