LOCAL EXISTENCE AND UNIQUENESS FOR A TWO–DIMENSIONAL SURFACE GROWTH EQUATION WITH SPACE–TIME WHITE NOISE

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Abstract. We study local existence and uniqueness for a surface growth model with space–time white noise in 2D. Unfortunately, the direct fixed-point argument for mild solutions fails here, as we do not have sufficient regularity for the stochastic forcing. Nevertheless, one can give a rigorous meaning to the stochastic PDE and show uniqueness of solutions in that setting. Using spectral Galerkin method and any other types of regularization of the noise, we obtain always the same solution.

1. Introduction

We study local existence and uniqueness of the following equation

$$\partial_t h + \Delta^2 h + \Delta|\nabla h|^2 = \eta \tag{1.1}$$

subject to periodic boundary conditions on $[0, 2\pi]^2$ and with space–time white noise η. This equation arises in the theory of amorphous surface growth, see for example [17, 18], and it has been considered also in the theory of ion sputtering. The equation is simplified in the sense that we have left out lower order terms that can easily be handled and do not present any obstacle in the theory of local existence.

A thorough analysis of the one–dimensional version of the problem has been given in [3], where the general theory introduced in [8] has allowed to prove the existence of Markov solutions to the equation. Moreover each of these solutions converges to a unique equilibrium distribution.

In order to complete the same program for the (physically relevant) two–dimensional case, there are several problems that need to be faced.

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- When dealing with space–time white noise it turns out that the expected smoothness of the solution is not enough to define the non–linear term.
- In contrast with the one–dimensional case, existence of global weak solutions is harder, due to the lack of reasonable energy estimates. Existence of weak solutions without noise has been proved in [21], using a–priori bounds derived from the estimate of $\int e^h\,dx$, which cannot be used for any approximation by Galerkin methods.
- The weak–strong uniqueness principle (i.e. uniqueness of local solutions in the class of weak solutions) fails, and this is a serious obstacle to the application of the same ideas used in [2].

In this paper we address the problem of proving the existence of local solutions for the equation forced by space–time white noise, as well as the issue of low regularity for the non–linear term. A preliminary existence result of local in time solution has been already given in [4], based on the ideas introduced in [15]. Nevertheless, these methods where not able to treat the physically relevant case of space-time white noise.

Here we follow an approach similar to the one used in [7] for a similar singular two–dimensional problem with space–time white noise. One key difference is that we cannot rely on an explicitly given invariant measure. The idea is to decompose the solution in a rough part having the low regularity dictated by the forcing and a remainder, slightly smoother. The non–linearity for the rough term is then defined as the limit of cut–offs via spectral Galerkin methods, thanks to the underlying Gaussian nature of the processes involved. This, roughly speaking, corresponds to a re–normalization of the non–linearity, but without any additional term, due to the fact that the Laplace in front of the gradient squared kills the infinite constant. The method works even for rougher noise at the price, of a lower level of regularity. For even rougher noise the remainder fails to be regular enough, and then we need an additional term in the expansion of the solution.

To be more precise, we interpret the solution as $h = z + v$, where z solves the corresponding linear stochastic equation (where the non–linearity is dropped) and v is the remainder. The term z is a function valued process. Due to regularization of the bi–Laplacian it is even continuous in space and time, but it fails to have a derivative. The remainder v is then given by the mild solution

$$v(t) = \mathrm{e}^{-tA}\,h_0 - \int_0^t \mathrm{e}^{-(t-s)\Delta^2}\,\Delta(|\nabla v|^2 + 2\nabla v\cdot\nabla z + |\nabla z|^2)\,dt + z(t).$$

Since v is regular enough, the double product $\nabla v\cdot\nabla z$ is well defined. The problems originate from the "squared distribution" $|\nabla z|^2$. Once this term is properly defined as a limit of spectral approximations, for instance, we can work out a fixed point argument similar to [9] (see also [3]).

Recently there is an interest in the analysis of non–linear PDEs that, like the one presented here, are forced by noise rough enough so that the non–linear term in principle is not well–defined. The meaning of the non–linearity is then recovered through probability. We refer to [11, 13]. Two recent papers [10, 12] have proposed general and powerful methods that apply to our equation (1.1) as well as to more difficult problems, where for instance the re–normalized infinite constant shows up in the equation. The method we have used here, based on Fourier expansion, works very well for our problem (1.1) and we believe it is, at least for this problem, neat and simple.

1.1. **Notations.** Let $\mathbb{T}_2$ be the two dimensional torus, understood as $\mathbb{T}_2 = [-\pi,\pi]^2$ with identification of the borders. Consider the complex Fourier basis of $L^2(\mathbb{T}_2;\mathbf{C})$, defined by $e_k = (2\pi)^{-1}\mathrm{e}^{\mathrm{i}k\cdot x}$, for $z \in \mathbf{Z}^2$. Notice that if $u \in L^2(\mathbb{T}_2)$ is real valued and $u = \sum u_k e_k$, then $u_k = \bar{u}_{-k}$. Define $\mathbf{Z}^2_\star = \mathbf{Z}^2 \setminus \{0\}$, $\mathbf{Z}^2_+ = \{k \in \mathbf{Z}^2 : k_1 > 0 \text{ or } k_1 = 0, k_2 > 0\}$ and $\mathbf{Z}^2_- = -\mathbf{Z}^2_+$.

For $s \in \mathbf{R}$ and $p \geq 1$, denote by $W^{s,p}_\wp$ the space of 2π–periodic $W^{s,p}$ functions, and by $\mathring{W}^{s,p}_\wp$ its sub–space of functions with zero mean. We will use the notation $H^s_\wp$, $\mathring{H}^s_\wp$ and $\mathring{L}^2$ when $p = 2$. Such spaces can be defined either as the closure in the corresponding norm of smooth periodic functions, or as spaces on $\mathbf{R}^2$ with weight, or by interpolation. We refer to [19, Chapter 3] for details on the definition as well as for their properties. In particular, we will use that the dual $(W^{s,p}_\wp)'$ is $W^{-s,q}_\wp$, where q is the Hölder conjugate exponent of p. Moreover the standard Sobolev embeddings hold, namely $W^{s,q}_\wp \subset W^{r,p}_\wp$ if $r \leq s$ and $r - \frac{d}{p} \leq s - \frac{d}{q}$.

Denote by $(S(t))_{t\geq 0}$ the semigroup generated in $\mathring{L}^2$ by $-\Delta^2$ with domain of definition $\mathring{H}^4_\wp$. Set for real valued functions u_1, u_2,

$$\mathcal{B}(u_1,u_2) = \Delta(\nabla u_1 \cdot \nabla u_2)\ .$$

It will be useful, for our purposes, to extend the definition of $\mathcal{B}$ over complex valued functions as $\mathcal{B}(u_1,u_2) = \Delta(\nabla u_1 \cdot \overline{\nabla u_2})$. With this position $\mathcal{B}$ coincides with the previous definition for real valued functions and it is Hermitian, namely $\mathcal{B}(u_2,u_1) = \overline{\mathcal{B}(u_1,u_2)}$.

In the rest of the paper we shall adopt the sloppy habit to use the same symbol for numbers that depend only on universal constants and that can change from line to line of an inequality.

2. Main results

2.1. **Existence of mild solutions.** Our first main results shows existence of a local solution of (1.1). The solution is interpreted in the mild formulation

$$h(t) = S(t)h_0 - \int_0^t S(t-s)\mathcal{B}(h,h)\,ds - \int_0^t S(t-s)\,dW_s, \tag{2.1}$$

for a suitable initial condition h_0. Denote by $z(t) = \int_0^t S(t-s)\,dW_s$ the stochastic convolution. The above mild formulation can be recast in terms of $v = h - z$ as

$$v(t) = S(t)h_0 - \int_0^t S(t-s)\mathcal{B}(v,v)\,ds - \int_0^t S(t-s)\big(\mathcal{B}(z,z) + 2\mathcal{B}(z,v)\big)\,ds.$$

Since we do not expect the term $\mathcal{B}(z,z)$ to be well defined, given the regularity of the stochastic convolution (see Lemma 3.1, we replace it by a suitable extension $\widetilde{\mathcal{B}}(z,z)$ defined in Section 3 as the limit of spectral Galerkin approximations of z.

In conclusion a mild solution h of (1.1) is a random process such that $v = h - z$ is a mild solution in the sense that

$$v(t) = S(t)h_0 - \int_0^t S(t-s)\mathcal{B}(v,v)\,ds - \int_0^t S(t-s)\big(\widetilde{\mathcal{B}}(z,z) + 2\mathcal{B}(z,v)\big)\,ds. \tag{2.2}$$

Given $\rho > 0$, $\epsilon > 0$ and $T > 0$, define

$$\|u\|_{\epsilon,T} := \sup_{t\le T} t^{\frac{\epsilon}{4}}\|u(t)\|_{H^{1+\epsilon}}.$$

We will find a solution of (2.2) by means of a fixed point argument in the space

$$\mathcal{X}(\epsilon,\rho,T) := \{v \in C([0,T];\mathring{L}^2(\mathbb{T}_2)) : \|v\|_{\epsilon,T} \le \rho\}$$

for suitable ρ, T.

Theorem 2.1. *Let $h_0 \in \mathring{H}^1_\wp$ and $\epsilon \in (0,\frac{1}{2})$. Given a cylindrical Wiener process W on $\mathring{L}^2(\mathbb{T}_2)$, there exist a stopping time τ_{h_0} and a solution h of the mild formulation (2.1) defined on $[0,\tau_{h_0})$, such that $h \in C([0,\tau_{h_0});\mathring{L}^2(\mathbb{T}_2))$ and $h - z \in C((0,\tau_{h_0});\mathring{H}^{1+\epsilon}_\wp)$. Moreover, $\mathbb{P}[\tau_{h_0} > 0] = 1$.*

The proof is given later in Section 2.1. Note that solutions are unique up to the minimum of both their stopping times in the space $\mathcal{X}(\epsilon,\rho,T)$. Moreover, by standard methods one can continue uniquely the solution as continuous $\mathring{H}^{1+\epsilon}_\wp$-valued solutions, until they blow up.

2.2. **Other regularizations.** Here we consider regularizing methods different from the spectral Galerkin method used to define $\widetilde{\mathcal{B}}(z,z)$. We give an abstract criterion and examples of its application.

We first define what we mean by regularization of z. Let Φ be a bounded operator on $\mathring{L}^2(\mathbb{T}_2)$ and define

$$z^\Phi(t) = \int_0^t S(t-s)\Phi\,dW(s) = \sum_{k\in\mathbf{Z}^2_\star}\int_0^t \mathrm{e}^{-(t-s)|k|^4}\,d\beta_k^\Phi(s)e_k,$$

where the $\beta_k^\Phi(t) = \langle \Phi W(t), e_k\rangle$ are (not necessarily independent) Brownian motions with variance $\|\Phi e_k\|^2_{L^2}$. We suppose that the regularized process z^Φ defined above is smooth enough in order to define $B(z^\Phi,z^\Phi)$ uniquely. A sufficient condition that ensures this statement is given in the following lemma.

Lemma 2.2. *Let Φ and z^Φ as above. Assume that for every $k \in \mathbf{Z}^2_\star$,*

$$\sum_{m+n=k} \frac{\|\Phi e_m\|_{L^2}\|\Phi e_n\|_{L^2}}{|m|\,|n|} < \infty, \tag{2.3}$$

then $B(z^\Phi, z^\Phi)$ is well–defined as an element of $\mathring{H}^{-2-\gamma}_\wp$ for every $\gamma > 0$. In particular (2.3) holds if $\sum_m |m|^{-2}\|\Phi e_m\|^2_{L^2} < \infty$.

Having approximations in mind, we turn to sequences $(\Phi_N)_{N\in\mathbf{N}}$ of bounded operators satisfying (2.3) and we analyse under which conditions they provide a "good" approximation of the process z. By "good" we mean that the quantities involved in the definition of $\widetilde{\mathcal{B}}(z, z)$ and in the proof of Theorem 2.1 should be well approximated by the corresponding quantities for the sequence $(z^{\Phi_N})_{N\geq 1}$. The first result gives sufficient conditions that ensure convergence in $L^p_{\text{loc}}([0,\infty); W^{s,p})$. The technical assumptions on Φ_N basically states, that they converge in a weak sense to the identity, and that the off-diagonal terms of the operators are not too large.

