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RESEARCH ARTICLE

Amplitude Equations for SPDEs with Cubic Nonlinearities

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For a quite general class of SPDEs with cubic nonlinearities we derive rigorously amplitude equations describing the essential dynamics using the natural separation of time-scales near a change of stability. Typical examples are the Swift-Hohenberg equation, the Ginzburg-Landau (or Allen-Cahn) equation and some model from surface growth.

We discuss the impact of degenerate noise on the dominant behavior, and see that additive noise has the potential to stabilize the dynamics of the dominant modes. Furthermore, we discuss higher order corrections to the amplitude equation.

Keywords: amplitude equations, Swift-Hohenberg equation, slow-fast system, multi-scale analysis, higher order corrections, stabilization by noise

AMS Subject Classification: 60H15, 60H10

1. Introduction

Stochastic partial differential equations (SPDEs) with cubic nonlinearity appear in several applications, for instance the Swift-Hohenberg equation, which was first used as a toy model for the convective instability of fluids in the Rayleigh-Bénard problem (see [6] or [10]). The simplest example is the well known real valued Ginzburg-Landau equation, which depending on the underlying application is also called Allen-Cahn, Chaffee-Infante or nonlinear heat equation. Moreover, we briefly discuss a model from surface growth proposed by Lai & Das Sarma (cf. [11] and see also [12]).

All equations considered in this article are parabolic nonlinear SPDEs perturbed by additive forcing. Near a change of stability, we can rely on the natural separation of time-scales, in order to derive simpler equations for the evolution of the dominant pattern. As these equations describe the amplitudes of dominant pattern, they are referred to as amplitude equations. When the order of the noise strength is comparable to the order of the distance from the change of stability, the impact of noise can be seen on the dominant pattern. The first result for the Swift-Hohenberg equation was [4], which was extended significantly by [2] later. For a review see [1] and the references therein.

Recently, the impact of degenerate noise not acting directly on the dominant pattern was studied for equations of Burgers type, first formally in [13] and later rigorously [3]. Here noise is transported via nonlinear interaction to the dominant modes. An interesting related approach in a slightly different setting is [5], where

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the impact of noise acting on the boundary was studied for a reaction-diffusion system.

Our current research was initiated by an observation of Axel Hutt and collaborators [7–9]. Using numerical simulations and a formal argument based on center manifold theory, they showed that noise constant in space leads to a deterministic amplitude equation, which is stabilized by the impact of additive noise. This leads to a significant shift of the first pattern forming instability. The aim of this paper is to make these results rigorous.

Moreover, we want to study higher order corrections to the amplitude equation, in order to see the fluctuations induced by the impact of the noise on the dominant pattern. Related results in this direction are discussed by Roberts & Wang [14]. Nevertheless their setting is slightly different, and they use averaging techniques that do not lead to explicit error estimates.

The general prototype of equations under consideration is of the type

$$du(t) = \big[\mathcal{A}u(t) + \varepsilon^2\mathcal{L}u(t) + \mathcal{F}(u(t))\big]\, dt + \varepsilon dW(t), \tag{1}$$

where $\mathcal{A}$ is non-positive self-adjoint operator with finite dimensional kernel, $\varepsilon^2\mathcal{L}u$ is a small deterministic perturbation, $\mathcal{F}$ is a cubic nonlinearity, and W is some finite dimensional Gaussian noise with small noise strength $\varepsilon > 0$. Note that the small deterministic part, that reflects the distance from bifurcation, scales with ε. Different scalings are possible, but the one chosen here, is exactly the one where noise and linear instability will interact in an interesting way. For simplicity of presentation, we will work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. Other norms like the supremum-norm or the L^p-norm would lead to similar results.

Our aim of this paper is to establish rigorously an amplitude equation and their higher order corrections for this quite general class of SPDEs with cubic nonlinearities given by (1). In the examples we demonstrate that additive degenerate noise leads to a stabilization of the solutions.

The paper is organized as follows. In the next section, we discuss the formal derivation of our results, while giving the precise assumptions and statements of the main results in Section 3. Section 4 gives bounds on the non-dominant modes, while Section 5 provides averaging results, in order to remove the impact of the higher modes on the dominant ones. In Section 6, we study the approximation via amplitude equations, which is in the final Section 7 extended to higher order corrections.

2. Formal Derivation

Before we proceed to give detailed assumptions, we present a short formal derivation and motivation of the main results. We will denote the kernel of $\mathcal{A}$ by $\mathcal{N} := \ker \mathcal{A}$. These are the dominant modes or the pattern that change stability. By $S = \mathcal{N}^\perp$ we denote the orthogonal complement in $\mathcal{H}$. Furthermore, denote by P_c the orthogonal projection $P_c : \mathcal{H} \to \mathcal{N}$ onto $\mathcal{N}$. We define the complementary projection $P_s := \mathcal{I} - P_c$, where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

In the following we always study the behavior of solutions u of (1) on the natural slow time-scale of order ε^{-2}, given by the distance from bifurcation. So we rescale u to the slow time and split it into the dominant part $a \in \mathcal{N}$ and the orthogonal part $\psi \in S$.

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon\psi(\varepsilon^2 t)\ . \tag{2}$$

For the slow time-scale $T = \varepsilon^2 t$, this leads to the following system of equations:

$$da = \left[\varepsilon^{-2}\mathcal{A}_c a + \mathcal{L}_c a + \mathcal{L}_c \psi + \mathcal{F}_c(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_c \tag{3}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s \psi + \mathcal{L}_s a + \mathcal{L}_s \psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \ , \tag{4}$$

where $\tilde{W}(T) := \varepsilon W(\varepsilon^{-2}T)$ is a rescaled version of the driving Wiener process W. For short-hand notation, we use the subscripts c and s for projection onto $\mathcal{N}$ and $\mathcal{S}$, i.e., $\mathcal{A}_c = P_c\mathcal{A}$ and $\mathcal{A}_s = P_s\mathcal{A}$, for short.

Let us suppose that the projections P_c and P_s commute not only with $\mathcal{A}$, but also with $\mathcal{L}$. Moreover, suppose that the noise is degenerate and acts only on S. Then the system (3)-(4) takes the form:

$$da = \left[\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\right] dT \tag{5}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s \psi + \mathcal{L}_s \psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \ . \tag{6}$$

Formally, we immediately see that in first approximation ψ is a fast Ornstein-Uhlenbeck process (OU, for short) given by the linear equation

$$d\psi = \varepsilon^{-2}\mathcal{A}_s \psi dT + \varepsilon^{-1} d\tilde{W}_s \ .$$

The rigorous statement of this approximation can be found in Lemma 4.1.

Thus we can eliminate ψ in Equation (5) by explicitly averaging over the fast modes. In order to derive error estimates this procedure will be based on the Itô-Formula (see Lemma 5.1). Usually, in most applications of averaging, we can only hope for weak convergence in law without any error bound.

2.1 The Impact of Noise

Let us discuss the averaging and the impact of the noise in some more detail here. Consider here for simplicity of the argument instead of the fast OU-process ψ some simple real valued fast Z given by

$$Z(T) := \alpha\varepsilon^{-1}\int_0^T e^{-\varepsilon^{-2}\lambda(T-\tau)} d\tilde{\beta}(\tau), \tag{7}$$

where $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ denotes a rescaled version of a Brownian motion β on the fast time-scale.

We apply Itô formula to Z and Z^2, in order to obtain

$$Z dT = \frac{\alpha\varepsilon}{\lambda} d\tilde{\beta} - \frac{\varepsilon^2}{\lambda} dZ$$

and

$$Z^2 dT = \frac{\alpha^2}{2\lambda} dT + \frac{\varepsilon\alpha}{\lambda} Z d\tilde{\beta} - \frac{\varepsilon^2}{2\lambda} dZ^2 \ .$$

Thus, on the slow time-scale T we can suppose that in integrals the process Z is small due to averaging, and a square of Z can be replaced by the constant $\alpha^2/2\lambda$. See Lemma 5.1 for the rigorous statement of this averaging result. Note that the next order corrections (of order ε) are always stochastic integrals and thus martingales.

We see later in Lemma 4.2 that the fast OU-processes Z is nearly order 1. To be more precise $Z = \mathcal{O}(\varepsilon^{-\kappa_0})$ for every arbitrarily small $\kappa_0 > 0$. Thus we obtain formally that Z is a white noise on the slow time scale:

$$Z(T) = \varepsilon\frac{\alpha}{\lambda}\partial_T\tilde{\beta}(T) + \text{error},$$

where this error is small only in the sense of distributions, for example in the space $\mathcal{H}^{-1}(0,T)$, as it contains derivatives of Z.

2.2 Amplitude Equation

One main result of the paper is the following approximation by amplitude equations. It can be derived from (5) by expanding the cube and applying the averaging result. Suppose for simplicity that the initial condition is sufficiently small. Then we obtain for the solution u of (1)

$$u(t) = \varepsilon b(\varepsilon^2 t) + \varepsilon\mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}), \tag{8}$$

where $\mathcal{Z}$ is a fast OU-process and b is the solution of the amplitude equation on the slow time-scale

$$\partial_T b = \mathcal{L}_c b + \mathcal{F}_c(b) + \sum_{k=n+1}^{N}\frac{3\alpha_k^2}{2\lambda_k}\mathcal{F}_c(b, e_k, e_k)\ . \tag{9}$$

The precise form of the additional linear terms is discussed later. In (8) the OU process $\mathcal{Z}$ is noise of order ε, as discussed in Section 2.1 before

To illustrate this approximation result stated later in Theorem 3.9, we discuss here (similar to [9] the Swift-Hohenberg equation subject to periodic boundary conditions on $[0, 2\pi]$ forced by spatially constant noise:

$$\partial_t u = -(1+\partial_x^2)^2 u + \nu\varepsilon^2 u - u^3 + \varepsilon\alpha\partial_t\beta. \tag{10}$$

Rescaling the solution u of (10) to the slow time-scale by $u(t) = \varepsilon v(\varepsilon^2 t)$, our main theorem in this case (cf. Theorem 3.9) states that v is of the type

$$v \simeq \gamma_1 \sin + \gamma_{-1}\cos + \varepsilon\frac{\alpha}{\sqrt{2\pi}}\partial_T\tilde{\beta} + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solutions of the amplitude equations

$$\partial_T\gamma_i = (\nu - \tfrac{3\alpha^2}{4\pi})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \ \text{ for } i = \pm 1.$$

We note that, if α is large compared to ν, then $(\nu - \frac{3\alpha^2}{4\pi})$ is negative. In this case the degenerate additive noise stabilizes the dynamics of the dominant modes.

Let us finally remark, that the precise value of the constants depends on the choice of the basis for $\mathcal{N}$ and the basis functions used for the noise.

2.3 Higher order Corrections

The second main result studies the higher order correction of (8) for the approximation of solutions of equation (1). As indicated in Section 2.1, we obtain from the averaging result over the fast OU-process additional Martingale terms of order ε in (9). These terms depend both on b and the fast OU-process. Further averaging arguments for stochastic integrals are thus necessary. This leads to an equation for the higher order correction containing additive noise, which strength depends on the first order approximation.

Unfortunately, as we rely for the additional averaging on a Martingale representation argument of [3], we are limited in the final argument to one-dimensional dominant spaces, i.e. $\dim \mathcal{N} = 1$. Nevertheless, using standard averaging techniques it is possible to carry over our results to higher dimensional $\mathcal{N}$, if we only ask for weak convergence of the approximation without any error estimate.

