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# MODULATION EQUATION FOR STOCHASTIC SWIFT–HOHENBERG EQUATION*

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**Abstract.** The purpose of this paper is to study the influence of large or unbounded domains on a stochastic PDE near a change of stability, where a band of dominant pattern is changing stability. This leads to a slow modulation of the dominant pattern. Here we consider the stochastic Swift–Hohenberg equation and derive rigorously the Ginzburg–Landau equation as a modulation equation for the amplitudes of the dominating modes. We verify that small global noise has the potential to stabilize the modulation equation, and thus to destroy the dominant pattern.

**Key words.** modulation equation, Swift–Hohenberg equation, stabilization by noise, Ginzburg–Landau equation, additive noise, multiscale analysis

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## 1. Introduction.

We consider the stochastic Swift–Hohenberg equation on an unbounded domain near its change of stability. This equation has been used as a toy model for the convective instability in the Rayleigh–Bénard problem (see [8] or [10]). Now it is one of the celebrated models in the theory of pattern formation. For a scalar field $U(t,x)$ it takes the form

$$\partial_t U = \mathcal{L}U + \varepsilon^2 \nu U - U^3 + \varepsilon\sigma\partial_t\beta, \tag{SH}$$

where the linear differential operator is $\mathcal{L} = -(1+\partial_x^2)^2$ and its eigenvalues are $-\lambda_k = -(1-k^2)^2$ for $k \in \mathbb{R}$ corresponding to eigenfunctions $e^{ikx}$. The noise is the derivative of a standard Brownian motion $\{\beta(t)\}_{t\geq 0}$ in $\mathbb{R}$. In this paper we restrict ourselves to the case of noise constant in space, because on one hand, this is the case where we are able to study the stabilization effect. On the other hand, noise in space and time may lead to spatially unbounded solutions of (SH). So, this result is only the starting point for modulation equations on unbounded domains. The stochastic Swift–Hohenberg model was first studied in the context of amplitude equations with nondegenerate noise in [5] and later in [3].

For (SH) on the whole real line with degenerate additive noise, Hutt and collaborators [11], [12] used a formal argument based on center manifold theory. They showed that noise constant in space leads to a deterministic amplitude equation, which is stabilized by the impact of additive noise. The aim of this paper is to make their results rigorous.

We show that the solution $U(t,x)$ of (SH) is well approximated by

$$U(t,x) \simeq \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + \varepsilon\bar{A}(\varepsilon^2 t, \varepsilon x)e^{-ix} + \varepsilon\mathcal{Z}_\varepsilon(\varepsilon^2 t)\ ,$$

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where the complex amplitude $A(T,X)$ is the solution of the Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + (\nu - \tfrac{3}{2}\sigma^2)A - 3|A|^2 A\,, \tag{GL}$$

and

$$\mathcal{Z}_\varepsilon(T) = \varepsilon^{-1}\sigma \int_0^T e^{-\varepsilon^{-2}(T-\tau)} d\tilde{\beta}(\tau) \tag{1.1}$$

is a fast Ornstein–Uhlenbeck (OU) process with $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ being a rescaled version of the Brownian motion.

In a previous paper [6] we considered a similar but much simpler setting. We studied the stochastic Swift–Hohenberg equation (SH) near its change of stability on bounded domains. While on the unbounded domain we deal with whole bands of eigenvalues, in case of bounded domains only two eigenvalues change sign, and we can rely on Fourier series expansion. The evolution is well approximated by an ODE for the amplitudes of the dominating pattern. With degenerate additive noise (i.e., the noise does not act directly on the dominant modes) we established rigorously an amplitude equation of the form (for a noise constant in the space and periodic boundary conditions)

$$\partial_T a = (\nu - \frac{3}{2}\sigma^2)a - 3|a|^2 a,$$

where $a$ is the complex-valued amplitude of the dominating modes in $\ker \mathcal{L} = \mathrm{span}\{e^{ix}, e^{-ix}\}$. We approximated the solution of (SH) by

$$U(t) = \varepsilon a(\varepsilon^2 t)e^{ix} + c.c. + \varepsilon \mathcal{Z}_\varepsilon(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}),$$

where "*c.c.*" denotes the complex conjugate.

Blömker, Hairer, and Pavliotis [4] considered the stochastic Swift–Hohenberg equation (SH) near its change of stability on a large domain $[-L/\varepsilon, L/\varepsilon]$ with additive noise, where the noise is assumed to be real-valued homogeneous space-time noise. They showed that, under appropriate scaling, its solutions can be approximated by the solution $A$ of the stochastic Ginzburg–Landau equation

$$U(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + c.c.$$

One severe problem is that solutions of stochastic PDEs are not very regular in space and time. They are at most Hölder continuous, and for (SH) we have only one spatial derivative. In [4] the amplitude $A(T)$ was shown to split into a more regular $H^1$-part and a Gaussian.

For the deterministic Swift–Hohenberg equation on an unbounded domain (i.e., for $\sigma = 0$) Kirrmann, Schneider, and Mielke [13] approximated solutions of the Swift–Hohenberg equation via the Ginzburg–Landau equation

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A,$$

but this method of approximation depends on high regularity of the modulation equation, as they needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$, which means one bounded derivative in time and four bounded spatial derivatives. For more results on the deterministic Swift–Hohenberg equation, see, for instance, [7], [15], [16], and [18].