Theorem 2.3. *Let $(\Phi_N)_{N\geq 1}$ be a sequence of bounded operators on $\mathring{L}^2(\mathbb{T}_2)$ such that*

- *for every $m, n \in \mathbf{Z}^2_\star$,*
$$\langle \Phi_N^\star \Phi_N e_m, e_n\rangle \longrightarrow \delta_{m,n},$$
where the Kronecker-Delta is given by $\delta_{m,n} = 1$ if $m = n$ and zero otherwise,
- *there is $\gamma \in (0, 1)$ such that*

$$\sum_{m,n} \sup_{N\in\mathbf{N}} \Big\{ \frac{|\langle(\Phi_N - I)e_m, (\Phi_N - I)e_n\rangle|}{(|m| + |n|)^{4-2\gamma}} \Big\} < \infty. \tag{2.4}$$

Then for every $s \in (0, \gamma)$, $p \geq 1$ and $T > 0$,

$$\mathbb{E}[\|z^{\Phi_N} - z\|^p_{L^p([0,T];W^{s,p})}] \longrightarrow 0, \qquad N \to \infty.$$

Our second result gives some sufficient conditions that ensure that different approximations give the same limit non–linearity. The particular choice of the Galerkin truncations operators yields conditions for the limit of a generic sequence $(\Phi_N)_{N\in\mathbf{N}}$ to the limit non–linearity defined in Section 3.

Theorem 2.4. *Let $(\Phi_N)_{N\in\mathbf{N}}$ and $(\Psi_N)_{N\in\mathbf{N}}$ be two sequences of regularizing operators such that for $N \to \infty$*

$$\langle\Phi_N e_m, \Phi_N e_n\rangle \to \delta_{m,n}, \qquad \langle\Psi_N e_m, \Psi_N e_n\rangle \to \delta_{m,n}, \qquad \langle\Psi_N e_m, \Phi_N e_n\rangle \to \delta_{m,n}, \tag{2.5}$$

for every $m, n \in \mathbf{Z}^2_\star$. Let

$$c_{mn} = \sup_{N\in\mathbf{N}} \{ |\langle\Phi_N e_m, \Phi_N e_n\rangle| + |\langle\Psi_N e_m, \Psi_N e_n\rangle| + |\langle\Phi_N e_m, \Psi_N e_n\rangle| \}$$

for $m, n \in \mathbf{Z}^2_\star$, and assume that for some $\gamma > 0$,

$$\sum_{k\in\mathbf{Z}^2_\star} |k|^{-2\gamma} \sum_{m+n=k} \frac{c_{mn}}{|n|^3|m|^3} < \infty, \tag{2.6}$$

and

$$\sum_{k\in\mathbf{Z}^2_\star} |k|^{-2\gamma} \sum_{\substack{m_1+n_1=k\\ m_2+n_2=k}} \frac{c_{m_1m_2}c_{n_1n_2}}{|m_1|^3|m_2|^3|n_1|^3|n_2|^3} < \infty. \tag{2.7}$$

Then for all $T > 0$ and $q \geq 1$,

$$\mathbb{E}\big[\|\mathcal{B}(z^{\Phi_N}, z^{\Phi_N}) - \mathcal{B}(z^{\Psi_N}, z^{\Psi_N})\|^q_{L^q([0,T],H^{-2-\gamma})}\big] \longrightarrow 0.$$

With the above results at hand, we verify that a convergence of a regularization $(\Phi_n)_{n\in\mathbf{N}}$ that leads to the convergence of z^{Φ_n} to z and of $\mathcal{B}(z^{\Phi_n}, z^{\Phi_n})$ to $\tilde{\mathcal{B}}(z, z)$ result in the solution $v^{\Phi_n} = h^{\Phi_n} - z^{\Phi_n}$ of the regularized problem converging to $v = h - z$, the solution given by Theorem 2.1, in probability. We give only one possible version of the result. Other versions may be obtained by working on different function spaces.

Given an initial condition $h_0 \in \mathring{H}^1_\wp(\mathbb{T}_2)$, let v be the process given by the mild formulation (2.2). Define for every $R > 0$ the stopping time

$$\tau^R = \inf\{t > 0 : \ \|v(t) - S(t)h_0\|_{H^{1+\epsilon}} \geq R\},$$

and $\tau^R = \infty$ if the above set is empty. By its definition, it is immediate to see that $\tau^R \leq \tau_{h_0}$, where τ_{h_0} is the life–span of v.

Theorem 2.5. *Let $h_0 \in \mathring{H}^1_\wp(\mathbb{T}_2)$. Let Φ_N be a sequence of regularizing operators such that $z^{\Phi_N} \in C^0([0,\infty), \mathring{H}^{1+\epsilon}_\wp)$ for all $N \in \mathbf{N}$ and fix*

$$\epsilon \in (0, \tfrac{1}{2}), \quad \alpha = 1 - \frac{\epsilon}{2}, \quad q > \frac{4}{\epsilon}, \quad \beta \in (2, 3-\epsilon), \ \text{and } q' > \frac{4}{3-\beta-\epsilon}\,.$$

If

$$\mathbb{E}\|z^{\Phi_N} - z\|_{L^q([0,1],W^{\alpha,q})} + \mathbb{E}\|\mathcal{B}(z^{\Phi_N}, z^{\Phi_N}) - \tilde{\mathcal{B}}(z,z)\|_{L^{q'}([0,1],H^{-\beta})} \to 0, \tag{2.8}$$

as $N \to \infty$, then

$$\sup_{[0,1\wedge\tau_R]} \|v - v^{\Phi_N}\|_{H^{1+\epsilon}} \longrightarrow 0,$$

in probability.

The proof of these results is given in Section 5.2. Here we illustrate examples of applications of the results presented above.

Remark 2.6. For every $N \geq 1$ define $\Phi_N e_m = e_m$ if $|m| \leq N$, and 0 otherwise. The spectral truncations Φ_N are clearly regularizing and all assumptions of Theorems 2.3, and 2.4 hold true. Indeed, the two theorems find a non–trivial application when one needs to control the off–diagonal terms.

Example 2.7. Given a non–negative smooth function q with support contained in a small neighbourhood of the origin and such that $\int q(x)\,dx = 1$, let q_N be the periodic extension on $\mathbb{T}_2$ of $z \mapsto q(Nz)$. Let

$$\Phi_N f(x) = \int_{[0,2\pi]^2} N^2 q_N(x-y) f(y)\,dy$$

The operators Φ_N are self–adjoint and diagonal in the Fourier basis. Denote by ϕ_k^N the eigenvalues of Φ_N. These are (up to constant) given by the Fourier coefficients of $z \mapsto N^2 q(Nz)$.

Assume that $q \in H^\eta$ for some $\eta > 0$ and that $\phi_k^N \to 1$ as $N \to \infty$, which is easy to verify. Then it is straightforward to check that the assumptions of Lemma 2.2 and of Theorems 2.3 and 2.4 are verified. We comment in more details in Example 5.1.

Example 2.8. Here we study a non-diagonal case, which is for instance given by noise not homogeneous in space [1]. Define

$$\Phi_N f(x) = \int_{[0,2\pi]^2} N^2 q_N\big(x,y)\big) f(y)\,dy$$

with a kernel q_N which determined by a non-negative, smooth q such that the support of q is contained in a small neighbourhood of the diagonal $x = y$ such that $\int_{\mathbb{T}_2\times\mathbb{T}_2} q(x,y)dxdy = 1$. The kernel q_N is the periodic extension on $\mathbb{T}_2 \times \mathbb{T}_2$ of $(\xi,\eta) \mapsto q(N\xi, N\eta)$.

We can now again check all assumptions of Lemma 2.2 and Theorem 2.3 and 2.4. As before, let $\Psi^{(N)} = \pi_N$ be the projection onto the first Fourier modes. Again, (2.3) is true, once q is sufficiently smooth.

The bounds in (2.6) and (2.7) are easy to establish, as we verify later that $c_{m.n}$ is uniformly bounded. The crucial condition is (2.4), which requires some work and does not seem hold for arbitrary kernels. We comment on all these assumptions in detail later in Example 5.2.

2.3. **Rougher noise.** As it is apparent by the previous sections, space–time white noise is the borderline case between the standard theory for mild solutions and the additional work summarized by Theorem 2.1. It is then possible to consider rougher noise.

In view of the computations needed to define $\widetilde{\mathcal{B}}(z,z)$ (Lemma 3.4) it is reasonable to consider a simplified case, namely when the covariance operator we apply to white noise is diagonal in the Fourier basis. Consider a bounded linear operator Φ on $\mathring{L}^2(\mathbb{T}_2)$ and assume for the rest of this section the following properties,

- $\Phi\,\mathrm{e}_k = \phi_k e_k$ for every $k \in \mathbf{Z}^2_\star$,
- there is $\beta > 0$ such that $|\phi_k|^2 \le c|k|^\beta$ for every $k \in \mathbf{Z}^2_\star$.

This situation is similar to Example 2.7 before, when we consider kernels q given by a distribution instead of a function.

The value $\beta = 0$ is morally the space–time white noise. Moreover, the definition of $\widetilde{\mathcal{B}}(z,z)$ imposes a structural restriction that limits the range of possible values of β to $\beta < 1$ (see Remark 6.4).

The same ideas of Section 2.1, when slightly modified to take into account the parameter β, lead to the following result.

Theorem 2.9. *Assume $\beta \in (0,\frac{2}{3})$ and let $h_0 \in \mathring{H}^1_\wp$ and $\epsilon \in (\frac{\beta}{2},(1-\beta)\wedge\frac{1}{2})$. Given a cylindrical Wiener process W on $\mathring{L}^2(\mathbb{T}_2)$, there exist a stopping time τ_{h_0} and a solution h of (6.1) understood as $h = v + z$, where z is given by (6.2) and $h - z$ satisfies the mild formulation (2.2) on $[0,\tau_{h_0})$. Moreover, $h \in C([0,\tau_{h_0});\mathring{L}^2(\mathbb{T}_2))$, $h - z \in C((0,\tau_{h_0});\mathring{H}^{1+\epsilon}_\wp)$ and $\mathbb{P}[\tau_{h_0} > 0] = 1$.*

The restriction $\beta < \frac{2}{3}$ is due to the term $\mathcal{B}(v,z)$ in the mild formulation (2.2). When the noise is too rough, the auxiliary function v is not enough regular to ensure that the product $\mathcal{B}(v,z)$ is well–defined.

Assume now $\beta \in [\frac{2}{3},1)$. To overcome the difficulty caused by the poor regularity of both v and z, we add a term in the second Wiener chaos in the decomposition of h, namely $h = u + \zeta + z$, where ζ solves

$$\dot{\zeta} + A\zeta + \widetilde{\mathcal{B}}(z,z) = 0, \qquad \zeta(0) = 0,$$

and u is the mild solution of

$$\dot{u} + Au + \mathcal{B}(u,u) + 2\mathcal{B}(u,z) + 2\mathcal{B}(u,\zeta) + 2\widetilde{\mathcal{B}}(\zeta,z) + \mathcal{B}(\zeta,\zeta) = 0,$$

with initial condition $u(0) = h(0)$. To this end we need to suitably define $\widetilde{\mathcal{B}}(\zeta,z)$ as we have already done for $\widetilde{\mathcal{B}}(z,z)$, by exploiting the cancellations in the expectations of these processes. This gives no gain for ζ (we have already "used" the cancellation to define $\widetilde{\mathcal{B}}(z,z)$) but it is effective both in improving the regularity of $\mathcal{B}(\zeta,\zeta)$ (with respect to what we would get from standard multiplication theorems in Sobolev spaces), and in defining $\widetilde{\mathcal{B}}(\zeta,z)$.

Actually, the approach through the higher Wiener chaos expansion of h can be used for any value of $\beta \in (0,1)$. Indeed, it is sufficient to define $\widetilde{\mathcal{B}}(\zeta,z) = \mathcal{B}(\zeta,z)$ whenever the latter term is well defined (see Remark 6.8). We are able then to prove the following result.

Theorem 2.10. *Assume $\beta \in (0,1)$ and let $h_0 \in \mathring{H}^1_\wp$ and $\epsilon \in (\frac{\beta}{2},\frac{1}{2})$. Given a cylindrical Wiener process W on $\mathring{L}^2(\mathbb{T}_2)$, there exist a stopping time τ_{h_0} and a solution h of (6.1) understood as $h = u + \zeta + z$, where z is given by (6.2), ζ by (6.4) and $u = h - z - \zeta$ satisfies the mild formulation (6.3) on $[0,\tau_{h_0})$. Moreover, $h \in C([0,\tau_{h_0});\mathring{L}^2(\mathbb{T}_2))$, $h - z - \zeta \in C((0,\tau_{h_0});\mathring{H}^{1+\epsilon}_\wp)$ and $\mathbb{P}[\tau_{h_0} > 0] = 1$.*

3. The stochastic convolution

Let z be the stochastic convolution

$$z(t) = \int_0^t S(t-s)\,dW_s,$$

namely the solution of

$$dz + Az\,dt = dW,$$

with initial condition $z(0) = 0$ and zero mean. The stochastic convolution can be expanded in the complex Fourier basis,

$$z = \sum_{k\in\mathbf{Z}^2_\star} z_k e_k, \qquad z_k(t) = \int_0^t \mathrm{e}^{-|k|^4(t-s)}\, d\beta_k(s), \tag{3.1}$$

where $\beta_k = \langle W_t, e_k\rangle$, $\beta_{-k} = \bar\beta_k$, and $(\beta_k)_{k\in\mathbf{Z}^2_+}$ is a sequence of independent complex–valued standard Brownian motions.