We improve the approximation of (1) from (8) by including a higher order term:

$$u(t) \simeq \varepsilon b_1(\varepsilon^2 t) + \varepsilon^2 b_2(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{3-}), \tag{11}$$

where b_1 is again the solution of the amplitude equation (9). Later we will see that b_2 is the solution of

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(b_2, b_1, b_1) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{b_1}, \tag{12}$$

where $\tilde{M}_{b_1}(T)$ is a martingale defined by

$$\tilde{M}_{b_1}(T) = \int_0^T \Big(\sum_{k=2}^{N} g_k(b_1) \Big)^{\frac{1}{2}} dB(s) \;, \tag{13}$$

where the integration is against a one-dimensional Brownian motion B arising from a martingale representation argument (cf. Lemma 7.9). The g_k's are polynomials of degree 4 in b_1 given later in (75).

3. Assumptions and main results

This section summarizes all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (1) on the Hilbert-space $\mathcal{H}$ we assume the following:

Assumption 3.1 Suppose $\mathcal{A}$ is a non-positive operator on the space $\mathcal{H}$ with eigenvalues $0 \le \lambda_1 \le \ldots \le \lambda_k \le \ldots$ and $\lambda_k \ge Ck^m$ for all sufficiently large $k \in \mathbb{N}$. Assume $\{e_k\}_{k=1}^{\infty}$ is a corresponding complete orthonormal system of eigenvectors such that $\mathcal{A}e_k = -\lambda_k e_k$. Finally, suppose that $\mathcal{N} := \ker(\mathcal{A})$ has finite dimension n with basis $(e_1, \ldots, e_n)$, which means $\lambda_n = 0 < \lambda_{n+1}$.

As before, we denote by P_c the orthogonal projection onto $\mathcal{N}$ and by P_s the orthogonal projection onto the orthogonal complement $\mathcal{S} = \mathcal{N}^{\perp}$.

Definition 3.2: For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^{\alpha}$ as

$$\mathcal{H}^{\alpha} = \Big\{ \sum_{k=1}^{\infty} \gamma_k e_k \;:\; \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \Big\} \text{ with norm } \Big\| \sum_{k=1}^{\infty} \gamma_k e_k \Big\|_{\alpha}^2 = \Big(\sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} \Big)^{1/2}.$$

It is well known, that the operator $\mathcal{A}$ given by Assumption 3.1 generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ defined by

$$e^{\mathcal{A}t}\Big(\sum_{k=1}^{\infty}\gamma_k e_k\Big) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\gamma_k e_k \qquad \forall\ t \geq 0\ .$$

It has the following property that for all $t>0$, $\beta\geq\alpha$, $0<c\leq\lambda_{n+1}$ and all $u\in\mathcal{H}^{\beta}$

$$\left\|e^{t\mathcal{A}}P_s u\right\|_{\alpha} \leq Mt^{-\frac{\alpha-\beta}{m}}e^{-ct}\left\|P_s u\right\|_{\beta}\ , \tag{14}$$

where M depends only on α, β and c.

Assumption 3.3 Let $\mathcal{L}:\mathcal{H}^{\alpha}\to\mathcal{H}^{\alpha-\beta}$ for some $\beta\in[0,m)$ be a linear continuous mapping that commutes with P_c and P_s.

Assumption 3.4 Assume that $\mathcal{F}:(\mathcal{H}^{\alpha})^3\to\mathcal{H}^{\alpha-\beta}$ with β as in Assumption 3.3 is trilinear, symmetric, and satisfies for some constant $C>0$ the following conditions

$$\|\mathcal{F}(u,v,w)\|_{\alpha-\beta} \leq C\|u\|_{\alpha}\|v\|_{\alpha}\|w\|_{\alpha} \qquad \forall\ u,v,w\in\mathcal{H}^{\alpha}, \tag{15}$$

$$\langle\mathcal{F}_c(u),u\rangle \leq 0 \qquad \forall\ u\in\mathcal{N}, \tag{16}$$

and

$$\langle\mathcal{F}_c(u,u,w),w\rangle \leq 0 \qquad \forall\ u,w\in\mathcal{N}. \tag{17}$$

We use $\mathcal{F}(u)=\mathcal{F}(u,u,u)$ and $\mathcal{F}_c=P_c\mathcal{F}$ for shorthand notation throughout the paper. For the noise we suppose:

Assumption 3.5 Let W be a Wiener process in $\mathcal{H}$ over some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Suppose for $t\geq 0$,

$$W(t)=\sum_{k=n+1}^{N}\alpha_k\beta_k(t)e_k \ \text{ for some } N\geq n+1,$$

where the $(\beta_k)_{k\in\{n+1,\dots,N\}}$ are independent, standard Brownian motions in $\mathbb{R}$ and the $(\alpha_k)_{k\in\{n+1,\dots,N\}}$ are real numbers.

We define the fast OU processes $\mathcal{Z}$ and its coefficients $\mathcal{Z}_k(T)$ by

$$\mathcal{Z}_k(T) := \alpha_k\varepsilon^{-1}\int_0^T e^{-\varepsilon^{-2}\lambda_k(T-\tau)}d\tilde{\beta}_k(\tau), \tag{18}$$

for $k\in\{n+1,\dots,N\}$ and

$$\mathcal{Z}(T) := \sum_{k=n+1}^{N}\mathcal{Z}_k(T)e_k\ , \tag{19}$$

where the processes $\tilde{\beta}_k(T) := \varepsilon\beta_k(\varepsilon^{-2}T)$ are rescaled versions of the original Brownian motion.

Remark 3.6 We take $N < \infty$ in the above assumption for simplicity of presentation. Nevertheless most results are still true for $N = \infty$, using the same method of proof. We only need to control the convergence of various infinite series, which is possible if the noise is not too irregular, which means for α_k decaying sufficiently fast for $k \to \infty$.

To be more precise, consider the OU-process as an infinite series expansion. For instance for the averaging, we can expand all terms containing the fast OU-process using series and use the averaging results (cf. Lemma 5.1 and 5.2) on the coefficients. We thus obtain for the additional linear term in (9) an infinite series, which needs to be finite. Similar problems arise, for example, when we try to derive bounds for the OU-process $\mathcal{Z}$.

For our result we rely on a cut off argument. This is implemented using a stopping time. We consider only solutions $u = (a, \psi)$ that are not too large, as given by the next definition.

Definition 3.7: For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined in (2) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{12})$, the stopping time τ^* as

$$\tau^* := T_0 \wedge \inf\left\{T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa}\right\}. \tag{20}$$

Definition 3.8: For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say that $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{21}$$

We use also the analogous notation for time-independent random variables.

The main theorem for the first approximation result is:

Theorem 3.9 : (Approximation) *Under Assumptions 3.1, 3.3, 3.4 and 3.5 let u be a solution of (1) with initial condition $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$, where $a(0) \in \mathcal{N}$ and $(0) \in S$. Suppose b is a solution of the amplitude equation (9) with $b(0) = a(0)$.*

Then for all $p > 1$ and $T_0 > 0$ and all $\kappa \in (0, \frac{1}{12})$, there exists $C > 0$ such that for $\|u(0)\|_\alpha \leq \delta_\varepsilon \varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ we have

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \leq C\varepsilon^p, \tag{22}$$

where

$$\mathcal{Q}(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \mathcal{Z}(T), \tag{23}$$

with $\mathcal{Z}(T)$ defined in (19).

The proof will be given in Section 6 later. Let us first discuss the additional error term $\mathcal{Q}$ in (23). We see that the first part of $\mathcal{Q}$ decays exponentially fast on the fast time-scale $\mathcal{O}(\varepsilon^2)$. The second part is an OU-process $\mathcal{Z}$, which is a small noise term, as discussed in the formal derivation.

An immediate consequence is the following corollary.

Corollary 3.10: *Under the Assumptions of Theorem 3.9 for arbitrary initial* *condition $u(0)$ we obtain*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \le C\varepsilon^p + \mathbb{P}(\|u(0)\|_\alpha > \delta_0\varepsilon). \quad (24)$$

The proof is straightforward. It is given at the end of Section 6.1 for completeness. For the higher order correction the main result is:

Theorem 3.11: (Higher order correction) *Under Assumptions 3.1, 3.3, 3.4 and 3.5 with all $\alpha_k = \sigma$ and $n = 1$. Let u be a solution of (1) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$ where $a(0) \in \mathcal{N}$ and $\psi(0) \in \mathcal{S}$. Suppose b_1 and b_2 are solutions of the amplitude equation (9) and its higher order corrections (12), respectively, with $b_1(0) = a(0)$ and $b_2(0) = 0$.*

Then for all $p > 1$, $T_0 > 0$, and $\kappa \in (0, \frac{1}{7})$, there exists $C > 0$ such that for $\|u(0)\|_\alpha \le \delta_\varepsilon\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ we have

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b_1(\varepsilon^2 t)-\varepsilon^2 b_2(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa}\Big) \le C\varepsilon^p \quad (25)$$

for all $\varepsilon > 0$ sufficiently small.

The proof of this theorem will be given in Section 7 later. Again, with the same proof as the previous corollary, we obtain:

Corollary 3.12: *Under the Assumptions of Theorem 3.11 for arbitrary initial condition $u(0)$ we obtain*

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\big\|u(t)-\varepsilon b_1(\varepsilon^2 t)-\varepsilon^2 b_2(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa}\Big)& \\ \le \mathbb{P}(\|u(0)\|_\alpha > \delta_\varepsilon\varepsilon) + C\varepsilon^p.&\end{aligned} \quad (26)$$

4. Bounds for the high modes

In this section we verify that the non-dominant modes ψ of u are well approximated by a fast OU-process. As the initial condition $\psi(0)$ itself is not too small, we necessarily need to include an exponentially fast decaying term depending on the initial condition. This is already half of the approximation result of (25).

Lemma 4.1: *Under Assumption 3.1, 3.3 and 3.4, for $\kappa > 0$ from the definition of τ^* and $p \ge 1$, there is a constant $C > 0$ such that,*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\| \quad (T) - \mathcal{Q}(T)\Big\|_\alpha^p \le C\varepsilon^{2p-3p\kappa}, \quad (27)$$

where $\mathcal{Q}(T)$ is defined in (23). I.e., $\psi = \mathcal{Q} + \mathcal{O}(\varepsilon^{2-3\kappa})$.

Proof: The mild solution of (6) is for $T \le \tau^*$

$$(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \int_0^T e^{\varepsilon^{-2}(T-\tau)\mathcal{A}_s}\left[\mathcal{L}_s\psi + \mathcal{F}_s(a+\psi)\right](\tau)\,d\tau + \mathcal{Z}(T).$$

Using triangle inequality

$$\begin{aligned}\Big\| \quad (T) - \mathcal{Q}(T)\Big\|_\alpha &\leq \Big\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} \mathcal{L}_s \psi\,(\tau)\,d\tau \Big\|_\alpha \\ &\qquad + \Big\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} \mathcal{F}_s(a\,(\tau) + \psi\,(\tau))d\tau \Big\|_\alpha \\ &:= I_1 + I_2\ .\end{aligned}$$

We now bound these two terms separately. For the first term, we obtain by using (14) for the semigroup

$$\begin{aligned}I_1 &\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|\mathcal{L}_s\psi(\tau)\right\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|\psi(\tau)\right\|_{\alpha} d\tau \\ &\leq C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \left\|\psi(\tau)\right\|_\alpha \int_0^{\varepsilon^{-2}cT} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\ &\leq C\varepsilon^{2-\kappa}\end{aligned}$$

for $T \leq \tau^*$. For the second term, we obtain by using Assumption 3.4 for $\mathcal{F}$

$$\begin{aligned}I_2 &\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|\mathcal{F}_s(a\,(\tau) + \psi\,(\tau))\right\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \left\|a\,(\tau) + \psi(\tau)\right\|_{\alpha}^3 d\tau \\ &\leq C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \left\|a\,(\tau) + \psi(\tau)\right\|_\alpha^3 \int_0^{\varepsilon^{-2}cT} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\ &\leq C\varepsilon^2 \Big(\sup_{[0,\tau^*]} \|a\|_\alpha^3 + \sup_{[0,\tau^*]} \|\psi\|_\alpha^3 \Big) \\ &\leq C\varepsilon^{2-3\kappa},\end{aligned}$$

where we used again the definition of τ^*. Combining all results, yields (27). □

Let us now provide bounds on $\mathcal{Z}$ and thus later on ψ. These are also used to show that ψ is not too large, even at time τ^*. The following lemma establishes that $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$ for any $\kappa_0 > 0$. With some more effort one can prove that the bound is actually logarithmic in ε. But the result presented here is sufficient for our purposes.