Our method of approximation relies on very low regularity of the modulation equation, which is necessary when turning to spatial noise. Unfortunately, we still need too much regularity for $A$ if we apply full space-time white noise, as we need $A \in C^0([0,T],\mathcal{H}^{1/2+})$, but as a solution of the stochastic Ginzburg–Landau equation $A$ would be at most Hölder continuous with any exponent less than 1/2.

The remainder of this paper is organized as follows. In the next section we define the standard fractional Sobolev space $\mathcal{H}^\alpha$. We also state and prove the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C^0(\mathbb{R})$. In section 3 we give a formal derivation of the modulation equation and state the main result. In section 4 we recall the Green's functions $\mathcal{G}_t(x)$ of the Swift–Hohenberg operator and give estimates on it. In section 5 we bound the OU process $\mathcal{Z}_\varepsilon(T)$. Finally, in section 6 we give the proof of the main result.

**2. The $\mathcal{H}^\alpha$-spaces.** In this section we define the well-known Sobolev space $\mathcal{H}^\alpha$, where we rely on weighted $L^2$-norms of Fourier transforms. We also recall the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C^0(\mathbb{R})$ by stating the Sobolev embedding theorem.

DEFINITION 2.1. *For $\alpha \in \mathbb{R}$ we define the space $\mathcal{H}^\alpha$ by*

$$\mathcal{H}^\alpha = \left\{ u : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} \left(1+y^2\right)^\alpha |\mathcal{F}(u)(y)|^2 dy < \infty \right\},$$

*with norm*

$$\|u\|_\alpha^2 = \int_{-\infty}^{\infty} \left(1+y^2\right)^\alpha |\mathcal{F}(u)(y)|^2\, dy,$$

*where $\mathcal{F}(u)$ is the Fourier transform of $u$, which takes the form*

$$\mathcal{F}(u)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(k) e^{-iky} dk.$$

Note that in the space $\mathcal{H}^\alpha$ functions still decay to 0 at $\infty$. Thus if $A \in \mathcal{H}^\alpha$, we are still in a setting where the solutions of (SH) and the amplitude $A$ decay to 0 for $|x| \to \infty$.

Let us now consider semigroups in the space $\mathcal{H}^\alpha$.

LEMMA 2.2. *Let $\mathcal{A}$ be a nonpositive operator with eigenvalues $P(k)$ such that $P(k) \le 0$ defined by $\mathcal{F}(\mathcal{A}u) = P(\cdot)\mathcal{F}(u)$. Then for $t \ge 0$ and $u \in \mathcal{H}^\alpha$,*

$$\left\| e^{t\mathcal{A}} u \right\|_\alpha \le \|u\|_\alpha. \tag{2.1}$$

It is well known that $e^{t\mathcal{A}}$ defined by $\mathcal{F}(e^{t\mathcal{A}}u) = e^{tP}\mathcal{F}(u)$ generates a contraction semigroup. Nevertheless, we give a proof for completeness of presentation.

*Proof.* We note from Definition 2.1 that (as $e^{-2tP(k)} \le 1$)

$$\left\| e^{t\mathcal{A}} u \right\|_\alpha^2 = \int_{-\infty}^{\infty} \left(1+y^2\right)^\alpha \left| e^{-tP(y)} \mathcal{F}(u)(y) \right|^2 dy \le \|u\|_\alpha^2. \qquad \square$$

The next lemma states the relation between the norm $\|\cdot\|_\alpha$ and the supremum-norm in $C^0(\mathbb{R})$.

LEMMA 2.3. *For $\alpha > \frac{1}{2}$ there is a constant $C > 0$ such that*

$$\|u\|_\infty \le C\|u\|_\alpha \quad \textit{for all } u \in \mathcal{H}^\alpha. \tag{2.2}$$

*Proof.* This follows directly from embedding theorems (see [2] or Theorem 5.4 in [1]). ∎

The following lemma is necessary in order to estimate the nonlinearity.

LEMMA 2.4. *For $\alpha > \frac{1}{2}$ and $m \in \mathbb{N}$ there exist a constant $C > 0$ such that*

$$\|u^m\|_\alpha \leq C\|u\|_\alpha^m \quad \text{for } u \in \mathcal{H}^\alpha. \tag{2.3}$$

*Proof.* See the proof of Theorem 4 in [17]. ∎

**3. Formal derivation and the main result.** In this section let us discuss a formal derivation of the amplitude equation or modulation equation corresponding to Equation (SH). This is based on the approach in [13] and uses high regularity of the amplitude $A$. First we need to define what we mean exactly when we write "order of" or its abbreviation $\mathcal{O}()$.