Due to the bi–Laplace operator, the stochastic convolution is function–valued. On the other hand the stochastic convolution is not sufficiently regular to define the non–linear term $\Delta|\nabla z|^2$ as a function (and neither as a distribution), see Lemma 3.1 below. It turns out that, suitably defined, the term $\Delta|\nabla z|^2$ makes sense.

Lemma 3.1. *For every $t > 0$,*

$$\mathbb{E}[\|z(t)\|^2_{H^1}] = \infty$$

and $z \notin \mathring{H}^1_\wp$ for all times, almost surely.

Proof. Using the explicit representation of z in Fourier series,

$$\mathbb{E}[\|\nabla z(t)\|^2_{L^2}] = \sum_{k\in\mathbf{Z}^2_\star} |k|^2 \int_0^t \mathrm{e}^{-2|k|^4(t-s)}\, ds = \sum_{k\in\mathbf{Z}^2_\star} \frac{1}{2|k|^2}(1 - \mathrm{e}^{-2|k|^4 t}) = \infty.$$

The almost sure statement follows from Gaussianity [6, Theorem 2.5.5]. □

3.1. **Regularity in Sobolev spaces.** Lemma 3.1 above shows that the gradient of z is not defined. On the other hand, z has a "fractional" derivative of any order smaller than one.

Proposition 3.2. *For every $p \geq 1$ and $s \in (0, 1)$,*

$$\sup_{t>0} \mathbb{E}\big[\|z(t)\|^p_{W^{s,p}}\big] < \infty.$$

Proof. Use the Fourier representation of z to get,

$$\begin{aligned}\mathbb{E}[|z(t,x) - z(t,y)|^2] &\leq c \sum_{k\in\mathbf{Z}^2_\star} \mathbb{E}[|z_k(t)|^2]|e_k(x) - e_k(y)|^2 \leq \\ &\leq c \sum_{k\in\mathbf{Z}^2_\star} \frac{1 \wedge |k\cdot(x-y)|^2}{|k|^4} \leq c|x-y|^2 \log(8\pi|x-y|^{-1}),\end{aligned}$$

where to estimate the last sum on the right hand side of the formula above one can split in the two parts $|k| \geq |x-y|^{-1}$ and $|k| \leq |x-y|^{-1}$. By Gaussianity, for every $p \geq 1$, $\mathbb{E}[|z(t,x) - z(t,y)|^p] \leq c_p|x-y|^p \log^{p/2}(8\pi|x-y|^{-1})$. Therefore,

$$\mathbb{E}\Big[\iint \frac{|z(t,x) - z(t,y)|^p}{|x-y|^{2+sp}}\, dx\, dy\Big] \leq c \iint \frac{\log^{p/2}(8\pi|x-y|^{-1})}{|x-y|^{2-(1-s)p}}\, dx\, dy < \infty. \qquad \square$$

Remark 3.3. The regularity in time stated in the previous proposition can be improved, with standard arguments, to L^∞ or even Hölder, but we will not use this fact in the paper.

3.2. **The non–linearity for the stochastic convolution.** If $u = \sum_{k\in\mathbf{Z}^2} u_k e_k$ and $v = \sum_{k\in\mathbf{Z}^2} v_k e_k$ are real valued, the non–linear term can be formally written in terms of the Fourier coefficients as

$$\mathcal{B}(u,v) = \sum_{k\in\mathbf{Z}^2_\star} |k|^2 \Big(\sum_{m+n=k} m\cdot n\ u_m v_n\Big) e_k.$$

Consider the stochastic convolution z and set for every $k \in \mathbf{Z}^2_\star$,

$$J_k(t) = \sum_{m+n=k} m\cdot n\ z_m(t) z_n(t). \tag{3.2}$$

Formally, $\mathcal{B}(z,z) = \sum_k |k|^2 J_k e_k$. Lemma 3.1 immediately tells us that $J_0(t) = -\|\nabla z(t)\|_{L^2} = \infty$ almost surely. Likewise, an investigation of absolute convergence of $J_k(t)$ for $k \neq 0$ yields

$$\mathbb{E}\Big[\sum_{m+n=k} |m\cdot n|\,|z_m(t)\,z_n(t)|\Big] \geq \sum_{\substack{m+n=k\\ m\neq n}} |m\cdot n| \mathbb{E}[|z_m(t)|]\mathbb{E}[|z_n(t)|] \geq \sum_{\substack{m+n=k\\ m\neq n}} \frac{c_t |m\cdot n|}{|m|^2|n|^2} = \infty.$$

Following [7], we extend the definition of the non–linearity $\mathcal{B}$ so that the terms $J_k(t)$, for $k \neq 0$, are convergent. This is possible due to cancellations, since the z_m are centred Gaussians. The term $J_0(t) = -\|\nabla z(t)\|_{L^2}$ should be the most problematic, since there is no hope to exploit any cancellation. On the other hand it is constant in the space variable and it is cancelled by the Laplace operator.

Given $N \geq 1$, let H_N be the linear sub–space of $L^2(\mathbb{T}_2)$ spanned by $(e_k)_{0<|k|\leq N}$. Let π_N be the projection of $L^2(\mathbb{T}_2)$ onto H_N and define

$$\mathcal{B}_N(u,v) = \mathcal{B}(\pi_N u, \pi_N v).$$

We extend the operator $\mathcal{B}$ on the non–differentiable function z as the limit of the sequence $(\mathcal{B}_N(z,z))_{N\geq 1}$ in suitable function spaces.

Lemma 3.4. *Let z be the stochastic convolution. Then $(\mathcal{B}_N(z,z))_{N\geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathring{H}^{-2-\gamma}_\wp)$ for every $\gamma > 0$. In particular, the limit $\mathcal{B}(z,z)$ is well–defined as an element of $\mathring{H}^{-2-\gamma}_\wp$.*

Proof. Let J_k^N be the term analogous to J_k for $\pi_N z$. If $N \leq N'$,

$$\mathbb{E}[|J_k^N(t) - J_k^{N'}(t)|^2] = \sum_{m_1+n_1=k}^{N\leftrightarrow N'} \sum_{m_2+n_2=k}^{N\leftrightarrow N'} m_1\cdot n_1\ m_2\cdot n_2 \mathbb{E}[z_{m_1} z_{n_1} \bar{z}_{m_2} \bar{z}_{n_2}],$$

where by the symbol $N \leftrightarrow N'$ in the sum over m, n we mean that the sum is extended only over those indices m, n that satisfy $N < |m| \vee |n| \leq N'$.

The sequence $(z_m)_{m\in\mathbf{Z}^+}$ is a family of independent centred Gaussian random variables. Moreover $\bar z_m = z_{-m}$. A few elementary considerations show that $\mathbb{E}[z_{m_1} z_{n_1} \bar z_{m_2} \bar z_{n_2}]$ is non–zero only if $m_1 = m_2$ and $n_1 = n_2$, or if $m_1 = n_2$ and $m_2 = n_1$. Therefore

$$\mathbb{E}[|J_k^N(t) - J_k^{N'}(t)|^2] = 2 \sum_{m+n=k}^{N\leftrightarrow N'} (m\cdot n)^2 \mathbb{E}[|z_m|^2 |z_n|^2] \leq$$

$$\leq c \sum_{m+n=k}^{N\leftrightarrow N'} \frac{(m\cdot n)^2}{|m|^4|n|^4}(1-\mathrm{e}^{-2|m|^4 t})(1-\mathrm{e}^{-2|n|^4 t}) \leq c \sum_{m+n=k}^{N\leftrightarrow N'} \frac{1}{|m|^2|n|^2}.$$

The last series above can be estimated with Lemma A.4, indeed since $|m|\vee|n| \geq N$,

$$\sum_{m+n=k}^{N\leftrightarrow N'} \frac{1}{|m|^2|n|^2} \leq \frac{2}{N^\gamma} \sum_{\substack{m+n=k\\ |n|\leq|m|}}^{N\leftrightarrow N'} \frac{1}{|m|^{2-\gamma}|n|^2} \leq \frac{2}{N^\gamma} \sum_{m+n=k}^{N\leftrightarrow N'} \frac{1}{|m|^{2-\gamma}|n|^2} \leq \frac{c}{N^\gamma |k|^{2-\gamma}}.$$

In conclusion,

$$\mathbb{E}[\|\widetilde{\mathcal{B}}_N(z,z) - \widetilde{\mathcal{B}}_{N'}(z,z)\|_{H^{-2-\gamma}}^2] = \sum_{k\in\mathbf{Z}_\star^2} |k|^{-2\gamma} \mathbb{E}[|J_k^N - J_k^{N'}|^2] \leq \frac{c}{N^\gamma} \sum_{k\in\mathbf{Z}_\star^2} \frac{1}{|k|^{2+\gamma}},$$

and the term on the right hand side converges to zero as $N, N' \to \infty$. □

Remark 3.5. In order to define $\widetilde{\mathcal{B}}$ we have chosen Galerkin projections as regularizations of the underlying Wiener–process and passed to the limit in order to define the solution. We will see that other regularizations, for example convolution operators, yield exactly the same result.

We shall need higher moments of $\widetilde{\mathcal{B}}(z,z)$ for our considerations on the non–linear problem. We shall derive the claim from hyper–contractivity of Gaussian measures [16, 20].

Proposition 3.6. *Given $\gamma > 0$ and $p > 1$, there is a constant $c > 0$ such that*

$$\sup_{t>0} \mathbb{E}\big[\|\widetilde{\mathcal{B}}(z(t), z(t))\|_{H^{-2-\gamma}}^p\big] \leq c.$$

Proof. For the second moment we can proceed as in the previous lemma, using again the elementary estimate of Lemma A.4,

$$\mathbb{E}[|J_k^N|^2] = \sum_{\substack{m+n=k\\ |m|,|n|\leq N}} (m\cdot n)^2 \mathbb{E}[|z_m|^2|z_n|^2] \leq c \sum_{m+n=k} \frac{1}{|m|^2|n|^2} \leq \frac{c}{|k|^2}\log(1+|k|).$$

Thus

$$\mathbb{E}\big[\|\mathcal{B}_N(z(t),z(t))\|_{H^{-2-\gamma}}^2\big] \leq \sum_k |k|^{-2\gamma}\mathbb{E}[|J_k^N|^2] \leq c\sum_k \frac{\log(1+|k|)}{|k|^{2+2\gamma}}$$

and the second moment is finite. To prove that all moments are finite, consider an integer $p \geq 1$. Theorem I.22 of [20] yields that

$$\mathbb{E}[|J_k^N|^{2p}] \leq (2p-1)^{2p}\big(\mathbb{E}[|J_k^N|^2]\big)^p,$$

hence by the Hölder inequality,

$$\begin{aligned}\mathbb{E}[\|\mathcal{B}_N(z(t),z(t))\|^{2p}_{H^{-2-\gamma}}] = \mathbb{E}\Big[\Big(\sum_k |k|^{-2\gamma}|J_k^N|^2\Big)^p\Big] \leq \\ \leq \Big(\sum_k |k|^{-2\gamma}\big(\mathbb{E}[|J_k^N|^{2p}]\big)^{\frac{1}{p}}\Big)^p \leq c_p,\end{aligned}$$

and the moment of order $2p$ is uniformly bounded in time. □

4. Proof of Theorem 2.1

Fix $\epsilon \in (0, \frac{1}{2})$ and $h_0 \in \mathring{H}^1_\wp$. Let $\mathcal{T}$ be the operator that takes as its value the right–hand side of (2.2), namely

$$\mathcal{T}v(t) := S(t)h_0 - \int_0^t S(t-s)\mathcal{B}(v,v)\,ds - \int_0^t S(t-s)\big(\widetilde{\mathcal{B}}(z,z) + 2\mathcal{B}(z,v)\big)\,ds.$$

We use the standard contraction fixed point theorem. To this end we show that by choosing ρ, T suitably, $\mathcal{T}$ maps $\mathcal{X}(\epsilon,\rho,T)$ into itself. By possibly taking a smaller value of ρ, $\mathcal{T}$ is also a contraction.