Lemma 4.2: *Under Assumption 3.1 and 3.5, there is a constant* $C > 0$, *depending on* $p > 1,\ \alpha_k,\ \lambda_k,\ \kappa_0 > 0$ *and* T_0, *such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \leq C\varepsilon^{-\kappa_0},$$

and

$$\mathbb{E} \sup_{T\in[0,T_0]} \|\mathcal{Z}(T)\|_\alpha^p \le C\varepsilon^{-\kappa_0},$$

where $\mathcal{Z}_k(T)$ and $\mathcal{Z}(T)$ are defined in (18) and (19), respectively.

Proof: In order to prove the first part, we define

$$\delta(T) = e^{-\lambda_\varepsilon T} \quad \text{and} \quad \gamma(T) = \int_0^T e^{2\lambda_\varepsilon \tau} d\tau = \frac{1}{2\lambda_\varepsilon}(\delta(T)^{-2} - 1),$$

where $\lambda_\varepsilon = \varepsilon^{-2}\lambda_k$, and

$$Y(T) := \alpha_k \varepsilon^{-1}\delta(T) \cdot \beta\left(\gamma(T)\right).$$

Note that $\mathcal{Z}_k(T)$ and $Y(T)$ are Gaussian stochastic process with

$$\mathbb{E}\mathcal{Z}_k(T) = \mathbb{E}Y(T) = 0,$$

and

$$\mathbb{E}\mathcal{Z}_k(T)\mathcal{Z}_k(S) = \mathbb{E}Y(T)Y(S) = \alpha_k^2\varepsilon^{-2}\delta(T+S)\gamma(S).$$

Thus $\mathcal{Z}_k(T)$ is a version of $Y(T)$, and

$$\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p &= \mathbb{E} \sup_{T\in[0,T_0]} |Y(T)|^p = \left(\alpha_k\varepsilon^{-1}\right)^p \mathbb{E} \sup_{T\in[0,T_0]} |\delta(T)\cdot\beta\left(\gamma(T)\right)|^p \\
&\le \left(\alpha_k\varepsilon^{-1}\right)^p \sum_{i=0}^{n-1} \mathbb{E} \sup_{T\in[T_i,T_{i+1}]} |\delta(T)|^p \left|\beta\left(\gamma(T)\right)\right|^p,
\end{aligned}$$

where $(T_i)_{i=0}^n$ is an equidistant decomposition of $[0,T_0]$. Using Doob's theorem, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p &\le C_{p,\alpha_k}\varepsilon^{-p} \sum_{i=0}^{n-1} \delta(T_i)^p \gamma(T_{i+1})^{\frac{p}{2}} \\
&\le C_{p,\alpha_k}\varepsilon^{-p}\lambda_\varepsilon^{-p/2} \sum_{i=0}^{n-1} \left[\frac{\delta(T_i)}{\delta(T_{i+1})}\right]^p \\
&= C_{p,\alpha_k}\lambda_k^{-p/2} \sum_{i=0}^{n-1} e^{p\lambda_\varepsilon h} = C_{p,\alpha_k}\lambda_k^{-p/2}\frac{T_0}{h}e^{p\lambda_\varepsilon h},
\end{aligned}$$

where $h = T_{i+1} - T_i$. Taking $h = \frac{1}{\lambda_\varepsilon}$, we obtain

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \le C\varepsilon^{-2}. \tag{28}$$

By Hölder inequality we derive for all $p \ge 1$ and sufficiently large $q > \frac{2}{\kappa_0}$

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \le \left(\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^{pq}\right)^q \le C\varepsilon^{-\kappa_0}.$$

In order to prove the second part,

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p &\le C_p\Big(\mathbb{E}\sup_{T\in[0,T_0]}\sum_{k=n+1}^{N}k^{2\alpha}\mathcal{Z}_k^2(T)\Big)^{p/2}\\ &\le C_p\Big(\sum_{k=n+1}^{N}k^{2\alpha}\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T)\Big)^{p/2}.\end{aligned} \tag{29}$$

Using Hölder inequality for all q and (28), to obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T)\le\Big(\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^{2q}(T)\Big)^{1/q}\le C\varepsilon^{-2/q}.$$

Hence, for q sufficiently large

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p\le C\varepsilon^{-p/q}\le C\varepsilon^{-\kappa_0}\ .$$

□

The following corollary states that the non-dominant modes $\psi(T)$ are with high probability much smaller than $\varepsilon^{-\kappa}$ as asserted in the Definition 3.7 for $T\le\tau^*$. To be more precise, $\psi=\mathcal{O}(\delta_\varepsilon+\varepsilon^{-\kappa_0})$ for every $\kappa_0>0$ and $\delta_\varepsilon\in(0,\varepsilon^{-\frac{1}{3}\kappa})$. We will use this later to show that $\tau^*\ge T_0$ with high probability (cf. Remark 6.5 and proof of Theorem 3.9).

Corollary 4.3: *Under the assumptions of Lemmas 4.1 and 4.2 with $\kappa<\frac{2}{3}$. For $p>0$ and for $\kappa_0>0$ there exist a constant $C>0$ such that for $\|\psi(0)\|_\alpha\le\delta_\varepsilon$ one has*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big)\le C(\delta_\varepsilon^p+\varepsilon^{-\kappa_0})\ . \tag{30}$$

Proof: From (27) we obtain by triangle inequality and Lemma 4.2

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big)\le C\delta_\varepsilon^p+C\varepsilon^{-\kappa_0}+C\varepsilon^{2p-3p\kappa}\ .$$

For $\kappa<\frac{2}{3}$ we obtain (30). □

Lemma 4.4: *If Assumption 3.1 holds, then for $q\ge1$ there exists a constant $C>0$ such that for $\|\psi(0)\|_\alpha\le\delta_\varepsilon$ one has*

$$\int_0^T\Big\|e^{\tau\varepsilon^{-2}\mathcal{A}_s}\psi(0)\Big\|_\alpha^q d\tau\le C\delta_\varepsilon^q\varepsilon^2.$$

Proof: Using (14) we obtain

$$\int_0^T\Big\|e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi(0)\Big\|_\alpha^q d\tau\le c\int_0^T e^{-q\varepsilon^{-2}c\tau}\,\|\psi(0)\|_\alpha^q\,d\tau\le\frac{\varepsilon^2}{qc}\,\|\psi(0)\|_\alpha^q\,.$$

□

5. Averaging over the fast OU-process

Let us now turn to the averaging result for the OU-process $\mathcal{Z}$. First in Lemma 5.1, we provide the first order approximation. It states that even powers of a real valued OU-process average to a constant, while odd powers are small of order $\mathcal{O}(\varepsilon)$. Later in Lemma 5.2 we collect all terms of order ε.

Lemma 5.1: *Let X be a real valued stochastic process such that for some $r \geq 0$ we have $X(0) = \mathcal{O}(\varepsilon^{-r})$. Fix any $\kappa_0 > 0$. If $dX = GdT$ with $G = \mathcal{O}(\varepsilon^{-r})$, then, for any non-negative integers n_1, n_2, n_3 not all zero and for all triples of different indices k_1, k_2, $k_3 \in \{n+1, \ldots, N\}$, we obtain*

$$\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}d\tau = \sum_{i=1}^{3}\frac{n_i(n_i-1)\alpha_{k_i}^2}{2(n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3})}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-2}d\tau$$
$$+\mathcal{O}(\varepsilon^{1-r-(n_1+n_2+n_3)\kappa_0}), \tag{31}$$

where the fast OU-process $\mathcal{Z}_k$ is defined in (18).

Proof: We note first that

$$\mathbb{E}\sup_{[0,T_0]}|X|^p \leq C\mathbb{E}\sup_{[0,T_0]}|G|^p \leq C\varepsilon^{-pr}.$$

Applying Itô formula to the term $X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}$ and integrating from 0 to T in order to obtain

$$(n_1\lambda_k + n_2\lambda_l + n_3\lambda_j)\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tau$$
$$= -\varepsilon^2 X(T)\mathcal{Z}_k^{n_1}(T)\mathcal{Z}_l^{n_2}(T)\mathcal{Z}_j^{n_3}(T) + \varepsilon^2\int_0^T \mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}Gd\tau$$
$$+n_1\alpha_k\varepsilon\int_0^T X\mathcal{Z}_k^{n_1-1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tilde{\beta}_k + n_2\alpha_l\varepsilon\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2-1}\mathcal{Z}_j^{n_3}d\tilde{\beta}_l$$
$$+n_3\alpha_j\varepsilon\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3-1}d\tilde{\beta}_j + \frac{n_1(n_1-1)\alpha_k^2}{2}\int_0^T X\mathcal{Z}_k^{n_1-2}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tau$$
$$+\frac{n_2(n_2-1)\alpha_l^2}{2}\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2-2}\mathcal{Z}_j^{n_3}d\tau + \frac{n_3(n_3-1)\alpha_j^2}{2}\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3-2}d\tau\ ,$$

where we used that the β's are independent, and thus $d\beta_k d\beta_l = 0$ if $k \neq l$. Taking the absolute value and using Burkholder-Davis-Gundy theorem yields (31). $\square$

Lemma 5.2: *Under the assumptions of Lemma 5.1, we have*

$$\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}d\tau = \sum_{i=1}^{3}\frac{n_i(n_i-1)\alpha_{k_i}^2}{2(n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3})}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-2}d\tau$$
$$+\varepsilon\sum_{i=1}^{3}\frac{n_i\alpha_{k_i}}{n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3}}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-1}d\tilde{\beta}_{k_i}$$
$$+\mathcal{O}(\varepsilon^{2-r-(n_1+n_2+n_3)\kappa_0}).$$

Proof: We follow the same proof, as in the previous Lemma. □

Remark 5.3 In case of X being a stochastic process in $\mathcal{N}$ or $\mathbb{C}$, both Lemmas above are still true.

6. First order estimates

This section is devoted to the proof of the first main result of Theorem 3.9. In the second part of this section we give some applications for this approximation result.

6.1 Proof of the main result

Let us first establish that we can apply the averaging lemma to (5).