DEFINITION 3.1. *Let $X_\varepsilon$ with $\varepsilon > 0$ be a family of stochastic processes with paths $X_\varepsilon^\omega \in C^0([0,T_0],\mathcal{H}^\alpha)$, and let $f(\varepsilon)$ be a function of $\varepsilon$. Then $X_\varepsilon$ is of order $f(\varepsilon)$, which we abbreviate by*

$$X_\varepsilon = \mathcal{O}(f(\varepsilon)),$$

*if and only if for every pth moment of $\sup_{t\in[0,T_0]}\|X_\varepsilon(t)\|_\infty$ there is a constant $C_p$ such that the following is valid for all $\varepsilon > 0$:*

$$\mathbb{E}\left(\left|\sup_{t\in[0,T_0]}\|X_\varepsilon(t)\|_\infty\right|^p\right) \leq C_p|f(\varepsilon)|^p.$$

Now let us rescale (SH). If we assume that

$$U(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x),$$

then (SH) takes the form

$$\partial_T u = \mathcal{L}_\varepsilon u + \nu u - u^3 + \varepsilon^{-1}\sigma\partial_T\tilde{\beta}(T), \tag{SH$_\varepsilon$}$$

with differential operator $\mathcal{L}_\varepsilon = -\varepsilon^{-2}(1+\varepsilon^2\partial_X^2)^2$ on the slow time $T = \varepsilon^2 t$ and the "slow" space $X = \varepsilon x$. Now define $w$ via

$$u(T,X) = w(T,X) + \mathcal{Z}_\varepsilon(T), \tag{3.1}$$

where $\mathcal{Z}_\varepsilon$ is as defined in (1.1). Plugging (3.1) into (SH$_\varepsilon$), we obtain

$$\partial_T w = \mathcal{L}_\varepsilon w + \nu w - w^3 - 3w^2\mathcal{Z}_\varepsilon - 3w\mathcal{Z}_\varepsilon^2 + \nu\mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3. \tag{3.2}$$

Leaving out the error term for simplicity of presentation, we make the following ansatz:

$$w_A(T,X) = A(T,X)e^{ix} + \varepsilon^2 B(T,X)e^{2ix} + \varepsilon^2 H(T,X)e^{3ix} + \varepsilon^2 J(T,X) + c.c., \tag{3.3}$$

The higher order terms of order $\mathcal{O}(\varepsilon^2)$ are used to cancel various terms that appear due to the nonlinearity. We assume that all functions are sufficiently smooth.

Plugging (3.3) into (3.2) and using the relation

$$\begin{aligned}\mathcal{L}_\varepsilon\big(f(X)e^{i\frac{n}{\varepsilon}X}\big) = &-[\varepsilon^{-2}(1-n^2)^2 f + 4i\varepsilon^{-1}n(1-n^2)f' \\ &+ (2-6n^2)f'' + 4i\varepsilon n f''' + \varepsilon^2 f''''] \cdot e^{i\frac{n}{\varepsilon}X}\ ,\end{aligned} \tag{3.4}$$

we obtain

$$\begin{aligned}\partial_T A e^{ix} + c.c. =& [4A'' - 4i\varepsilon A'''] e^{ix} - [9B - 24i\varepsilon B'] e^{2ix} - [64H - 96i\varepsilon H'] e^{3ix} \\ & - J + \nu A e^{ix} - A^3 e^{3ix} - 3|A|^2 A e^{ix} - 3|A|^2 \bar{A} e^{-ix} - \bar{A}^3 e^{-3ix} \\ & - 3\mathcal{Z}_\varepsilon (A^2 e^{2ix} + 2|A|^2 + \bar{A}^2 e^{-2ix}) - 3\mathcal{Z}_\varepsilon^2 (A e^{ix} + \bar{A} e^{-ix}) \\ & + c.c. + \nu \mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3 + \mathcal{O}(\varepsilon^2) \,.\end{aligned}$$

Removing all unwanted $\mathcal{O}(1)$-terms by defining

$$B = -\frac{1}{3}\mathcal{Z}_\varepsilon A^2 \,, \quad H = -\frac{1}{64}A^3 \,, \text{ and } \ J = \nu \mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3 - 6\mathcal{Z}_\varepsilon |A|^2 \,, \tag{3.5}$$

we obtain

$$\begin{aligned}\partial_T A e^{ix} + c.c. =& [4A'' - 4i\varepsilon A''' + \nu A - 3|A|^2 A - 3\mathcal{Z}_\varepsilon^2 A] \cdot e^{ix} \\ & + 24i\varepsilon B' e^{2ix} + 96i\varepsilon H' e^{3ix} + c.c. + \mathcal{O}(\varepsilon^2) \,.\end{aligned} \tag{3.6}$$

Before we proceed with this formal derivation, let us state the following two lemmas on the approximation of $\mathcal{Z}_\varepsilon$. In the following we will rely on the important fact that due to averaging we can replace $\mathcal{Z}_\varepsilon^2$ approximately by the constant $\sigma^2/2$. Here we state the result in a way, which will be useful for the mild formulation later.

Lemma 3.2. *For every $\kappa_0 > 0$ and $p > 1$ there is a constant $C > 0$, depending only on $p$, $\sigma$, $\kappa_0$, and $T_0$, such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_\varepsilon(T)|^p \le C\varepsilon^{-\kappa_0} \,,$$

*where the fast OU $\mathcal{Z}_\varepsilon(T)$ is defined as in* (1.1).