The self–mapping property. Set $R_{h_0}(T) := \sup_{[0,T]} t^{\frac{\epsilon}{4}}\|S(t)h_0\|_{H^{1+\epsilon}}$, then it is easy to show ([9], see also [3, Lemma C.1]) that $R_{h_0}(T) \longrightarrow 0$ as $T \to 0$. Continuity of $\mathcal{T}v$ in $L^2(\mathbb{T}_2)$ is standard (see [3, Lemma C.1]). Moreover, by Lemma A.1, with $a \in [1-2\epsilon, 1-\epsilon)$, if $t \leq T$,

$$\begin{aligned}\Big\|\int_0^t S(t-s)\mathcal{B}(v,v)\,ds\Big\|_{H^{1+\epsilon}} &\leq \int_0^t \|A^{\frac{1}{4}(3+a+\epsilon)}S(t-s)A^{-\frac{1}{4}(a+2)}\mathcal{B}(v,v)\|_{L^2}\,ds \\ &\leq \int_0^t \frac{c}{(t-s)^{\frac{1}{4}(3+a+\epsilon)}}\|v(s)\|^2_{H^{1+\epsilon}}\,ds \\ &\leq ct^{-\frac{\epsilon}{4}}\|v\|^2_{\epsilon,T}t^{\frac{1}{4}(1-a-2\epsilon)}.\end{aligned}$$

The mixed term is estimated with the help of Corollary A.3, with $\gamma < 1-\epsilon$, $\alpha = 1-(\epsilon\wedge\gamma)/2$ and $q > 4/(\epsilon\wedge\gamma\wedge(1-\epsilon-\gamma))$, and the Hölder inequality,

$$\begin{aligned}\Big\|\int_0^t S(t-s)\mathcal{B}(v,z)\,ds\Big\|_{H^{1+\epsilon}} &\leq \int_0^t \|A^{\frac{1}{4}(3+\epsilon+\gamma)}S(t-s)A^{-\frac{1}{4}(2+\gamma)}\mathcal{B}(v,z)\|_{L^2}\,ds \\ &\leq \|v\|_{\epsilon,T}\int_0^t \frac{c\|z\|_{W^{\alpha,q}}}{(t-s)^{\frac{1}{4}(3+\epsilon+\gamma)}s^{\frac{\epsilon}{4}}}\,ds \\ &\leq ct^{-\frac{\epsilon}{4}}T^{e_1}\|v\|_{\epsilon,T}\Big(\int_0^T \|z\|^q_{W^{\alpha,q}}\,ds\Big)^{\frac{1}{q}}.\end{aligned}$$

where $e_1 = \frac{1}{4}(1-\epsilon-\gamma) - \frac{1}{q}$ is positive and the integrals are well defined by the choice of q.

Finally for the quadratic term in z we use Proposition 3.6. Let $2 < \gamma' < 3 - \epsilon$ and p' such that $\frac{1}{4}p'(1+\gamma'+\epsilon) < 1$, then by the Hölder inequality,

$$\begin{aligned}\Big\|\int_0^t S(t-s)\widetilde{\mathcal{B}}(z,z)\,ds\Big\|_{H^{1+\epsilon}} &\leq \int_0^t \big\|A^{\frac{1}{4}(1+\epsilon+\gamma')}S(t-s)\big(A^{-\frac{\gamma'}{4}}\widetilde{\mathcal{B}}(z,z)\big)\big\|_{L^2}\,ds\\ &\leq \Big(\int_0^t (t-s)^{-\frac{1}{4}(1+\epsilon+\gamma')p'}\,ds\Big)^{\frac{1}{p'}}\Big(\int_0^t \|\widetilde{\mathcal{B}}(z,z)\|_{H^{-\gamma'}}^{q'}\,ds\Big)^{\frac{1}{q'}}\\ &\leq ct^{-\frac{\epsilon}{4}}T^{e_2}\Big(\int_0^T \|\widetilde{\mathcal{B}}(z,z)\|_{H^{-\gamma'}}^{q'}\,ds\Big)^{\frac{1}{q'}},\end{aligned}$$

where q' is the Hölder conjugate exponent of p' and $e_2 = \frac{1}{p'} - \frac{1}{4}(\gamma'+1)$ is positive.

The three estimates together yield

$$\|\mathcal{T}v\|_{\epsilon,T} \leq R_{h_0}(T) + c_0\rho^2 + c_0\rho T^{e_1}Z_1(T) + c_0T^{e_2}Z_2(T), \tag{4.1}$$

where $Z_1(T)$ is the norm of z in $L^q(0,T;\mathring{W}^{\alpha,q}_{\wp})$, and $Z_2(T)$ is the norm of $B(z,z)$ in $L^{q'}(0,T;H^{-\gamma'}_{\wp})$. All the quantities in the displayed formula above converge to 0 as $T\to 0$, so for T small enough we can find a positive value of ρ that satisfies the self–mapping property.

The contraction property. The contraction property follows from similar estimates. Let $v_1, v_2 \in \mathcal{X}(\epsilon,\rho,T)$, then

$$\mathcal{T}v_1(t) - \mathcal{T}v_2(t) = \int_0^t S(t-s)\mathcal{B}(v_1+v_2, v_2-v_1)\,ds + 2\int_0^t S(t-s)\mathcal{B}(z, v_2-v_1).$$

We use Lemma A.1 for the first term and Corollary A.3 for the second term (with the same choice for the value of the parameters as the previous part),

$$\begin{aligned}\|\mathcal{T}v_1 - \mathcal{T}v_2\|_{\epsilon,T} &\leq c\|v_1+v_2\|_{\epsilon,T}\|v_1-v_2\|_{\epsilon,T} + cT^{e_1}Z_1(T)\|v_1-v_2\|_{\epsilon,T}\\ &\leq c_0\big(\rho + T^{e_1}Z_1(T)\big)\|v_1-v_2\|_{\epsilon,T}.\end{aligned} \tag{4.2}$$

and again by choosing T small enough the mapping is a contraction.

Given $a \in (0,1)$ and $b \in \big(0, \frac{a^2}{4c_0}\big)$, where c_0 is the constant appearing in (4.1) and (4.2), choose ρ such that $c_0\rho^2 - a\rho + b \leq 0$. Let

$$\tau_a^c := \inf\{t : c_0t^{e_1}Z_1(t) > 1-a\}, \qquad \tau_b^s := \inf\{t : R_{h_0}(t) + c_0t^{e_2}Z_2(t) > b\},$$

and choose $T < \tau_a^c \wedge \tau_b^s$. With these choices and positions, it is immediate to verify that the right–hand side of (4.1) is smaller or equal than ρ and that the Lipschitz constant of $\mathcal{T}$ appearing in (4.2) is smaller than 1. It turns out that $\tau_{h_0} \geq \tau_a^c \wedge \tau_b^s$ and, since $\tau_a^c > 0$ and $\tau_b^s > 0$ almost surely, the same holds for τ_{h_0}.

5. Other regularizations

Let Φ be a bounded operator on $\mathring{L}(\mathbb{T}_2)$ and consider the associated stochastic convolution,

$$z^\Phi(t) = \int_0^t S(t-s)\Phi\,dW(s) = \sum_{k\in\mathbf{Z}^2_\star} z^\Phi_k(t)e_k,$$

where

$$z^\Phi_k(t) = \int_0^t \mathrm{e}^{-(t-s)|k|^4}\,d\beta^\Phi_k, \qquad \text{and} \qquad \beta^\Phi_k(t) = \langle \Phi W(t), e_k\rangle,$$

and the β^Φ_k are Brownian motions. These are in general non independent, unless Φ is diagonal in the Fourier-basis e_k. We recall that Φ is regularizing if it satisfies the conclusions of Lemma 2.2, whose proof is given below.

Proof of Lemma 2.2. Condition (2.3) ensures that the terms J_k defined in (3.2) are a.s. absolutely convergent, hence the computations in Lemma 3.4 can be made rigorously for z^Φ without relying on the spectral truncation. □

5.1. **Good approximations.**

Here we prove Theorems 2.3 and 2.4. The proof of Theorem 2.3 is a straightforward modification of Proposition 3.2.

Proof of Theorem 2.3. Let $\gamma\in(0,1)$ be the value given in the statement of the theorem. If $x,y\in\mathbb{T}_2$,

$$\begin{aligned}
&\mathbb{E}\big[|z^{\Phi_N}(t,x) - z(t,x) - z^{\Phi_N}(t,y) + z(t,y)|^2\big]\\
&\quad= \sum_{m,n}\mathbb{E}\big[(z^{\Phi_N}_m(t) - z_m(t))\overline{(z^{\Phi_N}_n(t) - z_n(t))}\big](e_m(x)-e_m(y))\overline{((e_n(x)-e_n(y))}\\
&\quad\leq C\sum_{m,n}\frac{|\langle(\Phi_N-I)^\star(\Phi_N-I)e_m,e_n\rangle|}{|m|^4+|n|^4}(1\wedge|m(x-y)|)(1\wedge|n(x-y)|)\\
&\quad\leq C\sum_{m,n}\frac{|\langle(\Phi_N-I)^\star(\Phi_N-I)e_m,e_n\rangle|}{(|m|+|n|)^{4-2\gamma}}|x-y|^{2\gamma},
\end{aligned}$$

where we used that

$$\mathbb{E}\big[(\beta^{\Phi_N}_m(1)-\beta_k(1))\overline{(\beta^{\Phi_N}_n(1)-\beta_n(1))}\big] = \langle(\Phi_N-I)^\star(\Phi_N-I)e_m,e_n\rangle.$$

As in Proposition 3.2, Gaussianity and the definition of the norm in $W^{\alpha,p}$ yield

$$\sup_{t\in[0,T]}\mathbb{E}\|z^{\Phi_N}(t)-z(t)\|^p_{W^{\alpha,p}} \leq C\Big(\sum_{m,n}\frac{|\langle(\Phi_N-I)^\star(\Phi_N-I)e_m,e_n\rangle|}{(|m|+|n|)^{4-2\gamma}}\Big)^{p/2}.$$

The Lebesgue dominated convergence theorem for the double sum on the right hand side concludes the proof. □

Proof of Theorem 2.4. Consider two different regularizing operators Φ and Ψ. Of course we have in mind Φ_N and Ψ_N, but we omit the index N in the following. First

$$\mathcal{B}(z^\Phi, z^\Phi) - \mathcal{B}(z^\Psi, z^\Psi) = \mathcal{B}(z^\Phi + z^\Psi, z^\Phi - z^\Psi)\ .$$

Define for every $k \in \mathbf{Z}^2$,

$$B_k^\pm = \beta_k^\Phi \pm \beta_k^\Psi = \langle W(t), \Phi e_k \pm \Psi e_k\rangle \quad \text{and} \quad z_k^\pm = \int_0^t e^{-(t-s)|k|^4} dB_k^\pm\ .$$

Modify moreover the definition of J_k

$$\hat{J}_k(t) = \sum_{m+n=k} m \cdot n\ z_m^+(t) z_n^-(t).$$

Thus

$$\mathcal{B}(z^\Phi, z^\Phi) - \mathcal{B}(z^\Psi, z^\Psi) = \sum_{k\in\mathbf{Z}^2} |k|^2 \hat{J}_k e_k$$

and

$$\mathbb{E}\|\mathcal{B}(z^\Phi, z^\Phi) - \mathcal{B}(z^\Psi, z^\Psi)\|^2_{H^{-2-\gamma}} = \sum_{k\in\mathbf{Z}^2} |k|^{-2\gamma}\mathbb{E}|\hat{J}_k|^2 \tag{5.1}$$

By exchanging expectation and summation,

$$\mathbb{E}|\hat{J}_k|^2 = \sum_{\substack{m_1+n_1=k\\ m_2+n_2=k}} (m_1\cdot n_1)(m_2\cdot n_2)\mathbb{E} z_{m_1}^+ z_{n_1}^- \overline{z_{m_2}^+ z_{n_2}^-}.$$

Wick's formula [20, Proposition I.2] yields

$$\mathbb{E} z_{m_1}^+ z_{n_1}^- \overline{z_{m_2}^+ z_{n_2}^-} = \mathbb{E} z_{m_1}^+ z_{n_1}^- \mathbb{E}\overline{z_{m_2}^+ z_{n_2}^-} + \mathbb{E} z_{m_1}^+ \overline{z_{m_2}^+} \mathbb{E} z_{n_1}^- \overline{z_{n_2}^-} + \mathbb{E} z_{m_1}^+ \overline{z_{n_2}^-} \mathbb{E}\overline{z_{m_2}^+} z_{n_1}^-.$$

Hence, (using the symmetry of variables $n_2 \leftrightarrow m_2$ in the last term)

$$\begin{aligned}\mathbb{E}|\hat{J}_k|^2 = &\Big|\sum_{m+n=k}(m\cdot n)\mathbb{E} z_m^+ z_n^-\Big|^2\\ &+ \sum_{\substack{m_1+n_1=k\\ m_2+n_2=k}} (m_1\cdot n_1)(m_2\cdot n_2)\mathbb{E} z_{m_1}^+ \overline{z_{m_2}^+}\mathbb{E} z_{n_1}^- \overline{z_{n_2}^-}\\ &+ \sum_{\substack{m_1+n_1=k\\ m_2+n_2=k}} (m_1\cdot n_1)(m_2\cdot n_2)\mathbb{E} z_{m_1}^+ \overline{z_{m_2}^-}\mathbb{E}\overline{z_{n_2}^+} z_{n_1}^-.\end{aligned} \tag{5.2}$$

Since $\mathbb{E}\langle W(t), u\rangle\overline{\langle W(t), v\rangle} = \langle v, u\rangle$,

$$\mathbb{E} z_m^\pm \overline{z_\ell^\mp} = \int_0^t e^{-(t-s)(|m|^4+|\ell|^4)}\, ds \langle \Phi e_\ell \mp \Psi e_\ell, \Phi e_m \pm \Psi e_m\rangle.$$

Hence,

$$|\mathbb{E} z_m^\pm \overline{z_\ell^\mp}| \le \frac{1}{|m|^4+|\ell|^4}\big|\langle \Phi e_\ell \mp \Psi e_\ell, \Phi e_m \pm \Psi e_m\rangle\big|,$$

and similarly for other combinations of signs.