Lemma 6.1: *Assume that Assumption 3.3 and 3.4 hold. Let X be a stochastic process in $\mathcal{N}$ and $dX = GdT$. If $X = \mathcal{F}_c(a, e_k, e_l)$ or $X = \mathcal{F}_c(a, a, e_k)$, then $G = \mathcal{O}(\varepsilon^{-3\kappa})$ or $G = \mathcal{O}(\varepsilon^{-4\kappa})$, respectively.*

Proof: If $X = \mathcal{F}_c(a, e_k, e_k)$, then

$$dX = \mathcal{F}_c(da, e_k, e_l) = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), e_k, e_l)dT.$$

Let

$$G = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), e_k, e_l).$$

Taking the $\mathcal{H}^\alpha$ norm, using Assumption 3.4 and the fact all $\mathcal{H}^\alpha$-norms are equivalent on $\mathcal{N}$, to obtain

$$\begin{aligned}\|G\|_\alpha &\le C\|\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\|_\alpha \le C\|a\|_\alpha + C\|\mathcal{F}_c(a+\psi)\|_{\alpha-\beta}\\ &\le C\|a\|_\alpha + C\|a+\psi\|_\alpha^3 \le C\|a\|_\alpha + C\|a\|_\alpha^3 + C\|\psi\|_\alpha^3.\end{aligned}$$

Using the definition of τ^*, we obtain for $p>0$

$$\mathbb{E}\sup_{[0,\tau^*]}\|G\|_\alpha^p \le C\varepsilon^{-3p\kappa}.$$

Analogously, if $X = \mathcal{F}_c(a, a, e_k)$, then

$$dX = 2\mathcal{F}_c(da, a, e_k) = 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), a, e_k)dT.$$

Define

$$G := 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), a, e_k),$$

in order to obtain

$$\mathbb{E}\sup_{[0,\tau^*]} \|G\|_\alpha^p \le C\varepsilon^{-4p\kappa}.$$

□

Lemma 6.2: *If Assumptions 3.1, 3.3, 3.4 and 3.5 hold and $\|\psi(0)\|_\alpha \le \delta_\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with $\kappa \in (0, \frac{1}{12})$ from the definition of τ^*, then*

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}_c(a)d\tau + \sum_{k=n+1}^N \frac{3\alpha_k^2}{2\lambda_k}\int_0^T \mathcal{F}_c(a, e_k, e_k)d\tau + R(T), \quad (32)$$

where

$$R = \mathcal{O}(\varepsilon^{1-5\kappa}). \quad (33)$$

Proof: Recall Lemma 4.1, which states

$$= y_\varepsilon + \mathcal{Z} + \mathcal{O}(\varepsilon^{2-3\kappa}), \quad (34)$$

where

$$y_\varepsilon(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0).$$

Substituting from (34) into (5) we obtain for $\kappa < 2/3$ using the bounds for $a = \mathcal{O}(\varepsilon^{-\kappa})$, $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$, and $y_\varepsilon = \mathcal{O}(\delta_\varepsilon \varepsilon^2)$

$$\begin{aligned}
da &= [\mathcal{L}_c a + \mathcal{F}_c(a + y_\varepsilon + \mathcal{Z})]\, dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT \\
&= [\mathcal{L}_c a + \mathcal{F}_c(a) + 3\mathcal{F}_c(a,a,\mathcal{Z}) + 3\mathcal{F}_c(a,\mathcal{Z},\mathcal{Z}) + \mathcal{F}_c(\mathcal{Z}) \\
&\quad +3\mathcal{F}_c(a,a,y_\varepsilon) + 6\mathcal{F}_c(a,\mathcal{Z},y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon) \\
&\quad +3\mathcal{F}_c(a,y_\varepsilon,y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon) + \mathcal{F}_c(y_\varepsilon)]dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT.
\end{aligned}$$

Integrating from 0 to T yields for $T \le \tau^*$

$$\begin{aligned}
a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}_c(a)d\tau + 3\sum_{k=n+1}^N \int_0^T \mathcal{Z}_k \mathcal{F}_c(a,a,e_k)d\tau \\
&\quad +3\sum_{k=n+1}^N \int_0^T \mathcal{Z}_k^2 \mathcal{F}_c(a,e_k,e_k)d\tau + 3\sum_{k=n+1}^N \sum_{l\neq k}^N \int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{F}_c(a,e_k,e_l)d\tau \\
&\quad + \sum_{k,l,j=n+1}^N \int_0^T \mathcal{F}_c(\mathcal{Z}_k e_k, \mathcal{Z}_l e_l, \mathcal{Z}_j e_j)d\tau + R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}),
\end{aligned} \quad (35)$$

where

$$
\begin{aligned}
R_1 &= 3\int_0^T \mathcal{F}_c(a,a,y_\varepsilon)d\tau + 6\int_0^T \mathcal{F}_c(a,\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(a,y_\varepsilon,y_\varepsilon)d\tau \\
&\quad +3\int_0^T \mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(y_\varepsilon)d\tau \\
:= &\quad I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (36)
\end{aligned}
$$

Now we use Assumption 3.3, the definition of τ^*, and the equivalence of $\mathcal{H}^\alpha$-norms on $\mathcal{N}$ to bound R_1. We bound all terms in (36) separately. For the first term in (36)

$$
\|I_1\|_\alpha \le C\int_0^T \|a\|_\alpha^2 \|y_\varepsilon\|_\alpha \, d\tau \le C \sup_{[0,T_0]} \|a\|_\alpha^2 \int_0^T \|y_\varepsilon\|_\alpha \, d\tau.
$$

Using Lemma 4.4 for $q = 1$, we obtain

$$
I_1 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa}).
$$

Analogous results hold for all other terms. To be more precise:

$$
\begin{gathered}
I_2 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-\kappa-\kappa_0}),\ I_3 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa}),\ I_4 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa_0}), \\
I_5 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa_0}),\ I_6 = \mathcal{O}(\delta_\varepsilon^3 \varepsilon^2)\ .
\end{gathered}
$$

Collecting all results we obtain for $\kappa_0 \le \kappa$, where $\kappa_0 > 0$ is arbitrary from Lemma 4.2,

$$
R_1 = \mathcal{O}((1+\delta_\varepsilon^2)\varepsilon^{2-2\kappa}). \qquad (37)
$$

Finally, applying Lemmas 5.1 and 6.1 to (35), we obtain (32). □

Lemma 6.3: *Let Assumptions 3.1, 3.3 and 3.4 hold. Define b in $\mathcal{N}$ as the solution of (9). If the initial condition satisfies $\mathbb{E}\,|b(0)|^p \le \delta_\varepsilon^p$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, then for all $T_0 > 0$ there exists a constant $C > 0$ such that*

$$
\sup_{T\in[0,T_0]} \|b(T)\| \le C|b(0)|
$$

$$
\mathbb{E} \sup_{T\in[0,T_0]} |b(T)|^p \le C^p \delta_\varepsilon^p. \qquad (38)
$$

Proof: Taking the scalar product $\langle \cdot, b\rangle$ on both sides of (9) yields

$$
\frac{1}{2}\partial_T |b|^2 = \langle \mathcal{L}_c b, b\rangle + \langle \mathcal{F}_c(b), b\rangle + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(b, e_k, e_k), b\rangle\ .
$$

Using Cauchy-Schwarz inequality and Assumption 3.4, we obtain

$$\frac{1}{2}\partial_T |b|^2 \leq C\,|b|^2 .$$

We apply now Gronwalls lemma to deduce for all $T \in [0, T_0]$

$$|b(T)| \leq |b(0)|\, e^{CT_0}. \tag{39}$$

Taking expectation after supremum on both sides yields (38). □

In the following we are no longer able to calculate moments of error terms. Thus we restrict ourselves to a sufficiently large subset of Ω, where our estimates go through.

Definition 6.4: Given $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with $\kappa > 0$ from the definition of τ^*. Define the set $\Omega^* \subset \Omega$ of all $\omega \in \Omega$ such that all these estimates

$$\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha < C\varepsilon^{2-4\kappa}\ , \tag{40}$$

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa}\ , \tag{41}$$

$$\sup_{[0,\tau^*]} |R| < \varepsilon^{1-6\kappa}\ , \tag{42}$$

and

$$\sup_{[0,\tau^*]} |b| < \delta_0 \varepsilon^{-\frac{1}{2}\kappa}\ , \tag{43}$$

hold.

Remark 6.5 The set Ω^* has approximately probability 1. For this consider

$$\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha \geq \varepsilon^{2-4\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi\|_\alpha \geq \delta_\varepsilon + \varepsilon^{-\frac{1}{2}\kappa})$$
$$-\mathbb{P}(\sup_{[0,\tau^*]} |b| \geq \delta_\varepsilon \varepsilon^{-\frac{1}{2}\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} |R| \geq \varepsilon^{1-6\kappa}).$$

Using Chebychev inequality and Lemmas 4.1, 6.2, 6.3 and Corollary 4.3 with $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$, and some $\kappa_0 \leq \frac{1}{3}\kappa$, we obtain for sufficient large q

$$\mathbb{P}(\Omega^*) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q\kappa}] \geq 1 - C\varepsilon^{\frac{1}{6}q\kappa} \geq 1 - C\varepsilon^{p}. \tag{44}$$

Theorem 6.6: *Assume that Assumptions 3.1, 3.3, 3.4 and 3.5 hold and suppose $|a(0)| \leq \delta_\varepsilon$ and $\|\psi(0)\|_\alpha \leq \delta_\varepsilon$. Let b be a solution of the amplitude equation (9) and a as defined in (2). If the initial condition satisfies $a(0) = b(0)$, then*

$$\sup_{T\in[0,\tau^*]} |a(T) - b(T)| \leq C(1 + \delta_\varepsilon^2)\varepsilon^{1-12\kappa}, \tag{45}$$

and for $\kappa < \frac{1}{12}$

$$\sup_{T\in[0,\tau^*]} |a(T)| \le C(1+\delta_\varepsilon^2), \tag{46}$$

on Ω^*.

Proof: Define $\varphi := a - R$, where R is defined in (33). From (32) we obtain

$$\varphi(T) = a(0) + \int_0^T \mathcal{L}_c[\varphi + R]d\tau + \int_0^T \mathcal{F}_c(\varphi + R)d\tau + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\varphi + R, e_k, e_k)d\tau. \tag{47}$$

Subtracting (47) from the amplitude equation (9) and defining $h := b - \varphi$, we obtain

$$\begin{aligned} h(T) = &\int_0^T \mathcal{L}_c h d\tau - \int_0^T \mathcal{L}_c R d\tau + \int_0^T \left[\mathcal{F}_c(b) - \mathcal{F}_c(b-h+R)\right] d\tau \\ &+ \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(h-R, e_k, e_k)d\tau. \end{aligned}$$

Thus

$$\partial_T h = \mathcal{L}_c h - \mathcal{L}_c R + \mathcal{F}_c(b) - \mathcal{F}_c(b-h+R) + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \mathcal{F}_c(h-R, e_k, e_k). \tag{48}$$

Taking the scalar product $\langle \cdot, h\rangle$ on both sides of (48), we have

$$\begin{aligned} \frac{1}{2}\partial_T |h|^2 = \langle \partial_T h, h\rangle = &\langle \mathcal{L}_c h, h\rangle - \langle \mathcal{L}_c R, h\rangle + \langle \mathcal{F}_c(b) - \mathcal{F}_c(b-h+R), h\rangle \\ &+ \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(h, e_k, e_k), h\rangle - \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \left\langle \mathcal{F}_c(R, e_k, e_k), h\right\rangle. \end{aligned}$$

Using Cauchy-Schwarz inequality and Assumption 3.4, we obtain the following differential inequality

$$\partial_T |h|^2 \le C[|h|^2 + |h|^4] + C\left[|R|^4 + |b|^2|R|^2 + |b|^4|R|^2 + |b|^2|R|^4\right].$$

Using (42) and (43) in the definition of Ω^*, yields for $T \le \tau^*$

$$\partial_T |h|^2 \le C[|h|^2 + |h|^4] + C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa} \quad \text{on} \quad \Omega^*.$$

As long as $|h| \le 1$, we derive

$$\partial_T |h|^2 \le 2C|h|^2 + C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa}.$$

Using Gronwall's Lemma, we obtain for $T \le \tau^* \le T_0$

$$|h(T)|^2 \le C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa} \le 1$$

for $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$ with $\kappa < \frac{1}{12}$ and $\varepsilon > 0$ sufficiently small. Thus

$$\sup_{[0,\tau^*]} |h| \leq C(1+\delta_0^2)\varepsilon^{1-12\kappa} \quad \text{on} \quad \Omega^*. \tag{49}$$

We finish the first part by using (42), (49) and

$$\sup_{[0,\tau^*]} |a-b| = \sup_{[0,\tau^*]} |h-R| \leq \sup_{[0,\tau^*]} |h| + \sup_{[0,\tau^*]} |R|\,.$$

For the second part of the theorem consider

$$\sup_{[0,\tau^*]} |a| \leq \sup_{[0,\tau^*]} |a-b| + \sup_{[0,\tau^*]} |b|.$$

Using the first part and (43), we obtain (46) as $\kappa < \frac{1}{12}$. □

Now, we can use the results previously obtained to prove the main result of Theorem 3.9 for the approximation of the solution (8) of the SPDE (1).