Lemma 3.3. *Let $y$ be a complex function with $y = \mathcal{O}(\varepsilon^{-r})$ in $\mathcal{H}^\alpha$ and initial condition $\|y(0)\|_\infty = \mathcal{O}(\varepsilon^{-r})$ for some $r \ge 0$.*

*If $Y(T,s) = e^{4(T-s)\partial_X^2} y(s)$ and $dY(T,s) = e^{4(T-s)\partial_X^2} G(s)ds$ with $G = \mathcal{O}(\varepsilon^{-r})$ in $\mathcal{H}^\alpha$, then for any small $\kappa_0 \in (0,1)$,*

$$\int_0^T Y(T,s) \left\{ \mathcal{Z}_\varepsilon^2 - \frac{\sigma^2}{2} \right\} d\tau = \mathcal{O}(\varepsilon^{1-r-2\kappa_0}). \tag{3.7}$$

Note that $X = \mathcal{O}(\varepsilon^\rho)$ in $\mathcal{H}^\alpha$ if for $p \ge 1$ there is a $C > 0$ such that $\sup_{[0,T_0]} \|X\|_\alpha^p \le C\varepsilon^{p\rho}$. Moreover, $X = \mathcal{O}(\varepsilon^{\rho-})$ if $X = \mathcal{O}(\varepsilon^{\rho-\kappa})$ for all $\kappa > 0$.

These two lemmas on averaging will be proved in section 5.

Now let us complete our formal derivation. Collecting all coefficients in front of $e^{ix}$ in (3.6) yields

$$\partial_T A = 4A'' + \nu A - 3|A|^2 A - 3\mathcal{Z}_\varepsilon^2 A + \mathcal{O}(\varepsilon).$$

Using the averaging result of Lemma 3.3, we obtain

$$\partial_T A = 4\partial_X^2 A + \left(\nu - \frac{3}{2}\sigma^2\right) A - 3|A|^2 A + \mathcal{O}(\varepsilon^{1-}).$$

Neglecting all small terms in $\varepsilon$ yields (GL).

The main result of this paper is the following approximation result for the stochastic Swift–Hohenberg equation (SH) through the Ginzburg–Landau equation (GL).

THEOREM 3.4 (approximation). *Let $U(t,x)$ be a solution of* (SH), *and let $w_A(T,X)$ be the formal approximation defined as*

$$w_A(T,X) = A(T,X)e^{iX\frac{1}{\varepsilon}} + c.c, \tag{3.8}$$

*where $A(T,X)$ is a solution of* (GL) *such that $A \in C^0([0,T_0],\mathcal{H}^\alpha)$ for $\alpha > \frac{1}{2}$. Suppose for the initial condition that $\|U(0) - \varepsilon A(0)e^{ix} - \varepsilon\bar{A}(0)e^{-ix}\|_\infty \le d\varepsilon^{1-3\kappa_0}\phi_\varepsilon$ for some fixed $d > 0$ and for $\kappa_0 \in (0,\frac{1}{8})$ such that $\varepsilon^{-8\kappa_0}\phi_\varepsilon^2 \to 0$ for $\varepsilon \to 0$.*

*Then for each $T_0 > 0$ and $p > 1$ there exists $C > 0$, depending on $\sup_{[0,T_0]}\|A\|_\alpha$, such that*

$$\mathbb{P}\left\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\left\|U(t,x) - \varepsilon w_A(\varepsilon^2 t,\varepsilon x) - \varepsilon\mathcal{Z}_\varepsilon(\varepsilon^2 t)\right\|_\infty > C\varepsilon^{1-4\kappa_0}\phi_\varepsilon\right\} \le C\varepsilon^p, \tag{3.9}$$

*where $\mathcal{Z}_\varepsilon(T)$ is the fast* OU *defined in* (1.1) *and*

$$\phi_\varepsilon^2 = \begin{cases} \varepsilon^2 & \text{if } \alpha > 3/2,\\ \varepsilon^2\ln(1/\varepsilon) & \text{if } \alpha = 3/2,\\ \varepsilon^{2\alpha-1} & \text{if } \alpha < 3/2. \end{cases} \tag{3.10}$$

**4. Green's function and semigroup estimation.** For the first part of this section we follow the ideas of Collet and Eckmann [7] which they apply to a slightly different operator. We define the Green's functions $\mathcal{G}_t(x)$ of the Swift–Hohenberg operator, and we give estimates on it.

DEFINITION 4.1. *Define the Green's function $\mathcal{G}_t(x)$ of the operator $\mathcal{L}$ for $t > 0$ and $x \in \mathbb{R}$ as*

$$\mathcal{G}_t(x) = \int_{-\infty}^{\infty} e^{ikx}e^{-t(1-2k^2+k^4)}dk. \tag{4.1}$$

The next lemma states that the Green's function $\mathcal{G}_t(x)$ is bounded with respect to the norm $\|\cdot\|_{L^1}$.

LEMMA 4.2. *There exists a constant $C > 0$ such that for all $t > 0$,*

$$\|\mathcal{G}_t\|_{L^1} \le C. \tag{4.2}$$

In order to prove this lemma, let us state and prove the following two lemmas.