Let us now consider again the sequences Φ_N and Ψ_N. We treat the diagonal terms with $m_1 = m_2$ and $n_1 = n_2$ and the off–diagonal terms differently. We have to assume some uniform summability of the off-diagonal terms. This is ensured by the bounds (2.6) and (2.7). Moreover, all summands go to 0 in (5.2) due to the convergence in (2.5). Thus from (5.1) by the Lebesgue dominated convergence theorem:

$$\sup_{t\geq 0} \mathbb{E}\|\mathcal{B}(z^{\Phi_N}, z^{\Phi_N}) - \mathcal{B}(z^{\Psi_N}, z^{\Psi_N})\|^2_{H^{-2-\gamma}} \to 0 \qquad \text{for } N \to 0.$$

As before, using hyper–contractivity as in the proof of Proposition 3.6, we can show that this holds for all moments and not only for the second. □

Let us come back to the examples given in Section 2.1.

Example 5.1 (Example 2.7 resumed). The operators Φ_N are diagonal and self–adjoint, denote by ϕ^N_k the eigenvalues of Φ_N. These numbers are (up to a constant) determined by the Fourier coefficients of $N^2 q_N$. Write

$$N^2 q_N(z) = \sum_{m\in\mathbf{Z}^2} q^N_m e_m(z),$$

then $\Phi_N e_k = q^N_{-k}$, thus $\phi^N_k = q^N_{-k}$. It is easy to check now that (2.3) is a decay condition on the eigenvalues and a sufficient condition is given by $|\phi^N_k| \lesssim |k|^{-\eta}$ for some $\eta > 0$, that is q belongs to H^η. The off–diagonal assumptions are clearly verified. It remains to check the convergence (2.5) when the Φ_N are combined with the Galerkin truncation operators π_N. To this end, it is sufficient to show that $\phi^N_k \to 1$ as $N \to \infty$. This can be checked using the fact that q is supported around 0,

$$q^N_k = N^2 \int_{[-\pi,\pi]^2} q_N(x) e_{-k}(x)\,dx = \int_{[-\pi,\pi]^2} q(z) e_{-k/N}(z)\,dz \longrightarrow 1.$$

Example 5.2 (Example 2.8 resumed). For simplicity of notation extend the operator to complex valued functions by

$$\Phi_N f(x) = \int_{[0,2\pi]^2} N^2 q_N(x,y))\overline{f(y)}\,dy = \langle N^2 q_N(x,\cdot), f\rangle\ .$$

Recall that q_N is given by a non-negative smooth q supported in a small neighbourhood of the diagonal $x = y$, such that q_N is a periodic extension of $q(Nx, Ny)$ on $\mathbb{T}_2 \times \mathbb{T}_2$. Denote by $q^{(N)}_{k,\ell}$ the Fourier coefficients of $N^2 q_N$, i.e.

$$N^2 q_N(x,y) = \sum_{k,\ell\in\mathbf{Z}^2} q^{(N)}_{k,\ell} e_k(x) e_\ell(y)\ .$$

This immediately implies, that

$$\langle \Phi_N e_m, e_n\rangle = \langle \sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} e_k, e_n\rangle = q_{n,m}^{(N)}$$

and

$$\langle \Phi_N e_m, \Phi_N e_n\rangle = \langle \sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} e_k, \sum_{k\in\mathbf{Z}^2} q_{k,n}^{(N)} e_k\rangle = \sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} q_{k,n}^{(N)} .$$

We can now again check all assumptions of Lemma 2.2 and Theorem 2.3 and 2.4. First (2.3) is true, once q is sufficiently smooth, for example in some H^ϵ, as this implies that $q_{k,m}^{(N)} \leq \|N^2 q_N\|_{H^\epsilon}(|k|^2+|\ell|^2)^{\epsilon/2}$ by Lemma A.4.

For the next steps, as before, let $\Psi^{(N)} = \pi_N$ be the projection onto the first Fourier modes. Now,

$$c_{m,n} \leq \sup_{N\in\mathbf{N}}\{|\sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} q_{k,n}^{(N)}| + \delta_{m,n} + q_{n,m}^{(N)}\}.$$

The bounds in (2.6) and (2.7) are easy to establish, as we can verify that $c_{m,n}$ is uniformly bounded together with the fact that, by Lemma A.4, $\sum_{m+n=k}|m|^{-3}|n|^{-3} \leq C(1+|k|)^{-3}$. In order to establish uniform bounds on $c_{m,n}$, consider

$$q_{n,m}^{(N)} = N^2 \int_{\mathbb{T}_2}\int_{\mathbb{T}_2} q_N(x,y) e_{-n}(x) e_{-m}(y)\, dx\, dy,$$

and, as $\langle q_N(x,\cdot,e_m\rangle = \sum_{\ell\in\mathbf{Z}^2} q_{\ell,m} e_k(x)$,

$$\sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} q_{k,n}^{(N)} = N^4 \int_{\mathbb{T}_2}\int_{\mathbb{T}_2} q_N(x,y) q_N(x,z) e_{-m}(x) e_{-n}(y)\, dx\, dy.$$

The bounds now follow immediately from substituting N and bounds on the support of q. With some more effort, one can also verify that

$$q_{n,m}^{(N)} = N^{-2}\int_{N\mathbb{T}_2}\int_{N\mathbb{T}_2} q(x,y) e_{-n/N}(x) e_{-m/N}(y) dx dy \to \delta_{n,m} \qquad \text{for} \quad N\to\infty .$$

We rely on a splitting on $N\mathbb{T}_2$ into in N translated copies of $\mathbb{T}_2$ here. The convergence of $\sum_{k\in\mathbf{Z}^2} q_{k,m}^{(N)} q_{k,n}^{(N)}$ to a Kronecker-Delta is more involved, and we skip details here. The crucial condition is (2.4), which does not seem to hold for arbitrary kernel q. First let us remark, that main problem in (2.4) are the terms with $m\neq n$, as the diagonal terms are easily summable under our assumption.

A weak sufficient condition for the off diagonal term would be to assume that for some $\xi > 1$,

$$|q_{k,\ell}^{(N)}| \leq C(2+|k|^2+|\ell|^2)^{-\xi}.$$

Now we can verify by comparison with integrals, that

$$|\langle(\Phi_N - I)e_n, (\Phi_N - I)e_m\rangle| \leq \sum_{k\in\mathbf{Z}^2} |q_{k,m}^{(N)} q_{k,n}^{(N)}| + |q_{m,n}^{(N)}| + \delta_{m,n} \leq C(1+|m|+|n|)^{-2\xi} .$$

Thus (2.4) is true, as long as $\gamma < \alpha$.

5.2. **Proof of the stability theorem.** In this section we prove Theorem 2.5. Let $h_0 \in \mathring{H}^1_\wp(\mathbb{T}_2)$, $\epsilon \in (0, \frac{1}{2})$ and let $(\Phi_N)_{N\in\mathbf{N}}$ be a sequence of regularizing operators satisfying the convergence property (2.8). In the following the index N is omitted, and we use a general regularizing operator Φ first.

Step 1: Existence. Fix $\epsilon \in (0, \frac{1}{2})$ from the proof of existence of solution in Theorem 2.1 and let $v = h - z$ be the solution from Theorem 2.1. We first establish:

Theorem 5.3. *Assume that the regularized operator Φ is such that $z^\Phi \in C^0([0,1], \mathring{H}^{1+\epsilon}_\wp)$. Then the equation with regularized noise z^Φ has a unique local solution v^Φ in $\mathcal{X}(\epsilon, \rho, T)$ for some small random $T > 0$.*

This is follows from [4] or analogous to the result presented in this paper in Theorems 2.1, 2.9, or 2.10.

It is straightforward to verify that $v^\Phi - S(t)h_0$ and $v - S(t)h_0$ are both continuous with values in $\mathring{H}^{1+\epsilon}_\wp$ locally close to 0. Moreover, we can continue all v^Φ by standard arguments as an $\mathring{H}^{1+\epsilon}_\wp$-valued continuous function, until they blow up in $H^{1+\epsilon}$.

Recall for a given large radius $R > 0$

$$\tau^R = \inf\{t > 0 : \ \|v(t) - S(t)h_0\|_{H^{1+\epsilon}} > R\}$$

Moreover, we can define $\tau^\Phi > 0$ as the maximal time of existence in $H^{1+\epsilon}$, at which v^Φ blows up.

Step 2: Bounding the error. We can define for all $t \in [0, \tau^\Phi \wedge \tau^R)$ the error

$$d^\Phi(t) = v^\Phi(t) - v(t) \ .$$

Define the stopping time, where the error exceeds 1:

$$\tau^\star = \inf\{t > 0 : \ \|d^\Phi\|_{H^{1+\epsilon}} > 1\} \wedge \tau^R \wedge 1$$

Obviously, $\tau^\Phi \geq \tau^\star > 0$. Using Itô-formula, we have

$$\begin{aligned} d^\Phi(t) &= \int_0^t S(t-s)[\mathcal{B}(v^\Phi, v^\Phi) - \mathcal{B}(v,v) + \mathcal{B}(v^\Phi, z^\Phi) - \mathcal{B}(v,z) + \mathcal{B}(z^\Phi, z^\Phi) - \tilde{\mathcal{B}}(z,z)]ds \\ &= \int_0^t S(t-s)[\mathcal{B}(d^\Phi, d^\Phi) + 2\mathcal{B}(v, d^\Phi) + \mathcal{B}(d^\Phi, z) \\ &\qquad\qquad + \mathcal{B}(d^\Phi, z^\Phi - z) + \mathcal{B}(v, z^\Phi - z) + \mathcal{B}(z^\Phi, z^\Phi) - \tilde{\mathcal{B}}(z,z)]ds \end{aligned}$$

Here, we rewrote all terms in a way that they depend only on v, z, d^Φ, $z - z^\Phi$, and $v - v^\Phi$. Thus, using the bounds from the proof of Theorem 2.1 with

$$\gamma = \epsilon, \quad \alpha = \frac{1}{2} - \epsilon, \quad q > \frac{4}{\epsilon}, \quad \beta \in (2, 3-\epsilon) \quad \text{and} \quad q' > 4/(3 - \beta - \epsilon)$$

yields

$$\|d^\Phi(t)\|_{H^{1+\epsilon}} \leq C\int_0^t (t-s)^{-\frac{1}{4}(4-\epsilon)}(\|d^\Phi\|^2_{H^{1+\epsilon}} + \|v\|_{H^{1+\epsilon}}\|d^\Phi\|_{H^{1+\epsilon}})ds$$
$$+ C\int_0^t (t-s)^{-\frac{1}{4}(3+2\epsilon)}\|d^\Phi\|_{H^{1+\epsilon}}(\|z\|_{W^{\alpha,q}} + \|z^\Phi - z\|_{W^{\alpha,q}})ds$$
$$+ C\int_0^t (t-s)^{-\frac{1}{4}(3+2\epsilon)}\|v\|_{H^{1+\epsilon}}\|z^\Phi - z\|_{W^{\alpha,q}}ds$$
$$+ C\|\mathcal{B}(z^\Phi, z^\Phi) - \tilde{\mathcal{B}}(z,z)\|_{L^{q'}([0,1],H^{-\beta})}\ .$$

The definition of $\tau^\star$ and using $t \in [0,1]$ yields

$$\|d^\Phi(t)\|_{H^{1+\epsilon}} \leq C\int_0^t (t-s)^{-\frac{1}{4}(4-\epsilon)}\|d^\Phi\|_{H^{1+\epsilon}}(1 + \|v\|_{H^{1+\epsilon}} + \|z\|_{W^{\alpha,q}} + \|z^\Phi - z\|_{W^{\alpha,q}})ds$$
$$+ C\int_0^t (t-s)^{-\frac{1}{4}(3+2\epsilon)}\|v\|_{H^{1+\epsilon}}\|z^\Phi - z\|_{W^{\alpha,q}}ds$$
$$+ C\|\mathcal{B}(z^\Phi, z^\Phi) - \tilde{\mathcal{B}}(z,z)\|_{L^{q'}([0,1],H^{-\beta})}\ .$$

Now define the random variable of the error

$$\mathcal{E}^\Phi = \|z^\Phi - z\|_{L^p([0,1],W^{\alpha,q})} + \|\mathcal{B}(z^\Phi, z^\Phi) - \tilde{\mathcal{B}}(z,z)\|_{L^{q'}([0,1],H^{-\beta})}\ .$$

This simplifies the previous estimate for $t \in [0, \tau^\star)$ to

$$\|d^\Phi(t)\|_{H^{1+\epsilon}} \leq C\int_0^t (t-s)^{-\frac{1}{4}(4-\epsilon)}\|d^\Phi\|_{H^{1+\epsilon}}(1 + R + \|z\|_{W^{\alpha,p}} + C\mathcal{E}^\Phi)ds + C\mathcal{E}^\Phi\ .$$

Step 3: Estimates in probability. Now define for $\delta \in (0,1)$ the set

$$\Omega_{\Phi,\delta} = \{\mathcal{E}_\Phi \leq \delta\}$$

which is a large set, in case if z^ϕ approximates z well, as $\mathbb{P}(\Omega_{\Phi,\delta}) \geq 1 - \mathbb{E}\mathcal{E}_\Phi/\delta$.