Proof: [Proof of Theorem 3.9] For the stopping time, we note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{ \sup_{T\in[0,T_0]} |a(T)| < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa}\} \supseteq \Omega^*,$$

where the last inclusion holds due to (41) and Theorem 6.6. Now let us turn to the approximation result. Using (2) and the triangle inequality yields on Ω^*

$$\begin{aligned}\sup_{T\in[0,\tau^*]} \Big\|u(\varepsilon^{-2}T) - \varepsilon b(T) - \varepsilon \mathcal{Q}(T)\Big\|_\alpha &\leq \varepsilon \sup_{[0,\tau^*]} \|a-b\|_\alpha + \varepsilon \sup_{[0,\tau^*]} \Big\| \quad - \mathcal{Q}\Big\|_\alpha \\ &\leq C(1+\delta_\varepsilon^2)\varepsilon^{2-12\kappa} + C\varepsilon^{3-4\kappa}\;.\end{aligned}$$

From (40) and (45), we obtain

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha &= \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \Big\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha \\ &\leq C\varepsilon^{2-\frac{38}{3}\kappa} \qquad \text{on} \quad \Omega^*.\end{aligned}$$

Thus

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \leq 1 - \mathbb{P}(\Omega^*).$$

Using (44), yields (22). □

Proof: [Proof of Corollary 3.10] Define $\Omega_0 \subset \Omega$ as

$$\Omega_0 = \{\omega \in \Omega \ : \ \|u(0)\|_\alpha \leq \delta_0 \varepsilon\},$$

and define

$$\hat{u}(0) = \begin{cases} 0 & \text{on} \;\; \Omega_0^c \\ u(0) & \text{on} \;\; \Omega_0. \end{cases}$$

Hence, the solutions u and $\hat{u}$ corresponding to the initial conditions $u(0)$ and $\hat{u}(0)$ coincide on Ω_0. Thus

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \\
&= \mathbb{P}\Big(\Big\{ \sup_{t\in[0,\varepsilon^{-2}T_0]} \|\hat{u}(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big\} \cap \Omega_0\Big) \\
&\qquad + \mathbb{P}\Big(\Big\{ \sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big\} \cap \Omega_0^c\Big) \\
&\le \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\hat{u}(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) + \mathbb{P}\Big(\Omega_0^c\Big) \\
&\le C\varepsilon^p + \mathbb{P}\left(\|u(0)\|_\alpha > \delta_\varepsilon \varepsilon\right),
\end{aligned}$$

where we used (22) for the solution $\hat{u}$. □

6.2 Applications

In the literature there are numerous examples of equations with cubic nonlinearities where our theory applies. Noteable examples are the Swift-Hohenberg equation, Ginzburg-Landau / Allen-Cahn equations and some surface growth model. In all these cases degenerate additive noise stabilizes the dynamics of the dominant modes. Moreover, the amplitude equation is always of the following type

$$\partial_T A = \nu A - C_\alpha A - C_{\mathcal{F}} A|A|^2,$$

where A is the amplitude of the dominant modes in $\mathcal{N}$. The constant C_α depends explicitly on the noise strength, while $C_{\mathcal{F}}$ depends only on the nonlinearity and the linear operators in the equation.

6.2.1 *Swift-Hohenberg equation*

The Swift-Hohenberg equation was already defined in the introduction (cf. (10)). It has been used as a toy model for the convective instability in Rayleigh-Bénard problem (see [6] or [10]). Now it is one of the celebrated models in the theory of pattern formation [6]. For this model note that

$$\mathcal{A} = -(1+\partial_x^2)^2, \quad \mathcal{L} = \nu\mathcal{I}, \ \mathcal{F}(u) = -u^3.$$

If we take the orthonormal basis

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k>0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k=0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k<0, \end{cases}$$

and the spaces

$$\mathcal{H} = L^2([0,2\pi]) \quad \text{and} \quad \mathcal{N} = \text{span}\{\sin,\ \cos\},$$

then the eigenvalues of $-\mathcal{A} = (1+\partial_x^2)^2$ are $\lambda_k = (1-k^2)^2$ for $k \in \mathbb{Z}$. So, it is easy to check that, after rearranging the indices, Assumption 3.1 is true with $m = 4$.

If we consider $u = u_1 \sin + u_{-1} \cos$ and $w = w_1 \sin + w_{-1} \cos \in \mathcal{N}$, then we can easily verify Assumption 3.4 as follows:

$$\langle \mathcal{F}_c(u), u \rangle = -\frac{3\pi}{4}(u_1^2 + u_{-1}^2)^2 \le 0,$$

where we used

$$\mathcal{F}_c(u) = -\tfrac{3}{4}(u_1^3 + u_1 u_{-1}^2) \sin - \tfrac{3}{4}(u_{-1}^3 + u_1^2 u_{-1}) \cos \,.$$

Moreover,

$$\langle \mathcal{F}_c(u,u,w), w \rangle = -\frac{3\pi}{4}\left(u_1^2 w_1^2 + w_1^2 u_{-1}^2 + w_{-1}^2 u_{-1}^2 + w_{-1}^2 u_1^2\right) \le 0.$$

With $\alpha = 1$ and $\beta = 0$ it holds that

$$\|\mathcal{F}(u,v,w)\|_{\mathcal{H}^1} = \|-uvw\|_{\mathcal{H}^1} \le C \|u\|_{\mathcal{H}^1} \|v\|_{\mathcal{H}^1} \|w\|_{\mathcal{H}^1} \,.$$

For Assumption 3.5 we consider several cases:
First case. The noise is a constant in the space (i.e. $W(t) = \frac{\alpha_0}{\sqrt{2\pi}}\beta_0(t)$).
Our main theorem states that the rescaled solution of (10)

$$u(t,x) = \varepsilon v(\varepsilon^2 t, x),$$

is of the type

$$v(T,x) = \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon \frac{\alpha_0}{\sqrt{2\pi}} \partial_T \tilde{\beta}_0(T) + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solutions of the following two-dimensional amplitude equation:

$$\partial_T \gamma_i = (\nu - \tfrac{3\alpha_0^2}{4\pi})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1 \,.$$

Second case. If the noise acts only on $\sin(kx)$ (or $\cos(kx)$) for one single $k \in \{2, 3, \ldots, N\}$, then the amplitude equations for (10) are

$$\partial_T \gamma_i = (\nu - \tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1.$$

Third case. If the noise takes the form $W(t) = \sum_{k=2}^N \frac{\alpha_k}{\sqrt{\pi}}\beta_k(t)\sin(kx)$, then the amplitude equations for (10) are

$$\partial_T \gamma_i = (\nu - \sum_{k=2}^N \tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1 \,.$$

Now our main theorem states that the rescaled solution of (10)

$$u(t,x) = \varepsilon v(\varepsilon^2 t, x),$$

is of the type

$$v(T,x) = \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\sum_{k=2}^{N}\frac{\alpha_k}{\sqrt{\pi}}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}).$$

In the following examples we always suppose that the noise takes the form $W(t) = \sum_{k=2}^{N}\sigma_k\beta_k(t)e_k$ with $\sigma_k = \delta\alpha_k$ for $k \in \{2,3,\ldots,N\}$ and δ defined later.

6.2.2 Ginzburg-Landau / Allen-Cahn equation

The second example is the Ginzburg-Landau or Allen-Cahn equation

$$\partial_t u = (\partial_x^2 + 1)u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W(t), \tag{50}$$

subject to Dirichlet boundary conditions on the interval $[0,\pi]$. We note that

$$\mathcal{A} = \partial_x^2 + 1, \quad \mathcal{L} = \nu\mathcal{I}, \ \mathcal{F}(u) = -u^3.$$

If we take

$$\mathcal{H} = L^2([0,\pi]), \quad e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx) = \delta\sin(kx) \quad \text{and} \ \ \mathcal{N} = \text{span}\{\sin\},$$

then the Assumption 3.1 is true. The eigenvalues of $-\mathcal{A} = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. The condition in (15) is satisfied for $\alpha = 1$ and $\beta = 0$. Furthermore, for $u = \gamma_1\sin \in \mathcal{N}$ and $w = \gamma_2\sin \in \mathcal{N}$ the condition in (16) is satisfied as follows:

$$\langle\mathcal{F}_c(u), u\rangle = -\frac{3\pi}{8}\gamma_1^4 \le 0,$$

where

$$\mathcal{F}_c(u) = -\frac{3}{4}\gamma_1^3\sin,$$

and

$$\langle\mathcal{F}_c(u,u,w), w\rangle = -\frac{3\pi}{8}\gamma_1^2\gamma_2^2 \le 0,$$

For Assumption 3.5, we consider two cases:

First case. The noise acts only on $\sin(2x)$.

In this case the amplitude equation (Landau equation) of (50) takes the form

$$\partial_T\gamma = \Big(\nu - \frac{\sigma^2}{4}\Big)\gamma - \frac{3}{4}\gamma^3. \tag{51}$$

Second case. The noise is acts on $\sin(2x),\ \sin(3x),\ \ldots,\ \sin(Nx)$.

In this case the amplitude equation of (50) takes the form

$$\partial_T\gamma = \Big(\nu - \frac{3}{4}\sum_{k=2}^{N}\frac{\sigma_k^2}{k^2-1}\Big)\gamma - \frac{3}{4}\gamma^3, \tag{52}$$

If we additionally assume that $\sigma_2 = \sigma_3 = \ldots = \sigma_N = \sigma$, then (52) takes the form

$$\partial_T \gamma = \Big(\nu - \frac{9\sigma^2}{16} + \frac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma - \frac{3}{4}\gamma^3,$$

where $\mathcal{F}_c\,(u, e_k, e_k) = -\frac{1}{\pi}u$.

The main theorem states that the rescaled solution of (50)

$$u(t) = \varepsilon v(\varepsilon^2 t),$$

takes the form

$$v(T) = \gamma(T)\sin + \varepsilon \sum_{k=2}^{N} \frac{\sigma_k}{k^2-1}\partial_T \tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}),$$

where γ is the solution of the amplitude equation (52).