LEMMA 4.3. *Define the function $g_\tau(y)$ as*

$$g_\tau(y) = \int_{-\infty}^{\infty} e^{imy}e^{-Q_1(m,\tau)}dm,$$

*where $Q_1(m,\tau) = \tau^{-2} - 2m^2 + \tau^2 m^4$. Then there exists a constant $C > 0$ such that for $0 < \tau \le 1$,*

$$\sup_{y\in\mathbb{R}}\left|(4+y^2)g_\tau(y)\right| \le C\ .$$

*Proof.* Using integration by parts, we obtain

$$\begin{aligned}(4+y^2)g_\tau(y) &= \int_0^{\infty} P_1(m,\tau)e^{imy}e^{-Q_1(m,\tau)}dm + \int_{-\infty}^{0} P_1(m,\tau)e^{imy}e^{-Q_1(m,\tau)}dm\\ &:= I_1 + I_2,\end{aligned}$$

where $P_1(m,\tau) = 12m^2\tau^2 - 16m^6\tau^4 + 32m^4\tau^2 - 16m^2$. For $m \geq 0$ and $0 < \tau \leq 1$ we note that

$$Q_1(m,\tau) = (m\tau - 1)^2 \underbrace{(m+\tau^{-1})^2}_{\geq \tau^{-2}} \geq (m-\tau^{-1})^2$$

and

$$P_1(m,\tau) = \tau^2 m^2 [12 - 16(m-\tau^{-1})^2(1+\tau m)^2].$$

Hence,

$$|P_1(m,\tau)| \leq C[1+(\tau m)^4][1+(m-\tau^{-1})^2].$$

Thus,

$$\left|P_1(m+\tau^{-1},\tau)\right| \leq C[1+(\tau m+1)^4][1+m^2] \leq C(1+m^6).$$

Now we bound $I_1$ and $I_2$ separately. For the first integral $I_1$ we obtain

$$\begin{aligned} I_1 &= \int_{-\tau^{-1}}^{\infty} P_1(r+\tau^{-1},\tau) e^{i(r+\tau^{-1})y} e^{-Q_1(r+\tau^{-1},\tau)} dr \\ &\leq \int_{-\tau^{-1}}^{\infty} P_1(r+\tau^{-1},\tau) e^{i(r+\tau^{-1})y} e^{-r^2} dr \ , \end{aligned}$$

where we substituted $r = m - \tau^{-1}$. Thus,

$$|I_1| \leq \int_{-\tau^{-1}}^{\infty} (c+cr^6) e^{-r^2} dr \leq \int_{-\infty}^{\infty} (c+cr^6) e^{-r^2} dr = C.$$

For the second integral $I_2$, we put $-m$ instead of $m$ to obtain

$$I_2 = \int_0^{\infty} P_1(m,\tau) e^{-imy} e^{-Q_1(m,\tau)} dm,$$

where $P_1$ and $Q_1$ are even polynomials in $m$. Analogously to the first integral, we derive

$$|I_2| \leq C.$$

Hence, from the bounds on $I_1$ and $I_2$ we obtain $\sup_{y\in\mathbb{R}} \left|(4+y^2)g_\tau(y)\right| \leq C$ for $0 < \tau \leq 1$. ∎

LEMMA 4.4. *Define the function $h_\eta(y)$ as*

$$h_\eta(y) = \int_{-\infty}^{\infty} e^{iky} e^{-Q_2(k,\eta)} dk,$$

*where $Q_2(k,\eta) = \eta^4 - 2\eta^2 k^2 + k^4$. Then there exists a constant $C > 0$ such that for $0 < \eta < 1$,*

$$\sup_{y\in\mathbb{R}} \left|(1+y^2)h_\eta(y)\right| \leq C \ .$$

*Proof.* Using integration by parts, we obtain

$$(1+y^2)h_\eta(y) = \int_1^\infty P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk + \int_{-\infty}^{-1} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk \\ + \int_{-1}^{1} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk := II_1 + II_2 + II_3 \,,$$

where $P_2(k,\eta) = 1 + 12k^2 - 4\eta^2 - 16k^6 + 32k^4\eta^2 - 16k^2\eta^4$. We note that for $k \geq 1$ and $0 < \eta < 1$,

$$Q_2(k,\eta) = (k-\eta)^2\underbrace{(k+\eta)^2}_{\geq 1} \geq (k-\eta)^2 \quad \text{and} \quad |P_2(k,\tau)| \leq c(1+k^6)\ .$$

We now bound all three terms separately. To bound $II_1$ and $II_2$, we follow exactly the same steps as in the case of Lemma 4.3. For the third term,

$$|II_3| \leq \int_{-1}^{1} |P_2(k,\eta)|\, dk \leq c\int_{-1}^{1}(1+k^6)dk = C.$$

Hence, combining all three estimates on $II_1$, $II_2$, and $II_3$, we obtain for $0 < \eta < 1$ that $\sup_{y\in\mathbb{R}} \big|(1+y^2)h_\eta(y)\big| \leq C$. ∎

*Proof of Lemma* 4.2. In order to prove (4.2), we consider two cases:

*First case.* $t \geq 1$. In this case we note that

$$\mathcal{G}_t(x) = \tau g_\tau(\tau x),$$

with $\tau = t^{-1/2}$ and

$$\|\mathcal{G}_t\|_{L^1} = \int_{-\infty}^{\infty} |\tau g_\tau(\tau x)|\, dx \leq \sup_{y\in\mathbb{R}} \big|(4+y^2)g_\tau(y)\big| \cdot \int_{-\infty}^{\infty} \frac{1}{4+\tilde{y}^2}d\tilde{y}\,,$$

where $y = \tau x$. Using Lemma 4.3, we obtain (4.2) for $t \geq 1$.