Thus on the set $\Omega_{\Phi,\delta}$ for $t \leq \tau^\star$ we obtain

$$\|d^\Phi(t)\|_{H^{1+\epsilon}} \leq C\int_0^t (t-s)^{-\frac{1}{4}(4-\epsilon)}\|d^\Phi\|_{H^{1+\epsilon}} \cdot ds(1 + \sup_{[0,1]}\|z\|_{W^{\alpha,p}}) + C\delta$$

Using Gronwall's inequality for $\frac{1}{\delta}\|d\|_{H^{1+\epsilon}}$ in the version of [14] yields the existence of a finite random constant $C(\omega)$ independent of δ such that

$$\sup_{t\in[0,\tau^\star]}\|d^\Phi(t)\|_{H^{1+\epsilon}} \leq C(\omega)\delta$$

Hence, we verified that for all $\delta_0 \in (0,1)$ on the set $\{C(\omega) < \delta_0/\delta\} \cap \Omega_{\Phi,\delta}$ we have $\tau^\star \geq 1 \wedge \tau_R$ and $\|d^\Phi\|_{H^{1+\epsilon}} \leq \delta_0$. Thus

$$\begin{aligned}\mathbb{P}(\sup_{[0,1\wedge\tau_R]} \|d^\Phi\|_{H^{1+\epsilon}} > \delta_0) &\leq \mathbb{P}(\{C(\omega) \geq \delta_0/\delta\} \cup \Omega^c_{\Phi,\delta}) \\ &\leq \mathbb{P}(\{C(\omega) \geq \delta_0/\delta\}) + \mathbb{E}\mathcal{E}^\Phi/\delta\end{aligned}$$

Fixing $\delta = \sqrt{\mathbb{E}\mathcal{E}^\Phi}$ yields

$$\mathbb{P}(\sup_{[0,1\wedge\tau_R]} \|d^\Phi\|_{H^{1+\epsilon}} \geq \delta_0) \leq \mathbb{P}(\{C(\omega) \geq \delta_0/\sqrt{\mathbb{E}\mathcal{E}^\Phi}\}) + \sqrt{\mathbb{E}\mathcal{E}^\Phi}$$

Step 4: Convergence. Now we can apply the results of the previous step to our sequence Φ_N. Here $\mathbb{E}\mathcal{E}^{\Phi_N} \to 0$ for $N \to \infty$. Thus $\sup_{[0,1\wedge\tau_R]} \|d^\Phi\|_{H^{1+\epsilon}} \to 0$ in probability.

6. Rougher noise

In this section we deal with a bounded linear operator Φ on $\mathring{L}^2(\mathbb{T}_2)$ and we assume that

- $\Phi\, \mathrm{e}_k = \phi_k e_k$ for every $k \in \mathbf{Z}^2_\star$,
- there is $\beta > 0$ such that for every $k \in \mathbf{Z}^2_\star$,

$$|\phi_k|^2 \leq c|k|^\beta.$$

Since Φ is real valued, we clearly have that $\bar{\phi}_k = \phi_{-k}$.

Remark 6.1. It is obvious that any additional information on the ϕ_k would in principle improve the results of this section. On the other hand our results are optimal once we know that $|\phi_k|^2 \approx |k|^\beta$, namely there are $c, c' > 0$ such that $c'|k|^\beta \leq |\phi_k|^2 \leq c|k|^\beta$.

We wish to find a mild solution of the following equation,

$$dh + \Delta^2 h + \mathcal{B}(h,h) = \Phi\, dW, \tag{6.1}$$

where W is a cylindrical Wiener process on $L^2(\mathbb{T}_2)$, again as $h = v + z$, where

$$z(t) = \int_0^t S(t-s)\Phi\, dW = \sum_{k\in\mathbf{Z}^2_\star} \phi_k z_k e_k, \tag{6.2}$$

and the z_k are defined as in (3.1). The following lemma can be easily proved as in Proposition 3.2.

Lemma 6.2. *Let $\beta < 2$ and let z be the process defined in (6.2). For every $p \geq 1$ and $s \in \big(0, 1 - \frac{\beta}{2}\big)$,*

$$\sup_{t>0} \mathbb{E}\big[\|z(t)\|^p_{W^{s,p}}\big] < \infty.$$

We turn to the definition of $\mathcal{B}(z,z)$. The counterpart of Lemma 3.4 and Proposition 3.6 is the following proposition.

Proposition 6.3. *Let $\beta < 1$ and let z be the stochastic convolution defined in (6.2). Then $(\mathcal{B}_N(z,z))_{N\geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathring{H}_\wp^{-2-\gamma})$ for every $\gamma > \beta$.*

Denote by $\widetilde{\mathcal{B}}(z,z)$ the limit of this sequence, which is well–defined as an element of $\mathring{H}_\wp^{-2-\gamma}$, for $\gamma > \beta$. Given $\gamma > \beta$ and $p \geq 1$, there is a constant $c > 0$ such that

$$\sup_{t>0} \mathbb{E}\big[\|\widetilde{\mathcal{B}}(z(t), z(t))\|^p_{H^{-2-\gamma}}\big] \leq c.$$

The proof of the above proposition follows the same lines of the above mentioned results (direct computations and hyper–contractivity).

Remark 6.4. The restriction $\beta < 1$ in the assumptions in the above proposition is necessary, at least in our simple approach. Denote by J_k the term $J_k = \sum_{m+n=k}(m\cdot n)\phi_m\phi_n z_m z_n$, and assume, as in Remark 6.1, that $|\phi_k|^2 \approx |k|^\beta$. Then it is easy to see that

$$\mathbb{E}[|J_k|^2] \approx \sum_{m+n=k} \frac{1}{|m|^{2-\beta}|n|^{2-\beta}},$$

which converges only if $\beta < 1$.

In order to manage the problem for $\beta \geq 1$, it is necessary to consider another renormalization of the non–linearity. While the first renormalization has been hidden by the Laplace term in front of the squared gradient (killing the infinite constant, see Lemma 3.1), it looks like one should need a more refined approach as in [12, 10] to proceed further, but this does not fit in our simple approach.

We have all ingredients to prove the existence of a local mild solution of 6.1, interpreted as $h = v + z$. Here v is the solution of the mild formulation (2.2), and z is given by (6.2).

Proof of Theorem 2.9. Let us look first at the "self–mapping property". We need to estimate the three terms $\mathcal{I}_1 = \int_0^t S(t-s)\mathcal{B}(v,v)\,ds$, $\mathcal{I}_2 = \int_0^t S(t-s)\mathcal{B}(v,z)\,ds$ and $\mathcal{I}_3 = \int_0^t S(t-s)\widetilde{\mathcal{B}}(z,z)\,ds$.

The estimate of $\mathcal{I}_1$ is the same of Theorem 2.1. For $\mathcal{I}_2$ we use Corollary A.3. Choose γ such that $\frac{\beta}{2} < \gamma < 1-\epsilon$, α such that $\frac{\beta}{2} < 1-\alpha < \epsilon\wedge\gamma$, and q large enough, then

$$\|\mathcal{I}_2\|_{H^{1+\epsilon}} \leq c\int_0^t \|A^{\frac{3+\epsilon+\gamma}{4}}S(t-s)A^{-\frac{\gamma+2}{4}}\mathcal{B}(v,z)\|_{L^2}\,ds \leq ct^{-\frac{\epsilon}{4}}T^{e_1}\Big(\int_0^T \|z\|^q_{W^{\alpha,q}}\Big)^{\frac{1}{q}}.$$

Finally, the estimate of $\mathcal{I}_3$ is the same of Theorem 2.1 and for $\beta < \gamma' < 1-\epsilon$ and q large enough,

$$\|\mathcal{I}_3\|_{H^{1+\epsilon}} \leq c\int_0^t \|A^{\frac{3+\epsilon+\gamma'}{4}}S(t-s)A^{-\frac{\gamma'+2}{4}}\widetilde{\mathcal{B}}(z,z)\|_{L^2}\,ds \leq ct^{-\frac{\epsilon}{4}}T^{e_2}\|\widetilde{\mathcal{B}}(z,z)\|_{L^{q'}(H^{-2-\gamma'})}.$$

The contraction property follows by the same inequalities, as in Theorem 2.1. □

Remark 6.5. The "troublemaker" in the proof of the above theorem is the last term, the one denoted by $\mathcal{I}_3$. Indeed, the computations for $\mathcal{I}_1$ and $\mathcal{I}_2$ work for any $\beta \in (0,1)$, given $\epsilon \in (0, \frac{1}{2})$ for the first term, and $\epsilon \in (\frac{\beta}{2}, 1 - \frac{\beta}{2})$ for the second term. The term $\mathcal{I}_3$ requires $\epsilon \in (0, 1-\beta)$, hence $\beta < \frac{2}{3}$.

6.1. **The second order expansion in Wiener chaos.** Assume now $\beta \in [\frac{2}{3}, 1)$ and consider the decomposition $h = u + \zeta + z$ of h, where where ζ solves

$$\dot{\zeta} + A\zeta + \widetilde{\mathcal{B}}(z,z) = 0, \qquad \zeta(0) = 0,$$

and u is the mild solution of

$$\dot{u} + Au + \mathcal{B}(u,u) + 2\mathcal{B}(u,z) + 2\mathcal{B}(u,\zeta) + 2\widetilde{\mathcal{B}}(\zeta,z) + \mathcal{B}(\zeta,\zeta) = 0, \tag{6.3}$$

with initial condition $u(0) = h(0)$, and $\widetilde{\mathcal{B}}(\zeta,z)$ is defined below in Lemma 6.7. We can write ζ as

$$\zeta(t) = -\int_0^t S(t-s)\widetilde{\mathcal{B}}(z,z)\,ds, \tag{6.4}$$

then by maximal regularity, $\zeta \in \mathring{H}^{1+a}_{\wp}$, for every $a < 1-\beta$. Notice that there is no additional gain in regularity if we try a direct computation in the style of Lemma 3.4. Roughly speaking, we have already "used" the effect of cancellations in the definition of $\widetilde{\mathcal{B}}(z,z)$.

Although we have gained additional regularity for the term $\mathcal{B}(\zeta,\zeta)$ appearing in the equation for the remainder u, this is not enough. Indeed a standard multiplication theorem in Sobolev spaces (Lemma A.1) yields that $\mathcal{B}(\zeta,\zeta) \in H^{-2-\gamma}_{\wp}$ for $\gamma > 2\beta - 1$, which by a quick computation allows, in the fixed point argument, for $\epsilon \in (\frac{\beta}{2}, 2 - 2\beta)$, thus $\beta < \frac{4}{5}$. Even worse, when dealing with $\mathcal{B}(\zeta, z)$, we see that we still have not enough regularity to give a meaning to this term. We shall solve both problems exploiting cancellations.

Before proceeding, we state a few preliminary remarks. First, we know that $z = \sum_k \phi_k z_k e_k$ and that

$$\zeta(t) = \int_0^t S(t-s)\widetilde{\mathcal{B}}(z,z)\,ds = \lim_N \int_0^t S(t-s)B(z^N, z^N)\,ds = \lim_N \zeta^N(t),$$

with obvious definition of ζ_N. If we write $\phi^N_k = \phi_k$ if $|k| \leq N$ and $\phi^N_k = 0$ otherwise, we have $z^N(t) = \sum_k \phi^N_k z_k(t) e_k$, and $\zeta^N(t) = \sum_{k\in\mathbf{Z}^2_\star} |k|^2 \mathcal{J}^N_k(t) e_k$ where we have set for every $k \in \mathbf{Z}^2_\star$,

$$J^N_k = \sum_{m+n=k} (m\cdot n)\phi^N_m \phi^N_n z_m z_n \qquad \text{and} \qquad \mathcal{J}^N_k(t) = \int_0^t \mathrm{e}^{-|k|^4(t-s)}\, J^N_k(s)\,ds.$$

Finally, we remark a simple computation that will be useful in the next sections. Let $m, n \in \mathbf{Z}^2_\star$, then $\mathbb{E}[z_m(t) z_n(s)] = 0$ unless $m + n = 0$. In the latter case,

$$\mathbb{E}[z_m(t) z_n(s)] = \mathbb{E}[z_m(t)\bar{z}_m(s)] = \frac{1}{|m|^4}\big(\mathrm{e}^{-|m|^4|t-s|} - \mathrm{e}^{-|m|^4(t+s)}\big) \leq \frac{1}{|m|^4}. \tag{6.5}$$

6.1.1. *The term $\mathcal{B}(\zeta,\zeta)$.* We shall prove the following result.