6.2.3 Surface growth model

The final example arising in the theory of surface growth is

$$\partial_t u = -\triangle^2 u - \mu\triangle u + \nabla\cdot(|\nabla u|^2\nabla u) + \varepsilon\partial_t W(t). \tag{53}$$

subject to periodic boundary conditions for simplicity only on the interval $[0, 2\pi]$. In order to get close to the change of stability, we consider $\mu = 1 + \varepsilon^2\nu$. Hence,

$$\mathcal{A} = -\triangle^2 - \triangle, \quad \mathcal{L} = -\nu\triangle \quad \text{and} \quad \mathcal{F}(u) = \nabla\cdot(|\nabla u|^2\nabla u).$$

Consider

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k>0,\\ \frac{1}{\sqrt{2\pi}} & \text{if } k=0,\\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k<0,\end{cases}$$

and

$$\mathcal{H} = L^2([0,2\pi]) \quad \text{and} \quad \mathcal{N} = \mathrm{span}\{1,\ \sin,\ \cos\}.$$

The eigenvalues of $-\mathcal{A}$ are $\lambda_k = k^4 - k^2$. So, Assumption 3.1 is true with $m = 4$. Moreover, if $u = \gamma_0 + \gamma_1\sin + \gamma_{-1}\cos \in \mathcal{N}$, then all conditions of Assumption 3.4 are satisfied as follows

$$\langle \mathcal{F}_c\,(u)\,, u\rangle = -\tfrac{3\pi}{4}\left(\gamma_1^2 + \gamma_{-1}^2\right)^2 \le 0\ ,$$

where

$$\mathcal{F}_c\,(u) = -\tfrac{3}{4}\left(\gamma_1^3 + \gamma_{-1}^2\gamma_1\right)\sin - \tfrac{3}{4}\left(\gamma_{-1}^3 + \gamma_1^2\gamma_{-1}\right)\cos,$$

and for $\alpha = \beta = 2$ we obtain

$$\|\mathcal{F}(u)\|_{L^2} = \|\partial_x(\partial_x u)^3\|_{L^2} \le C\|(\partial_x u)^3\|_{\mathcal{H}^1} \le C\|\partial_x u\|_{\mathcal{H}^1}^3 \le C\|u\|_{\mathcal{H}^2}^3.$$

For Assumption 3.5, we consider two cases:

First case. The noise acts only on $\sin(2x)$.

In this case the amplitude equation for (53) is a system of ordinary differential equations:

$$\partial_T \gamma_0 = 0,$$

$$\partial_T \gamma_i = \left(\nu - \tfrac{\sigma^2}{4}\right)\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1.$$

Second case. The noise acts on $\sin(2x), \sin(3x), \ldots, \sin(Nx)$.

In this case the amplitude equation for (53) is a system of ordinary differential equations:

$$\partial_T \gamma_0 = 0,$$

$$\partial_T \gamma_i = \Big(\nu - \tfrac{3}{4}\sum_{k=2}^{N} \frac{\sigma_k^2}{k^2 - 1}\Big)\gamma_i - \tfrac{3}{4}\gamma_i\left(\gamma_1^2 + \gamma_{-1}^2\right) \quad \text{for } i = \pm 1.$$

If we assume additionally that $\sigma_2 = \sigma_3 = \ldots = \sigma_N = \sigma$, then the amplitude equation for (53) takes the form

$$\partial_T \gamma_0 = 0,$$

$$\partial_T \gamma_i = \Big(\nu - \tfrac{9\sigma^2}{16} + \tfrac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma_i - \tfrac{3}{4}\gamma_i\left(\gamma_1^2 + \gamma_{-1}^2\right) \quad \text{for } i = \pm 1,$$

where $\mathcal{F}_c(\gamma_0 + \gamma_1 \sin + \gamma_{-1}\cos, e_k, e_k) = -\frac{k^2}{2}\frac{1}{\pi}(\gamma_1 \sin + \gamma_{-1}\cos)$.

7. Higher order correction

This section is devoted to the improvement of the approximation of (1) from the first order approximation (8) by discussing the higher order terms in (11) more carefully. In order to derive an equation for the higher order terms with explicit error bounds, we need to approximate martingale terms in the equation for a. We rely on the martingale approximation result of Lemma 6.1 from [3], which is based on the Levy-representation theorem, and thus limited in the final argument to $\dim \mathcal{N} = 1$. In the end of this section we give applications to the stochastic Swift-Hohenberg equation and Ginzburg-Landau equation.

7.1 Proof of the main result

For simplicity of presentation we assume for the whole section that $\alpha_k = \sigma$ for all $k \in \{n+1, \ldots, N\}$ in Assumption 3.5. This means the noise takes the form

$$W(t) = \sum_{k=n+1}^{N} \sigma \beta_k(t) e_k \quad \text{for } N \geq n+1. \tag{54}$$

Nevertheless, the proofs can easily be modified to the general case, where each β_k has a different σ_k.

In order to take higher order corrections into account we modify the stopping time in next definition as follows.

Definition 7.1: For the $\mathcal{N}\times S$-valued stochastic process (a,ψ) defined in (2) we split a into $a = a_1 + \varepsilon a_2$ with a_1 a solution of the amplitude equation (9) subject to initial condition $a_1(0) = a(0)$.

For some $T_0 > 0$ and $\kappa \in (0, \frac{1}{7})$ we define the stopping time $\tau^\sharp$ as

$$\tau^\sharp = T_0 \wedge \inf\big\{T > 0 : \|a_1(T)\|_\alpha > 2\varepsilon^{-\kappa} \text{ or } \|a_2(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa}\big\}\,. \tag{55}$$

First let us state bounds on stochastic integrals over fast OU-processes. Unfortunately, we can not prove explicit averaging results by using Itô's formula as in Lemma 5.1.

Lemma 7.2: *Consider X as in Lemma 5.1, then*

$$\int_0^T X\mathcal{Z}_k d\tilde\beta_l = \mathcal{O}(\varepsilon^{-r}), \tag{56}$$

and

$$\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde\beta_j = \mathcal{O}(1). \tag{57}$$

Proof: In order to prove (56) we rely on Burkholder-Davis-Gundy inequality

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde\beta_l\Big|^p \le C_p\mathbb{E}\Big(\int_0^{T_0} |X|^2\mathcal{Z}_k^2 d\tau\Big)^{\frac{p}{2}}.$$

Using Lemma 5.1 for some $\kappa_0 < \frac{1}{2}$ together with Hölder inequality, yields

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde\beta_l\Big|^p &\le C_p\mathbb{E}\Big(\frac{\alpha_k^2}{2}\int_0^{T_0} |X|^2 d\tau + \mathcal{O}(\varepsilon^{1-2r-2\kappa_0})\Big)^{\frac{p}{2}}\\ &\le C\varepsilon^{-pr}\Big(1+\varepsilon^{\frac{p}{2}(1-2\kappa_0)}\Big)\\ &\le C\varepsilon^{-pr}.\end{aligned}$$

In order to prove (57) we again use Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\left|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde\beta_j\right|^p \le C_p\mathbb{E}\Big(\int_0^{T_0} \mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big)^{\frac{p}{2}}. \tag{58}$$

Using Lemma 5.1, yields (57). □

To proceed, we first prove a technical lemma on ordinary differential equations.

Lemma 7.3: *Let X and R_δ be continuous functions from $[0,\tau]$ to $\mathcal{N}$ with $X(0) = R_\delta(0)$. If X is a solution of*

$$X(T) = \int_0^T Q_a(X)ds + \int_0^T Q_b(X)ds + R_\delta,$$

where Q_a and Q_b are linear and bounded operators on $\mathcal{N}$ such that

$$|Q_a(X)| \le C_a|X|, \ |Q_b(X)| \le C_b|X|, \tag{59}$$

and

$$\langle Q_b(X), X\rangle \le 0, \tag{60}$$

then

$$\sup_{[0,\tau]} |X|^2 \le [2 + C_0(C_a^2 + C_b^2)] \sup_{[0,\tau]} |R_\delta|^2 , \tag{61}$$

where $C_0 = \frac{1}{C_a+1} e^{2[C_a+1]T_0}$.

We note that later in the application of this lemma the constant C_b might grow with ε while C_a is independent of ε. Therefore, the condition in (60) is important, in order to have no ε-dependence in the exponent.

Proof: Define $Y = X - R_\delta$. Hence

$$\partial_T Y = Q_a(Y) + Q_a(R_\delta) + Q_b(Y) + Q_b(R_\delta).$$

Taking the scalar product $\langle \cdot, Y\rangle$ on both sides, we obtain

$$\frac{1}{2}\partial_T |Y|^2 = \langle Q_a(Y), Y\rangle + \langle Q_b(Y), Y\rangle + \langle Q_a(R_\delta), Y\rangle + \langle Q_b(R_\delta), Y\rangle .$$

Using Cauchy-Schwarz and Young inequalities and (60), yields

$$\partial_T |Y|^2 \le 2[C_a + 1] |Y|^2 + [C_a^2 + C_b^2] |R_\delta|^2 .$$

Applying Gronwall's lemma, yields for all $T \le \tau$

$$\begin{aligned} |Y(T)|^2 &\le [C_a^2 + C_b^2] \int_0^T |R_\delta|^2 e^{2[C_a+1](T-s)} ds \\ &\le C_0[C_a^2 + C_b^2] \sup_{[0,\tau]} |R_\delta|^2 . \end{aligned} \tag{62}$$

To prove (61) we use

$$|X|^2 = |Y + R_\delta|^2 \le 2 |Y|^2 + 2 |R_\delta|^2 ,$$

and (62). □

Let us recall Lemma 6.2 and have a closer look at the various terms of order ε.

Lemma 7.4: *Under Assumptions 3.1, 3.3, 3.4 and 3.5 with all $\alpha_k = \sigma$ for $k \in \{n+1, \dots, N\}$, we obtain*

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau) d\tau + \int_0^T \mathcal{F}_c(a) d\tau + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(a, e_k, e_k) d\tau$$
$$+ \varepsilon M_a(T) + \tilde{R}(T), \tag{63}$$

where $M_a(T)$ is a martingale defined by

$$M_a(T) = \int_0^T \sum_{k=n+1}^{N} \circleddash_k(a) d\tilde{\beta}_k(s) \tag{64}$$

with stochastic integrand

$$\circleddash_k(a) = \frac{3\sigma}{\lambda_k} \mathcal{F}_c(a, a, e_k) + \sum_{l=n+1}^{N} \frac{6\sigma \mathcal{F}_c(a, e_k, e_l)}{\lambda_k + \lambda_l} \mathcal{Z}_l + \sum_{l=n+1}^{N} \frac{3\sigma^3 \mathcal{F}_c(e_k, e_l, e_l)}{\lambda_k(\lambda_k + 2\lambda_l)}$$
$$+ \sum_{l \neq k} \frac{6\sigma \mathcal{F}_c(e_k, e_k, e_l)}{\lambda_l + 2\lambda_k} \mathcal{Z}_k \mathcal{Z}_l + \sum_{l=n+1}^{N} \sum_{j=n+1}^{N} \frac{3\sigma \mathcal{F}_c(e_k, e_l, e_j)}{\lambda_k + \lambda_l + \lambda_j} \mathcal{Z}_l \mathcal{Z}_j \tag{65}$$

and remainder

$$\tilde{R} = R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}),$$

where $R_1 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-2\kappa})$ is defined in (37). All sums are from $n+1$ to N, if it is not explicitly stated otherwise.

Proof: In order to obtain (63) we use (35) and the Lemmas 5.2 and 7.2. □

Lemma 7.5: *Under Assumptions 3.1, 3.3, 3.4 and 3.5 with all $\alpha_k = \sigma$, consider some stochastic process $\xi = \mathcal{O}(\varepsilon^{-r})$ for $r \geq 0$. Then for all $p > 0$ there exists $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{T \in [0, \tau^\sharp]} |M_\xi(T)|^p \Big) \leq C \varepsilon^{-2pr}, \tag{66}$$

where M_ξ is defined in (64).

If ξ is bounded up to time T_0, then (66) holds with T_0 instead of $\tau^\sharp$.