*Second case.* $t \in (0,1)$. In this case we note that

$$\mathcal{G}_t(x) = \eta^{-1}h_\eta(\eta^{-1}x),$$

with $\eta = t^{\frac{1}{4}}$ and

$$\|\mathcal{G}_t\|_{L^1} = \int_{-\infty}^{\infty} \frac{1}{1+y^2}\, \big|(1+y^2)h_\eta(y)\big|\, dy \leq \sup_{y\in\mathbb{R}} \big|(1+y^2)h_\eta(y)\big| \int_{-\infty}^{\infty} \frac{1}{1+y^2}dy,$$

where $y = \eta^{-1}x$. Using Lemma 4.4, we obtain (4.2) for $t \in (0,1)$. ∎

LEMMA 4.5. *There exists a constant $C > 0$ such that,*

$$\|e^{t\mathcal{L}}u\|_\infty \leq C\|u\|_\infty \quad \textit{for all } t \geq 0 \textit{ and } u \in C^0(\mathbb{R}). \tag{4.3}$$

*Proof.* As

$$e^{t\mathcal{L}}u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)\mathcal{G}_t(x-y)dy, \tag{4.4}$$

we obtain

$$\|e^{t\mathcal{L}}u\|_\infty \leq C\,\|u\|_\infty\,\|\mathcal{G}_t\|_{L^1}.$$

Using Lemma 4.2, yields (4.3). ∎

COROLLARY 4.6. *There exists a constant $C > 0$ such that*

$$\left\|e^{T\mathcal{L}_\varepsilon}u\right\|_\infty \leq C\|u\|_\infty \quad \text{for all } T \geq 0 \text{ and } u \in C^0(\mathbb{R}).$$

*Proof.* We note that

$$e^{T\mathcal{L}_\varepsilon}u(X) = e^{\varepsilon^{-2}T(1+(\varepsilon\partial_X)^2)^2}u(X) = e^{\varepsilon^{-2}T(1+\partial_X^2)^2}u(\varepsilon X) = e^{t\mathcal{L}}u_\varepsilon(X),$$

where $u_\varepsilon(X) = u(\varepsilon X)$. Using Lemma 4.5, we obtain

$$\left\|e^{T\mathcal{L}_\varepsilon}u\right\|_\infty = \left\|e^{t\mathcal{L}}u_\varepsilon\right\|_\infty \leq C\left\|u_\varepsilon\right\|_\infty = C\|u\|_\infty. \qquad \square$$

The following lemma provides a result on how to change from semigroup $e^{T\mathcal{L}_\varepsilon}$ to $e^{4T\partial_X^2}$ when they are applied to $Ae^{iX\varepsilon^{-1}}$.

LEMMA 4.7. *There is a constant $C > 0$ such that for all $T > 0$ and all $A \in \mathcal{H}^\alpha$ with $\alpha > \frac{1}{2}$,*

$$\sup_{X\in\mathbb{R}}\left|e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}} - (e^{4T\partial_X^2}A)(X)e^{iX\varepsilon^{-1}}\right| \leq C\|A\|_\alpha\phi_\varepsilon,$$

*where $\phi_\varepsilon$ is defined as in* (3.10).

*Proof.* We write $e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}}$ as a convolution with the Green's function of $\mathcal{L}$, as in (4.4),

$$\begin{aligned}
e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(X-y)}e^{-T\varepsilon^{-2}\lambda_{\varepsilon k}}A(y)e^{iy\varepsilon^{-1}}\,dy\,dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k-\varepsilon^{-1})(\varepsilon x-y)}e^{-T\varepsilon^{-2}\lambda_{\varepsilon k}}A(y)dy\,dk\cdot e^{ix} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-T\varepsilon^{-2}\lambda_{\varepsilon k+1}}A(y)dy\,dk\cdot e^{ix},
\end{aligned}$$

where we used the substitution $k \to k + \varepsilon^{-1}$. Hence,

$$e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-T(\varepsilon k^2+2k)^2}A(y)dy\,dk\cdot e^{ix}. \tag{4.5}$$

Analogously, we can write $(e^{4T\partial_X^2}A)(X)\cdot e^{iX\varepsilon^{-1}}$ as

$$(e^{4T\partial_X^2}A)(X)e^{iX\varepsilon^{-1}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-4Tk^2}A(y)dy\,dk\cdot e^{ix}. \tag{4.6}$$

Let

$$\Theta = e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}} - (e^{4T\partial_X^2}A)(X)\cdot e^{iX\varepsilon^{-1}}.$$