Lemma 6.6. *Let $\beta\in(0,1)$. Then for every $\gamma>\beta-1$ and $p\geq 1$,*

$$\sup_{t\geq 0}\mathbb{E}[\|\mathcal{B}(\zeta,\zeta)\|^p_{H^{-2-\gamma}}]<\infty.$$

Since $\mathcal{B}(\zeta,\zeta)=\lim_N\mathcal{B}(\zeta^N,\zeta^N)$ in $H_\wp^{-2-\gamma}$ for every $\gamma>2\beta-1$, to prove additional regularity of $\mathcal{B}(\zeta,\zeta)$ it is enough to prove that $(\mathcal{B}(\zeta^N,\zeta^N))_{N\geq1}$ is uniformly bounded in $L^2(\Omega;H_\wp^{-2-\gamma})$ (and hence in $L^p(\Omega)$ for every $p\geq 1$ by hyper–contractivity, see Proposition 3.6) for every $\gamma>\beta-1$.

We have

$$\mathcal{B}(\zeta^N,\zeta^N)=\sum_{k\in\mathbf{Z}^2_\star}|k|^2\Big(\sum_{h+\ell=k}|\ell|^2|h|^2(h\cdot\ell)\mathcal{J}^N_h\mathcal{J}^N_\ell\Big)e_k,$$

hence we can estimate its norm, to obtain

$$\begin{aligned}\mathbb{E}[\|\mathcal{B}(\zeta^N,\zeta^N)\|^2_{H^{-2-\gamma}}]&=\sum_{k\in\mathbf{Z}^2_\star}|k|^{-2\gamma}\mathbb{E}\Big|\sum_{h+\ell=k}|\ell|^2|h|^2(h\cdot\ell)\mathcal{J}^N_h\mathcal{J}^N_\ell\Big|^2\\&\leq\sum_{k\in\mathbf{Z}^2_\star}|k|^{-2\gamma}\sum_{\substack{h_1+\ell_1=k\\h_2+\ell_2=k}}\frac{\sup\big|\mathbb{E}[J^N_{h_1}J^N_{\ell_1}\bar J^N_{h_2}\bar J^N_{\ell_2}]\big|}{|h_1|\,|h_2|\,|\ell_1|\,|\ell_2|}.\end{aligned}$$

Each $\mathbb{E}[J^N_{h_1}J^N_{\ell_1}\bar J^N_{h_2}\bar J^N_{\ell_2}]$ contains sums of terms like

$$\mathbb{E}[z_{m_1}(r_1)z_{n_1}(r_1)z_{a_1}(s_1)z_{b_1}(s_1)\bar z_{m_2}(r_2)\bar z_{n_2}(r_2)\bar z_{a_2}(s_2)\bar z_{b_2}(s_2)],\tag{6.6}$$

with $m_i+n_i=h_i$, $a_i+b_i=\ell_i$, $i=1,2$. Wick's formula yields 105 products of four expectations. Of these terms, 45 are zero by (6.5), since they contain terms like $\mathbb{E}[z_mz_n]$ with $m+n\neq 0$. By symmetry, we can collect the remaining terms in four classes as suggested in the picture[1] The first term gives no contribution,

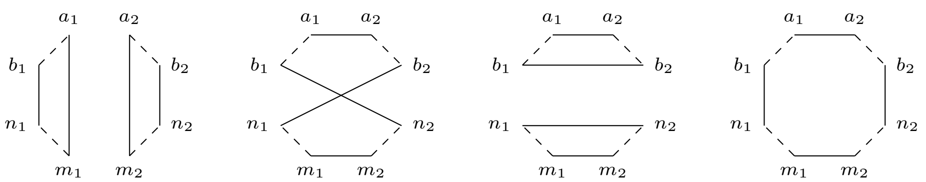

FIGURE 1. Graphical representation of classes of terms (6.6)

since it is non-zero only if $m_1+a_1=0$ and $b_1+n_1=0$, hence $k=h_1+\ell_1=m_1+n_1+a_1+b_1=0$. The second term contains terms like

$$\big|\mathbb{E}[z_{m_1}\bar z_{m_2}]\mathbb{E}[z_{n_1}\bar z_{b_2}]\mathbb{E}[z_{a_1}\bar z_{a_2}]\mathbb{E}[z_{b_1}\bar z_{n_2}]\big|\leq\frac{\delta_{m_1-m_2}\delta_{n_1-b_2}\delta_{a_1-a_2}\delta_{b_1-n_2}}{|m_1|^4|n_1|^4|a_1|^4|b_1|^4},$$

[1] A dashed line means that the sum of the two connected labels is one of the numbers h_i,ℓ_i, the continuous line means that the two labels are in the product of the same expectation, for instance the first picture reads $\mathbb{E}[z_{m_1}z_{a_1}]\mathbb{E}[z_{n_1}z_{b_1}]\mathbb{E}[\bar z_{m_2}\bar z_{a_2}]\mathbb{E}[\bar z_{n_2}\bar z_{b_2}]$.

hence $n_1 = b_2 = h_1 - m_1$, $b_1 = n_2 = h_2 - m_1$, $a_1 = a_2 = \ell_1 - h_2 + m_1$ and for these indices,

$$\sum_{\substack{h_1+\ell_1=k\\h_2+\ell_2=k}} \frac{\sup\bigl|\mathbb{E}[J^N_{h_1}J^N_{\ell_1}\bar{J}^N_{h_2}\bar{J}^N_{\ell_2}]\bigr|}{|h_1|\,|h_2|\,|\ell_1|\,|\ell_2|} \leq \sum_{\substack{h_1+\ell_1=k\\h_2+\ell_2=k}} \frac{c}{|h_1|\,|h_2|\,|\ell_1|\,|\ell_2|}\cdot$$
$$\cdot\Bigl(\sum_{m_1}\frac{c}{|m_1|^{2-\beta}|h_2-m_1|^{2-\beta}|h_1-m_1|^{2-\beta}|\ell_2-h_1+m_1|^{2-\beta}}\Bigr).$$

Change the order of sums, summing first over h_1, then over h_2 and finally over m_1, then by the Cauchy–Schwartz inequality and Lemma A.4,

$$\sum_{h_1+\ell_1=k}\frac{c}{|h_1|\,|\ell_1|\,|h_1-m_1|^{2-\beta}|\ell_2-h_1+m_1|^{2-\beta}} \leq$$
$$\leq \Bigl(\sum_{h_1}\frac{1}{|h_1|^2|\ell_1|^2}\Bigr)^{\frac12}\Bigl(\sum_{h_1}\frac{1}{|h_1-m_1|^{4-2\beta}|\ell_2-h_1+m_1|^{4-2\beta}}\Bigr)^{\frac12} \leq \frac{\sqrt{\log(1+|k|)}}{|k|\,|\ell_2|^{2-\beta}}.$$

If we plug this result in the sum over h_1 and we repeat the same estimate, we obtain that the sum over h_1 and h_2 is bounded by $|k|^{-2}|k-m_1|^{-(2-\beta)}\log(1+|k|)$. By applying Lemma A.4 to the sum over m_1, we finally obtain that the term arising from the fifth class is bounded by $|k|^{-(4-2\beta)}\log(1+|k|)$. The third and fourth classes can be estimated by similar but simpler considerations. In conclusion $\mathcal{B}(\zeta,\zeta) \in H^{-2-\gamma}$ for $\gamma > \beta - 1$ and the lemma is proved.

6.1.2. *The term* $\mathcal{B}(\zeta, z)$. Given the information we have on the regularity of z and ζ, we cannot use Corollary A.3 to give a meaning to $\mathcal{B}(\zeta, z)$ (and it may not have a unique meaning, as $\widetilde{\mathcal{B}}(z,z)$). Hence we resort on the same method we have used for $\widetilde{\mathcal{B}}(z,z)$.

Lemma 6.7. *Let $\beta \in [\frac23, 1)$ and let z, ζ, z^N, ζ^N be defined as before. Then $(\mathcal{B}(\zeta^N, z^N))_{N\geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathring{H}^{-2-\gamma}_\wp)$ for every $\gamma > \frac{\beta}{2}$.*

Denote by $\widetilde{\mathcal{B}}(\zeta, z)$ the limit of this sequence, which is well–defined as an element of $\mathring{H}^{-2-\gamma}_\wp$, for $\gamma > \frac{\beta}{2}$. Given $\gamma > \frac{\beta}{2}$ and $p \geq 1$, there is a constant $c > 0$ such that

$$\sup_{t>0}\mathbb{E}\bigl[\|\widetilde{\mathcal{B}}(\zeta(t), z(t))\|^p_{H^{-2-\gamma}}\bigr] \leq c.$$

Remark 6.8. When $\beta \in (0, \frac23)$, the term $\mathcal{B}(\zeta, z)$ is well–defined and hence the above lemma ensures additional regularity for $\mathcal{B}(\zeta, z)$. The argument is the same of Lemma 6.6.

As in the previous part,

$$\mathcal{B}(\zeta^N, z^N) = \sum_{k,\ell\in\mathbf{Z}^2_\star}|k|^2\bar{\phi}^N_\ell\bar{z}_\ell\mathcal{J}^N_k\mathcal{B}(e_k,e_\ell) =$$
$$= \sum_{k\in\mathbf{Z}^2_\star}|k|^2\Bigl(\sum_{h+\ell=k}|h|^2(h\cdot\ell)\phi^N_\ell z_\ell\mathcal{J}^N_h\Bigr)e_k = \sum_{k\in\mathbf{Z}^2_\star}|k|^2G^N_k e_k,$$

where G^N_k is the inner sum, and compute the norm of $\mathcal{B}(\zeta^N, z^N) - \mathcal{B}(\zeta^{N'}, z^{N'})$ in $H^{-2-\gamma}_\wp$ to obtain

$$\|\mathcal{B}(\zeta^N, z^N) - \mathcal{B}(\zeta^{N'}, z^{N'})\|^2_{H^{-2-\gamma}} = \sum_{k\in\mathbf{Z}^2_\star} |k|^{-2\gamma}\big|G^N_k - G^{N'}_k\big|^2.$$

Thus

$$\mathbb{E}[|G^N_k - G^{N'}_k|^2] \leq \sum_{\substack{h_1+\ell_1=k\\ h_2+\ell_2=k}}^{N\leftrightarrow N'} \frac{|\ell_1|\,|\ell_2|\,|\phi_{\ell_1}\bar\phi_{\ell_2}|}{|h_1||h_2|} \sup_{s_1,s_2}\big|\mathbb{E}\big[J_{h_1}(s_1)\bar J_{h_2}(s_2) z_{\ell_1}(t)\bar z_{\ell_2}(t)\big]\big|$$

The term $\mathbb{E}[J_{h_1}\bar J_{h_2} z_{\ell_1}\bar z_{\ell_2}]$ contains sums of terms like $\mathbb{E}[z_{\ell_1}\bar z_{\ell_2} z_{m_1} z_{n_1}\bar z_{m_2}\bar z_{n_2}]$ that, by Wick's formula, are the sum of 15 products of three pairwise expectations. Five of these terms are 0, by (6.5), since contain $\mathbb{E}[z_m z_n]$ with $m+n\neq 0$. By symmetry we can collect the terms in classes, whose representatives are

$$\mathbb{E}[z_{\ell_1}\bar z_{\ell_2}]\mathbb{E}[z_{m_1}\bar z_{m_2}]\mathbb{E}[z_{n_1}\bar z_{n_2}] \qquad \mathbb{E}[z_{\ell_1} z_{m_1}]\mathbb{E}[\bar z_{\ell_2}\bar z_{m_2}]\mathbb{E}[z_{n_1}\bar z_{n_2}]$$
$$\mathbb{E}[z_{\ell_1}\bar z_{m_2}]\mathbb{E}[\bar z_{\ell_2} z_{m_1}]\mathbb{E}[z_{n_1}\bar z_{n_2}].$$

We focus on the first class. Fix $\eta > 0$ small enough (depending on β and γ), then a simple computation, the assumption on the ϕ_k and Lemma A.4 yield

$$\begin{aligned}\sup\big|\mathbb{E}\big[J_{h_1}\bar J_{h_2} z_{\ell_1}\bar z_{\ell_2}\big]\big| &\leq \frac{c\delta_{h_1-h_2}\delta_{\ell_1-\ell_2}}{|h_1|^2|\ell|^{2-\beta}}\Big(\sum_{m+n=h_1}\frac{1}{|m|^{2-\beta}|n|^{2-\beta}}\Big) \leq\\ &\leq \frac{c\delta_{h_1-h_2}\delta_{\ell_1-\ell_2}}{|h_1|^{4-2\beta}|\ell|^{2-\beta}} \leq \frac{c\delta_{h_1-h_2}\delta_{\ell_1-\ell_2}}{N^\eta|h_1|^{4-2\beta-\eta}|\ell|^{2-\beta-\eta}}.\end{aligned}$$

By summing in h_1, h_2, we obtain the estimate $cN^{-\eta}|k|^{-(2-\beta-\eta)}$. The other two terms can be estimated by similar arguments giving the same bound. Hence the sequence is Cauchy in $H^{-2-\gamma}_\wp$ for every $\gamma > \frac{\beta}{2}$. The statement on moments follows by hyper–contractivity as in Proposition 3.6.