Proof: To prove (66) we take $|\cdot|^p$ and expectation after supremum on both sides of (65) and use Assumptions 3.4, Lemma 7.2, and Burkholder-Davis-Gundy inequality. □

Lemma 7.6: *Under Assumptions 3.1, 3.3, 3.4 and 3.5 with all $\alpha_k = \sigma$. If we define a as $a = a_1 + \varepsilon a_2$ such that a_1 is a solution of the amplitude equation*

$$da_1 = [\mathcal{L}_c a_1 + \mathcal{F}_c(a_1) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(a_1, e_k, e_k)] dT, \tag{67}$$

then a_2 is a solution of

$$da_2 = [\mathcal{L}_c a_2 + 3\mathcal{F}_c(a_1, a_1, a_2) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(a_2, e_k, e_k)]dT + dM_{a_1} + dR_2, \quad (68)$$

where

$$\begin{aligned} R_2 = \varepsilon^{-1}\tilde{R} + 3\varepsilon \int_0^T \mathcal{F}_c(a_1, a_2, a_2)d\tau + \varepsilon^2 \int_0^T \mathcal{F}_c(a_2)d\tau \\ + \varepsilon \sum_{k=n+1}^{N} \frac{6\sigma}{\lambda_k} \int_0^T \mathcal{F}_c(a_1, a_2, e_k)d\tilde{\beta}_k + \varepsilon^2 \sum_{k=n+1}^{N} \frac{3\sigma}{\lambda_k} \int_0^T \mathcal{F}_c(a_2, a_2, e_k)d\tilde{\beta}_k \\ + \varepsilon \sum_{k=n+1}^{N} \sum_{l=1}^{N} \frac{6\sigma}{\lambda_k + \lambda_l} \int_0^T \mathcal{F}_c(a_2, e_k, e_l)\mathcal{Z}_l d\tilde{\beta}_k, \end{aligned} \quad (69)$$

with

$$R_2 = \mathcal{O}(\varepsilon^{1-5\kappa}). \quad (70)$$

Proof: The equation for a_2 is a straightforward calculation using (63) and (67). To bound R_2, we take $\|\cdot\|_\alpha^p$ on both sides of (69) and use Assumption 3.4, Lemma 7.2, Burkholder-Davis-Gundy inequality, and the definition of $\tau^\sharp$ (cf. (55)). □

Lemma 7.7: *Under the assumptions of Lemma 7.6. Let a_1 be a solution of (67) with initial condition $a_1(0)$ such that $|a_1(0)| \le \delta_\varepsilon$. Define ζ in $\mathcal{N}$ as the solution of*

$$d\zeta = [\mathcal{L}_c\zeta + 3\mathcal{F}_c(a_1, a_1, \zeta) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\zeta, e_k, e_k)]dT + dM_{a_1}(T) \quad (71)$$

with $\zeta(0) = 0$.

If $|a_1(0)| \le \delta_\varepsilon$ for some $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, then for all $T_0 > 0$ and $p > 0$ there exist a constant $C > 0$ such that

$$\sup_{T\in[0,T_0]} |a_1(T)|^p \le C\delta_\varepsilon^p, \quad (72)$$

and

$$\sup_{T\in[0,T_0]} |\zeta(T)| \le C(1 + \delta_\varepsilon) \sup_{T\in[0,T_0]} |M_{a_1}(T)|. \quad (73)$$

Proof: The bound on a_1 follows directly from Lemma 6.3.

To bound ζ we define

$$Q_a(\zeta) = \mathcal{L}_c\zeta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\zeta, e_k, e_k) \quad \text{and} \quad Q_b(\zeta) = 3\mathcal{F}_c(a_1, a_1, \zeta) \; .$$

We obtain from Lemma 7.3

$$\sup_{T\in[0,T_0]} |\zeta(T)|^2 \le [2 + C(1 + \delta_\varepsilon^2)] \sup_{T\in[0,T_0]} |M_{a_1}(T)|^2.$$

Taking the square root on both sides, yields (73). □

Remark 7.8 Note that, from now on, we consider $\dim(\mathcal{N}) = 1$ and identify $\mathcal{N}$ with $\mathbb{R}$ using the natural isomorphism $\gamma \cdot e_1 \mapsto \gamma$. Thus for example $\mathcal{F}_c$ is defined as $\langle \mathcal{F}, e_1 \rangle$ and $\mathcal{F}_c^2$ is $\langle \mathcal{F}, e_1 \rangle^2$. Moreover it is easy to check that the quadratic variation of M_{a_1} as a real valued process $\langle M_{a_1}, e_1 \rangle$ is given by $\sum_{k=2}^N \int_0^T \ominus_k^2(a_1) d\tau$.

Before we prove the main result let us deduce the approximation g_k of the quadratic variation function $\ominus_k^2$.

Taking the square for both sides of (65) and using Lemma 5.1, we obtain for some small $\kappa_0 > 0$

$$\int_0^T \ominus_k^2(a_1) d\tau = \int_0^T g_k(a_1) d\tau + \mathcal{O}((1+\delta_\varepsilon^2)\varepsilon^{1-4\kappa_0}), \tag{74}$$

where

$$\begin{aligned} g_k(b_1) = & \frac{9\sigma^2}{\lambda_k^2} [\mathcal{F}_c(b_1, b_1, e_k)]^2 + \theta_1^{(k)} [\mathcal{F}_c(b_1, b_1, e_k)] \\ & + \sum_{l=2}^N \frac{18\sigma^4}{\lambda_l(\lambda_k+\lambda_l)^2} [\mathcal{F}_c(b_1, e_k, e_l)]^2 + \theta_2^{(k)}, \end{aligned} \tag{75}$$

with constants

$$\theta_1^{(k)} = \sum_{l=2}^N \frac{9\sigma^4 \mathcal{F}_c(e_k, e_l, e_l)}{\lambda_k^2 \lambda_l} ,$$

and

$$\begin{aligned} \theta_2^{(k)} = & \frac{11\sigma^6 \mathcal{F}_c^2(e_k)}{4\lambda_k^4} + \sum_{l \neq k}^N \frac{9\sigma^6 (3\lambda_k^2 + 4\lambda_l\lambda_k + 4\lambda_l^2) \mathcal{F}_c^2(e_k, e_l, e_l)}{4\lambda_k^2 \lambda_l (\lambda_k + 2\lambda_l)^2} \\ & + \sum_{l \neq k}^N \frac{9\sigma^6 \mathcal{F}_c^2(e_k, e_k, e_l)}{\lambda_k \lambda_l (\lambda_l + 2\lambda_k)^2} + \sum_{l \neq k}^N \sum_{j \notin \{l,k\}}^N \frac{9\sigma^6 \mathcal{F}_c^2(e_k, e_l, e_j)}{2\lambda_l \lambda_j (\lambda_k + \lambda_l + \lambda_j)^2} \\ & + \sum_{l \neq k}^N \frac{\sigma^6 (6\lambda_k^2 + 18\lambda_l + 3\lambda_k) \mathcal{F}_c(e_k, e_k, e_l) \mathcal{F}_c(e_k)}{2\lambda_l \lambda_k^3 (\lambda_k + 2\lambda_l)} \\ & + \sum_{l \neq k}^N \sum_{j \notin \{l,k\}}^N \frac{9\sigma^6 (4\lambda_l\lambda_j + \lambda_k^2 + \lambda_l\lambda_k) \mathcal{F}_c(e_k, e_l, e_l) \mathcal{F}_c(e_k, e_j, e_j)}{4\lambda_k^2 \lambda_l \lambda_j (\lambda_k + 2\lambda_l)(\lambda_k + 2\lambda_j)} . \end{aligned}$$

Let us state without proof Lemma 6.1 from [3] to bound $M_{a_1}(T) - \tilde{M}_{a_1}(T)$ where the martingale $M_{a_1}(T)$ is defined in (64) and the martingale $\tilde{M}_{a_1}(T)$ is defined in (13).

Lemma 7.9: *Let $M_{a_1}(T)$ be a continuous martingale with respect to some filtration $(\digamma_T)_{T\geq 0}$. Denote the quadratic variation of M_{a_1} by f and let g be an arbitrary $\digamma_T$-adapted increasing process with $g(0) = 0$. Then, there exists an enlarged filtration $\tilde{\digamma}_T$ with $\digamma_T \subset \tilde{\digamma}_T$ and a continuous $\tilde{\digamma}_T$-martingale $\tilde{M}_{a_1}(T)$ with quadratic* *variation g such that, for every $r_0 < \frac{1}{2}$ there exists a constant C with*

$$\mathbb{E} \sup_{T\in[0,T_0]} \Big| M_{a_1}(T) - \tilde{M}_{a_1}(T) \Big|^p \leq C(\mathbb{E}g(T_0)^{2p})^{1/4} \Big(\mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p \Big)^{r_0} + \mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^{p/2}.$$

Remark 7.10 Using the martingale representation theorem, there exists a Brownian motion B with respect to the filtration $\tilde{F}_T$ such that $\tilde{M}_{a_1}(T)$ is given as the stochastic integral in (13).

Lemma 7.11: *Under the conditions of Lemma 7.9, let $M_{a_1}(T)$ and $\tilde{M}_{a_1}(T)$ be martingales defined in (64) and (13), where the Brownian motion is given in Lemma 7.9 and Remark 7.10. Suppose $|a(0)| \leq \delta_\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$.*

Define $f(T) = \int_0^T \sum_{k=2}^N \circleddash_k^2(a_1)ds$ to be the quadratic variation of $M_{b_1}(T)$ and $g(T) = \int_0^T \sum_{k=2}^N g_k(a_1)ds$ to be the quadratic variation of the martingale $\tilde{M}_{a_1}(T)$. Then for $r_0 = \frac{1}{3}$ and $\kappa_0 \leq \kappa$ we obtain

$$\mathbb{E} \sup_{T\in[0,T_0]} \Big| M_{a_1}(T) - \tilde{M}_{a_1}(T) \Big|^p \leq C(1 + \delta_\varepsilon^{\frac{8}{3}p})\varepsilon^{\frac{1}{3}p - \frac{4}{3}p\kappa}. \tag{76}$$

Proof: From (74), we obtain

$$\begin{aligned} \mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p &= \mathbb{E} \sup_{T\in[0,T_0]} \Big| \int_0^T \sum_{k=2}^N [\circleddash_k^2(a_1) - g_k(a_1)]ds \Big|^p \\ &\leq C(1 + \delta_\varepsilon^2)^p \varepsilon^{p-4p\kappa_0} \ . \end{aligned}$$

Secondly, as the $\theta_i^{(k)}$ are constants, we derive

$$\begin{aligned} g(T_0)^{2p} \leq \sup_{T\in[0,T_0]} |g(T)|^{2p} &= \sup_{T\in[0,T_0]} \Big| \int_0^T \sum_{k=2}^N g_k(s)ds \Big|^{2p} \\ &\leq C \sup_{[0,T_0]} |a_1|^{8p} + C \sup_{[0,T_0]} |a_1|^{4p} \ . \end{aligned}$$

Using (72) we obtain

$$\mathbb{E}g(T_0)^{2p} \leq C\delta_\varepsilon^{8p} \ .$$

Applying Lemma 7.9 yields (76). □

Let us now turn to the proof of the main result.

Definition 7.12: Given $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with κ from the stopping time $\tau^\sharp$, we define the set $\Omega^{**} \subset \Omega$ as the set of all $\omega \in \Omega$ such that the following estimates hold:

$$\sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha < \varepsilon^{2-4\kappa}, \tag{77}$$

$$\sup_{[0,\tau^\sharp]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa} \,, \tag{78}$$

$$\sup_{[0,\tau^\sharp]} |R_2| < \varepsilon^{1-6\kappa}, \tag{79}$$

$$\sup_{[0,\tau^\sharp]} |M_{a_1}| < \varepsilon^{-\frac{1}{2}\kappa}, \tag{80}$$

and

$$\sup_{[0,\tau^\sharp]} |M_{a_1} - \tilde{M}_{a_1}| < (1 + \delta_\varepsilon^{\frac{8}{3}})\varepsilon^{\frac{1}{3}-\frac{7}{3}\kappa} \,. \tag{81}$$

We see later that the set Ω^{**} has approximately probability 1 and $\tau^\sharp = T_0$ on Ω^{**}. See the proof of Theorem 3.11, later.