Hence,

$$\begin{aligned}
\Theta &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(y)e^{ik(\varepsilon x-y)}\left[e^{-T(\varepsilon k^2+2k)^2} - e^{-4Tk^2}\right]dy\,dk\cdot e^{ix} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(A)(k)\left[e^{-T(\varepsilon k^2+2k)^2} - e^{-4Tk^2}\right]e^{i\varepsilon kx}dk\cdot e^{ix}.
\end{aligned}$$

Using the Cauchy–Schwarz inequality yields

$$|\Theta|^2 \le C\|A\|_\alpha^2 \int_{-\infty}^{\infty} \Psi(k)dk,$$

where

$$\Psi(k) = \frac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[e^{-T(\varepsilon^2k^4+4\varepsilon k^3)} - 1\right]^2.$$

In order to bound $\Theta$ it is sufficient to bound

$$\int_{-\infty}^{\infty} \Psi(k)dk = \int_0^{\frac{1}{2}\varepsilon^{-1}} \Psi(k)dk + \int_{-\frac{1}{2}\varepsilon^{-1}}^{0} \Psi(k)dk + \int_{\frac{1}{2}\varepsilon^{-1}}^{\infty} \Psi(k)dk + \int_{-\infty}^{-\frac{1}{2}\varepsilon^{-1}} \Psi(k)dk$$
$$:= I_1 + I_2 + I_3 + I_4,$$

where we consider all terms separately. For $I_1$, we note that $\varepsilon k^3(\varepsilon k+4)$ is nonnegative for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Thus, we can use the following inequality, which follows directly from the intermediate value theorem:

$$|e^x - 1| \le |x| \max\{1, e^x\}. \tag{4.7}$$

Hence,

$$\begin{aligned} I_1 &\le \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[\varepsilon Tk^3(\varepsilon k+4)\right]^2 dk \\ &\le C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} \left(Tk^2\right)^2 e^{-8Tk^2} dk, \end{aligned}$$

where we used $(\varepsilon k+4) < 5$ for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Now, using the fact

$$\sup_{z>0} \left\{z^m e^{-z}\right\} < \infty \quad \text{for all} \quad m \ge 0, \tag{4.8}$$

we get

$$I_1 \le C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} dk \le C\varepsilon^2 + C\varepsilon^2 \int_1^{\frac{1}{2}\varepsilon^{-1}} k^{2-2\alpha} dk \le C\phi_\varepsilon^2.$$

Let us now turn to $I_2$. Substituting $k = -k$ yields

$$I_2 = \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[e^{\varepsilon Tk^3(4-\varepsilon k)} - 1\right]^2 dk.$$

We note that $\varepsilon k^3(4-\varepsilon k)$ is nonnegative for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Using (4.7) yields

$$\begin{aligned} I_2 &\le \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[4\varepsilon Tk^3 e^{4\varepsilon Tk^3}\right]^2 dk \\ &\le \varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} \left(4Tk^2\right)^2 e^{-4Tk^2} dk, \end{aligned}$$

where we used $\varepsilon k \leq \frac{1}{2}$ for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Now (4.8) implies

$$I_2 \leq C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} dk \leq C\phi_\varepsilon^2.$$

To bound $I_3$, consider

$$\begin{aligned} I_3 &\leq C \int_{\frac{1}{2}\varepsilon^{-1}}^{\infty} \frac{1}{(1+k^2)^\alpha} \left[ e^{-T(\varepsilon k^2+2)^2} + e^{-8Tk^2} \right]^2 dk \\ &\leq C \int_{\frac{1}{2}\varepsilon^{-1}}^{\infty} \frac{1}{(1+k^2)^\alpha} dk \leq C\varepsilon^{2\alpha-1} \quad \text{for } \alpha > \frac{1}{2}. \end{aligned}$$

Analogously for $I_4$,

$$I_4 \leq C\varepsilon^{2\alpha-1} \quad \text{for } \alpha > \frac{1}{2}.$$

Collecting all four results together, we obtain$\|\Theta\|_\infty^2 \leq C\|A\|_\alpha^2 \phi_\varepsilon^2$. ∎

Let us now state a bound for the semigroup $e^{T\mathcal{L}_\varepsilon}$, when applied to $B(X)e^{inX\varepsilon^{-1}}$. The case $n = \pm 1$ was treated in Lemma 4.7.

LEMMA 4.8. *Let $n \in \mathbb{Z} \setminus \{\pm 1\}$ and $\alpha > \frac{1}{2}$. Then there are two constants $C > 0$ and $c_n > 0$, depending on $n$, such that, for $T > 0$ and $B \in \mathcal{H}^\alpha$,*

$$\sup_{X\in\mathbb{R}} \left| e^{T\mathcal{L}_\varepsilon} B(X) e^{inX\varepsilon^{-1}} \right|^2 \leq C\|B\|_\alpha^2 \{ e^{-c_n T\varepsilon^{-2}} + \varepsilon^{2\alpha-1} \}. \tag{4.9}$$