6.1.3. *The local mild solution.* The additional regularity of $\mathcal{B}(\zeta,\zeta)$ and the interpretation of $\widetilde{\mathcal{B}}(\zeta, z)$ we have proved, finally allow to obtain a local solution of (6.1) as $h = u + \zeta + z$, where u is a mild solution of (6.3).

Proof of Theorem 2.10. We give a quick sketch of the estimate for the "self–mapping" property, the details are the same as Theorems 2.1 and 2.9. The terms arising from $\mathcal{B}(u,u)$ and $\mathcal{B}(u,z)$ in the mild formulation can be handled as $\mathcal{B}(v,v)$ and $\mathcal{B}(v,z)$, as well as $\mathcal{B}(u,\zeta)$, since ζ is smoother than z. For $\mathcal{B}(\zeta,\zeta)$ we use Lemma 6.6 and choose $q > 4$ to get

$$\Big\|\int_0^t S(t-s)\mathcal{B}(\zeta,\zeta)\,ds\Big\|_{H^{1+\epsilon}} \leq ct^{-\frac{\epsilon}{4}}T^{\frac14-\frac1q}\Big(\int_0^T \|\mathcal{B}(\zeta,\zeta)\|_{H^{-2}}\,ds\Big)^{\frac1q}.$$

Likewise, by Lemma 6.7, if $\epsilon \in (0, 1 - \frac{\beta}{2})$ and $\frac{\beta}{2} < \gamma < 1 - \epsilon$,

$$\Big\| \int_0^t S(t-s)\widetilde{\mathcal{B}}(\zeta, z)\, ds \Big\|_{H^{1+\epsilon}} \leq c t^{-\frac{\epsilon}{4}} T^{\frac{1}{4}(1-\gamma)-\frac{1}{q}} \Big(\int_0^T \|\mathcal{B}(\zeta, z)\|_{H^{-2-\gamma}}\, ds \Big)^{\frac{1}{q}},$$

for q large enough. □

Appendix A. Bounds on the non–linear operator

Lemma A.1. *If $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma \geq 1$ (with strict inequality if at least one of the numbers is equal to 1), then $\mathcal{B}$ maps $H^{1+\alpha}_\wp \times H^{1+\beta}_\wp$ continuously into $H^{-2-\gamma}_\wp$. In particular, there exists $c = c(\alpha, \beta, \gamma)$ such that*

$$\|\mathcal{B}(u_1, u_2)\|_{H^{-2-\gamma}} \leq c\|u_1\|_{H^{1+\alpha}} \|u_2\|_{H^{1+\beta}}.$$

Proof. Let $\phi \in H^{2+\gamma}_\wp$, then by integration by parts and the Hölder inequality,

$$\langle \phi, \mathcal{B}(u_1, u_2)\rangle = \int \Delta\phi \nabla u_1 \cdot \nabla u_2\, dx \leq \|\Delta\phi\|_{L^p} \|\nabla u_1\|_{L^q} \|\nabla u_2\|_{L^r},$$

with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Sobolev's embeddings yield $H^{1-2/p}_\wp \subset L^p$, hence

$$\langle \phi, \mathcal{B}(u_1, u_2)\rangle \leq c\|\phi\|_{H^{3-2/p}} \|u_1\|_{H^{2-2/q}} \|u_2\|_{H^{2-2/r}}.$$

Choose now p, q, r such that $\alpha \geq 1 - \frac{2}{q}$, $\beta \geq 1 - \frac{2}{r}$ and $\gamma \geq 1 - \frac{2}{p}$. The case where one number is 1 corresponds to the critical Sobolev embedding and needs a strict inequality. □

Proposition A.2. *Let $s \in (0, 1)$, $s_1, s_2 \in (s, 1)$ and $p, p_1, p_2 \geq 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then there is a constant $c > 0$ such that*

$$\|u_1 u_2\|_{W^{s,p}} \leq c\|u_1\|_{W^{s_1,p_1}} \|u_2\|_{W^{s_2,p_2}},$$

for all $u_1 \in W^{s_1,p_1}_\wp$ and $u_2 \in W^{s_2,p_2}_\wp$.

Proof. By definition

$$\|u_1 u_2\|^p_{W^{s,p}} = \int |u_1 u_2|^p\, dx + \iint \frac{|u_1(x)u_2(x) - u_1(y)u_2(y)|^p}{|x-y|^{2+sp}}\, dx\, dy.$$

By Hölder's inequality,

$$\int |u_1 u_2|^p\, dx \leq \|u_1\|^p_{L^{p_1}} \|u_2\|^p_{L^{p_2}}.$$

The second term in the norm above is split in the two terms

$$\iint |u_1(x)|^p \frac{|u_2(x) - u_2(y)|^p}{|x-y|^{2+sp}}\, dx\, dy + \iint |u_2(y)|^p \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{2+sp}}\, dx\, dy = \boxed{\mathsf{a}} + \boxed{\mathsf{b}}.$$

Using again the Hölder inequality,

$$\boxed{\text{a}} = \iint \frac{|u_1(x)|^p}{|x-y|^{\frac{2p}{p_1}-(s_2-s)p}} \Big(\frac{|u_2(x)-u_2(y)|^{p_2}}{|x-y|^{2+s_2p_2}}\Big)^{\frac{p}{p_2}}\,dx\,dy \leq$$

$$\leq \Big(\frac{|u_2(x)-u_2(y)|_2^p}{|x-y|^{2+s_2p_2}}\Big)^{\frac{p}{p_2}} \Big(\frac{|u_1(x)|_1^p}{|x-y|^{2-(s_2-s)p_1}}\Big)^{\frac{p}{p_1}} \leq c\|u_1\|_{L^{p_1}}\|u_2\|_{W^{s_2,p_2}}$$

since $|x-y|^{(s_2-s)p-2}$ is integrable. The term $\boxed{\text{b}}$ can be estimated similarly. □

Corollary A.3. *Let $\epsilon \in (0,1)$ and $\gamma > 0$. For every $\alpha \in (0,1)$ and $q > 2$ such that $1-\alpha+\frac{2}{q} < \epsilon \wedge \gamma$, there is a constant $c > 0$ such that*

$$\|\mathcal{B}(u_1,u_2)\|_{H^{-2-\gamma}} \leq c\|u_1\|_{W^{\alpha,q}}\|u_2\|_{H^{1+\epsilon}},$$

for every $u_1 \in W^{\alpha,q}_\wp$ and $u_2 \in H^{1+\epsilon}_\wp$.

Proof. Assume $\gamma \leq \epsilon$. By duality,

$$\|\mathcal{B}(u_1,u_2)\|_{H^{-2-\gamma}} \leq c\|\nabla u_1 \nabla u_2\|_{H^{-\gamma}} = c \sup_{\|\phi\|_{H^\gamma}} \langle \phi, \nabla u_1 \nabla u_2\rangle.$$

Let p be the conjugate exponent of q, then by integration by parts, duality and the previous proposition (with $p_1 = 2$, $s_1 = \gamma$ and $1-\alpha < s_2 \leq \epsilon - \frac{2}{q}$),

$$\langle \phi, \nabla u_1 \nabla u_2\rangle = -\langle u_1, \operatorname{div}(\phi\nabla u_2)\rangle \leq c\|u_1\|_{W^{\alpha,q}}\|\phi\nabla u_2\|_{W^{1-\alpha,p}} \leq$$

$$\leq c\|u_1\|_{W^{\alpha,q}}\|\phi\|_{H^\gamma}\|\nabla u_2\|_{W^{s_2,p_2}} \leq c\|u_1\|_{W^{\alpha,q}}\|u_2\|_{H^{1+\epsilon}},$$

since $H^\epsilon_\wp \subset W^{s_2,p_2}_\wp$ by the choice of s_2 and p_2.

If on the other hand $\epsilon \leq \gamma$, apply the previous proposition with $p_2 = 2$, $s_2 = \epsilon$, and $1-\alpha < s_2 \leq \gamma - \frac{2}{q}$, and use the Sobolev embedding $H^\gamma_\wp \subset W^{s_1,p_1}_\wp$. □

The following Lemma is stated in [5, Lemma 2.3].

Lemma A.4. *For all α, $\gamma > 0$ with $\alpha+\gamma > d$ there exists $C = C(\alpha,\gamma) < \infty$ so that, for all $k \in \mathbf{Z}^d$, with $k \neq \mathbf{0}$,*

$$\sum_{\substack{m+n=k\\ m\neq 0, n\neq 0}} \frac{1}{|m|^\alpha |n|^\gamma} \leq \begin{cases} C(1+|k|)^{-\beta}, & \text{if } \alpha \neq d \text{ and } \gamma \neq d,\\ C(1+|k|)^{-\beta}\log(1+|k|), & \text{if } \alpha = d \text{ or } \gamma = d,\end{cases}$$

where $\beta = \min\{\alpha,\gamma,\alpha+\gamma-d\}$.

References

1. Dirk Blömker, *Nonhomogeneous noise and Q-Wiener processes on bounded domains*, Stoch. Anal. Appl. **23** (2005), no. 2, 255–273. [MR2130349]
2. Dirk Blömker, Franco Flandoli, and Marco Romito, *Markovianity and ergodicity for a surface growth PDE*, Ann. Probab. **37** (2009), no. 1, 275–313. [MR2489166]
3. Dirk Blömker and Marco Romito, *Regularity and blow up in a surface growth model*, Dyn. Partial Differ. Equ. **6** (2009), no. 3, 227–252. [MR2569507]

4. ______, *Local existence and uniqueness in the largest critical space for a surface growth model*, NoDEA Nonlinear Differential Equations Appl. **19** (2012), no. 3, 365–381. [MR2926303]
5. Dirk Blömker, Marco Romito, and Roger Tribe, *A probabilistic representation for the solutions to some non-linear PDEs using pruned branching trees*, Ann. Inst. H. Poincaré Probab. Statist. **43** (2007), no. 2, 175–192. [MR2303118]
6. Vladimir I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. [MR1642391]
7. Giuseppe Da Prato and Arnaud Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal. **196** (2002), no. 1, 180–210. [MR1941997]
8. Franco Flandoli and Marco Romito, *Markov selections for the 3D stochastic Navier-Stokes equations*, Probab. Theory Related Fields **140** (2008), no. 3-4, 407–458. [MR2365480]
9. Hiroshi Fujita and Tosio Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315. [MR0166499]
10. Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski, *Paraproducts, rough paths and controlled distributions*, 2012, arXiv:1210.2684 [math.PR].
11. Martin Hairer, *Solving the KPZ equation*, 2011, arXiv:1109.6811 [math.PR], to appear on Ann. Math.
12. ______, *A theory of regularity structures*, 2013, arXiv:1303.5113 [math.PR].
13. Martin Hairer and Hendrik Weber, *Rough Burgers-like equations with multiplicative noise*, Probab. Theory Related Fields **155** (2013), no. 1-2, 71–126. [MR3010394]
14. Daniel Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981. [MR610244]
15. Herbert Koch and Daniel Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157** (2001), no. 1, 22–35. [MR1808843]
16. Edward Nelson, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227. [MR0343816]
17. Martin Raible, Stefan J. Linz, and Peter Hänggi, *Amorphous thin film growth: Minimal deposition equation*, Phys. Rev. E **62** (2000), 1691–1694.
18. Martin Raible, S. Mayr, M. Moske, Peter Hänggi, and K. Samwer, *Amorphous thin film growth: theory compared with experiment*, Europhysics Letters **50** (2000), 61–67.
19. Hans-Jürgen Schmeisser and Hans Triebel, *Topics in Fourier analysis and function spaces*, A Wiley-Interscience Publication, John Wiley & Sons Ltd., Chichester, 1987. [MR891189]
20. Barry Simon, *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton University Press, Princeton, N.J., 1974, Princeton Series in Physics. [MR0489552]
21. Michael Winkler, *Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth*, Z. Angew. Math. Phys. **62** (2011), no. 4, 575–608. [MR2822342]

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