The following theorem states that for solutions of (71) we have a good approximation through solutions of (68) by leaving out the error term R_2. We will take care of the martingale part later. Note that in this argument we could still work with $\dim(\mathcal{N}) \geq 1$.

Theorem 7.13: *We assume that Assumption 3.1, 3.3, 3.4 and 3.5 with all $\alpha_k = \sigma$ hold. Let a_1 be a solution of (67) and let ζ and a_2 are solution of (71) and (68), respectively. If the initial condition satisfies $a_2(0) = \zeta(0) = 0$ and if $\kappa < \frac{1}{7}$, then there is a constant $C > 0$ such that*

$$\sup_{T\in[0,\tau^\sharp]} |a_2(T) - \zeta(T)| \leq C\varepsilon^{1-7\kappa}, \tag{82}$$

and

$$\sup_{T\in[0,\tau^\sharp]} |a_2(T)| \leq C(1 + \delta_\varepsilon)\varepsilon^{-\frac{1}{2}\kappa} \tag{83}$$

on Ω^{**}.

Proof: To prove (82) subtract (68) from (71) and define $\eta := \zeta - a_2$ to obtain

$$d\eta = [\mathcal{L}_c\eta + 3\mathcal{F}_c(a_1, a_1, \eta) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k)]dT + dR_2.$$

If we take

$$Q_a(\eta) = \mathcal{L}_c\eta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k) \quad \text{and} \quad Q_b(\eta) = 3\mathcal{F}_c(a_1, a_1, \eta),$$

then we obtain from Lemma 7.3 using the bound on a_1 given by $\tau^\sharp$

$$\sup_{[0,\tau^\sharp]} |\eta|^2 \leq C\varepsilon^{-2\kappa} \sup_{[0,\tau^\sharp]} |R_2|^2 \quad \text{on} \quad \Omega^{**}. \tag{84}$$

From (79) we obtain

$$\sup_{[0,\tau^\sharp]} |\zeta - a_2| = \sup_{[0,\tau^\sharp]} |\eta| \le C\varepsilon^{1-7\kappa} \quad \text{on} \quad \Omega^{**}.$$

For the second part of the Theorem (cf. (83)), consider

$$\sup_{[0,\tau^\sharp]} |a_2| \le \sup_{[0,\tau^\sharp]} |\zeta - a_2| + \sup_{[0,\tau^\sharp]} |\zeta| \quad \text{on} \quad \Omega^{**}.$$

Using (80) together with (73) and (82), yields (83) for $\kappa < \frac{1}{7}$. □

In the following theorem we approximate the martingale part $\tilde{M}_{a_1}$, that still depends on the fast OU-process. Here we need $n = 1$, as otherwise we can only establish weak convergence of the approximation.

Theorem 7.14: *Under the assumptions of Theorem 7.13 let a_1 be a solution of (67) with $a_1(0) = a(0)$ such that $|a(0)| \le \delta_\varepsilon$, and let ζ be a solution of (71). Define b_2 in $\mathcal{N}$ as a solution of*

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(a_1, a_1, b_2) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{a_1}, \qquad (85)$$

where $\tilde{M}_{a_1}$ is defined in (13). If the initial condition satisfies $\zeta(0) = b_2(0) = 0$, then for every $p > 0$ and $\kappa \in (0, \frac{1}{7})$ from the definition of $\tau^\sharp$ there exists a constant $C > 0$ such

$$\sup_{T\in[0,\tau^\sharp]} |b_2(T) - \zeta(T)| \le C(1 + \delta_\varepsilon^{\frac{11}{3}})\varepsilon^{\frac{1}{3} - \frac{7}{3}\kappa}. \qquad (86)$$

Proof: Subtracting (71) from (85) and defining $\phi = b_2 - \zeta$ we obtain

$$\begin{aligned}\phi(T) = &\int_0^T \mathcal{L}_c \phi d\tau + 3\int_0^T \mathcal{F}_c(\phi, a_1, a_1)d\tau \\ &+ \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\phi, e_k, e_k)]d\tau + \tilde{M}_{a_1}(T) - M_{a_1}(T).\end{aligned}$$

If $\qquad Q_a(\phi) = \mathcal{L}_c\phi + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\phi, e_k, e_k) \qquad$ and $\qquad Q_b(\phi) = 3\mathcal{F}_c(a_1, a_1, \phi),$

then all conditions of Lemma 7.3 are satisfied as follows:

$$|Q_a(\phi)| \le C|\phi|, \; |Q_b(\phi)| \le |a_1|^2|\phi| \le C\delta_\varepsilon^2|\phi| \;\text{ on } \Omega^{**},$$

and from Assumption 3.4

$$\langle Q_b(\phi), \phi\rangle \le 0.$$

Hence, we apply Lemma 7.3 to obtain

$$\sup_{[0,\tau^\sharp]} |\phi|^2 \leq C(1+\delta_\varepsilon^2) \sup_{[0,\tau^\sharp]} |\tilde{M}_{a_1}(T) - M_{a_1}(T)|^2.$$

Using (81) finishes the proof. □

Finally, we use the results previously obtained to prove the main result of Theorem 3.11 for the approximation of the solution of the SPDE (1).

Proof: [Proof of Theorem 3.11] We note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$ we obtain

$$\begin{aligned}\Omega &\supseteq \{\tau^\sharp = T_0\} \\ &\supseteq \Big\{ \sup_{[0,T_0]} \|a_1\|_\alpha < 2\varepsilon^{-\kappa},\ \sup_{[0,T_0]} \|a_2\|_\alpha < \varepsilon^{-\kappa}, \sup_{[0,T_0]} \|\psi\|_\alpha < \varepsilon^{-\kappa} \Big\} \\ &\supseteq \Omega^{**},\end{aligned}$$

where the last inclusion holds due to (78) with Lemma 7.7 and Theorem 7.13. Moreover, $\Omega^* \supset \Omega^{**}$, as $a = a_1 + \varepsilon a_2$ by definition. Hence,

$$\begin{aligned}\mathbb{P}(\Omega^{**}) \geq 1 &- \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha \geq \varepsilon^{2-4\kappa} \Big) - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|\psi\|_\alpha \geq \varepsilon^{-\frac{1}{2}\kappa} \Big) \\ &- \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|R_2\|_\alpha \geq \varepsilon^{1-6\kappa} \Big) - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} |M_{a_1} - \tilde{M}_{a_1}| \geq \varepsilon^{\frac{1}{3}-\frac{29}{3}\kappa} \Big) \\ &- \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} |M_{a_1}| \geq \varepsilon^{-\frac{\kappa}{2}} \Big).\end{aligned}$$

Using Chebychev inequality and Lemmas 4.1, 7.5, 7.7, and 7.11, together with Corollary 4.3, we obtain for sufficiently small κ_0

$$\mathbb{P}(\Omega^{**}) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa}] \geq 1 - C\varepsilon^{\frac{1}{4}q\kappa} \geq 1 - C\varepsilon^p, \tag{87}$$

provided q is sufficiently large. Let us now turn to the approximation result. Using (2) and the triangle inequality, establishes

$$\begin{aligned}&\sup_{[0,\tau^\sharp]} \|u(\varepsilon^{-2}\cdot) - \varepsilon a_1 - \varepsilon^2 b_2 - \varepsilon\mathcal{Q}\|_\alpha \\ &= \sup_{[0,\tau^\sharp]} \|\varepsilon^2 a_2 - \varepsilon^2 b_2 + \varepsilon\psi - \varepsilon\mathcal{Q}\|_\alpha \\ &\leq \varepsilon^2 \sup_{[0,\tau^\sharp]} \|a_2 - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha \\ &\leq \varepsilon^2 \sup_{[0,\tau^\sharp]} \|a_2 - \zeta\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]} \|\zeta - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha \, ,.\end{aligned}$$

From (77), (82) and (86) we obtain

$$\begin{aligned}&\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha \\ &= \sup_{t\in[0,\varepsilon^{-2}\tau^\sharp]} \|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha \\ &\leq C\varepsilon^{\frac{7}{3}-7\kappa} \ \text{ on } \ \Omega^{**}.\end{aligned}$$

Thus

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon a_1(\varepsilon^2 t)-\varepsilon^2 b_2(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > C\varepsilon^{\frac{7}{3}-7\kappa}\Big)\leq 1-\mathbb{P}(\Omega^{**}).$$

Using (87), yields (25). □

7.2 Applications

To illustrate our main theorem, we consider two examples. The first one is the Swift-Hohenberg equation (10) but now with respect to Neumann boundary conditions on the interval $[0,\pi]$. The second one is the Ginzburg-Landau or Allen-Cahn equation (50). For both equations different cases of the noise are discussed.

7.2.1 Swift-Hohenberg equation

For Neumann boundary conditions we consider the orthonormal basis of eigenfunctions

$$e_k(x)=\begin{cases}\frac{1}{\sqrt{\pi}} & \text{if } k=0,\\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k>0.\end{cases}$$

The spaces are given by

$$\mathcal{H}=L^2([0,\pi]) \quad\text{and}\quad \mathcal{N}=\text{span}\{\cos\}\,.$$

In this case our main theorem states that the solution of (10) is

$$u(t,x)\simeq\varepsilon\gamma_1(\varepsilon^2 t)\cos(x)+\varepsilon^2\gamma_2(\varepsilon^2 t)\cos(x)+\varepsilon\sum_{k=2}^{N}\mathcal{Z}_k(\varepsilon^2 t)\cos(kx)+\mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of the amplitude equation given below. We discuss in the following three cases depending on the noise.

First case. If the noise is a constant in space, i.e.

$$W(t)=\sigma\beta_0(t)\ ,$$

then

$$\partial_T\gamma_1=\Big(\nu-\frac{3\sigma^2}{2}\Big)\gamma_1-\frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2=[\Big(\nu-\frac{3\sigma^2}{2}\Big)\gamma_2-\frac{3}{4}\gamma_1^2\gamma_2]dT+\frac{3\sigma^2}{\sqrt{2}}\gamma_1 dB.$$

Second case: If the noise acts on $\cos(kx)$ for only one $k\in\{2,4,5,6,\ldots,N\}$, then

$$\partial_T\gamma_1=\Big(\nu-\frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_1-\frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{2\sqrt{2}(k^2-1)^3}\gamma_1 dB.$$

Third case: If the noise takes the form

$$W(t) = \sigma\beta_3(t)\cos(3x)\ ,$$

then

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma}{256}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{32})}dB.$$

Note that in all three cases γ_1 converges to a constant, and thus the equation for γ_2 contains approximately additive white noise.

7.2.2 Ginzburg-Landau / Allen-Cahn equation

Our main theorem states that the solution of (50) takes the form

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\sin(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\sin(x) + \varepsilon\sum_{k=2}^{N}\mathcal{Z}_k(\varepsilon^2 t)\sin(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of the amplitude equations given below. Again, we discuss three cases depending on the noise.

First case. The noise acts on $\sin(kx)$ for only one $k \in \{2,4,5,\dots,N\}$. In this case

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{2\sqrt{2(k^2-1)^3}}\gamma_1 dB.$$

Second case. The noise acts only on $\sin(3x)$. In this case

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma}{32}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{16})}dB.$$

Third case. The noise is of the form

$$W(t) = \sigma\beta_2(t)e_2 + \sigma\beta_3(t)e_3\ .$$

In this case

$$\partial_T \gamma_1 = \Big(\nu - \frac{11\sigma^2}{32}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3$$

and

$$d\gamma_2 = [\Big(\nu - \frac{11\sigma^2}{32}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + d\tilde{M},$$

where

$$d\tilde{M} = \frac{3\sigma}{32}\left(\gamma_1^4 + \frac{1289\sigma^2}{128}\gamma_1^2 + \frac{89\sigma^4}{147}\right)^{1/2} dB.$$

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