*Proof.* Writing $e^{T\mathcal{L}}B(X)e^{inX\varepsilon^{-1}}$ as a convolution with the Green's function of $\mathcal{L}$ as in Lemma 4.7, we get

$$\begin{aligned} e^{T\mathcal{L}}B(X)e^{inX\varepsilon^{-1}} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(\varepsilon x - y)} e^{-T\varepsilon^{-2}\lambda_{\varepsilon k+n}} B(y) dy\, dk \cdot e^{inx} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(\varepsilon x - y)} e^{-T\varepsilon^{-2}(1-(\varepsilon k+n)^2)^2} B(y) dy\, dk \cdot e^{inX\varepsilon^{-1}}. \end{aligned}$$

Taking the absolute value $|\cdot|$ on both sides and using the Cauchy–Schwarz inequality yields

$$\left| e^{T\mathcal{L}_\varepsilon} B(X) e^{inX\varepsilon^{-1}} \right|^2 \leq C\|B\|_\alpha^2 \int_{-\infty}^{\infty} \frac{1}{(1+k^2)^\alpha} e^{-2T\varepsilon^{-2}(1-(\varepsilon k+n)^2)^2} dk. \tag{4.10}$$

It remains to bound the integral in (4.10):

$$\int_{-\infty}^{\infty} \Phi(k) dk \leq \int_0^{\frac{1}{2\varepsilon}} \Phi(k) dk + \int_{\frac{-1}{2\varepsilon}}^{0} \Phi(k) dk + 2\int_{\frac{1}{2\varepsilon}}^{\infty} \frac{1}{(1+k^2)^\alpha} dk$$

with

$$\Phi(k) = \frac{1}{(1+k^2)^\alpha} e^{-2T\varepsilon^{-2}q(k)} \quad \text{and} \quad q(k) = (1-(\varepsilon k+n)^2)^2.$$

Now, let us bound $q(k)$ on $[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}]$. We consider two cases depending on $n$;

*First case.* $n = 0$. In this case, as $|k| \le \frac{1}{2\varepsilon}$, we have

$$q(k) = (1 - \varepsilon^2 k^2)^2 \ge \frac{9}{16}.$$

*Second case.* $|n| \ge 2$. In this case, as $(\varepsilon k + n)^2 \ge 3/2$, we have

$$q(k) = (1 - (\varepsilon k + n)^2)^2 \ge \frac{1}{4}.$$

From this we deduce that

$$q(k) \ge c_n > 0.$$

Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(k) dk &\le 2 \int_0^{\frac{1}{2\varepsilon}} \frac{1}{(1+k^2)^\alpha} \cdot e^{-c_n T \varepsilon^{-2}} dk + 2 \int_{\frac{1}{2\varepsilon}}^{\infty} \frac{1}{(1+k^2)^\alpha} dk \\ &\le 2 e^{-c_n T \varepsilon^{-2}} \int_0^{\infty} \frac{1}{(1+k^2)^\alpha} dk + 2 \int_{\frac{1}{2\varepsilon}}^{\infty} k^{-2\alpha} dk \\ &\le C e^{-c_n T \varepsilon^{-2}} + C \varepsilon^{2\alpha - 1}. \end{aligned} \tag{4.11}$$

Plugging (4.11) into (4.10) yields (4.9). ∎

**5. General bounds on $\mathcal{Z}_\varepsilon$.** In this section, we prove Lemmas 3.2 and 3.3.

*Proof of Lemma* 3.2. See the first part of the proof of Lemma 14 in [6] with $\lambda_k = 1$. ∎

*Proof of Lemma* 3.3. First, we note from Lemmas 2.2 and 2.3 that

$$\|Y(T,s)\|_\infty^p \le C \sup_{[0,T_0]} \|y\|_\infty^p \le C \sup_{[0,T_0]} \|y\|_\alpha^p \le C \varepsilon^{-pr}.$$

Applying the Itô the formula to $Y \mathcal{Z}_\varepsilon^2$ yields

$$d\left(Y \mathcal{Z}_\varepsilon^2\right) = e^{4(T-s)\partial_X^2} G(s) \mathcal{Z}_\varepsilon^2 ds - 2\varepsilon^{-2} Y \mathcal{Z}_\varepsilon^2 ds + 2\varepsilon^{-1} \sigma \mathcal{Z}_\varepsilon Y d\tilde{\beta} + \varepsilon^{-2} \sigma^2 Y ds.$$

Integrating from 0 to $T$, taking $\|\cdot\|_\infty^p$ norms, and using the triangle inequality yields

$$\begin{aligned} \Big\| \int_0^T Y \left\{ \mathcal{Z}_\varepsilon^2 - \frac{\sigma^2}{2} \right\} ds \Big\|_\infty^p &\le c\varepsilon^{2p} \left\| Y \mathcal{Z}_\varepsilon^2 \right\|_\infty^p + c\varepsilon^{2p} \Big\| \int_0^T e^{4(T-s)\partial_X^2} G(s) \mathcal{Z}_\varepsilon^2 ds \Big\|_\infty^p \\ &\quad + c\varepsilon^p \Big\| \int_0^T Y \mathcal{Z}_\varepsilon d\tilde{\beta}(s) \Big\|_\infty^p \\ &\le C \varepsilon^{2p - pr} \sup_{[0,T_0]} |\mathcal{Z}_\varepsilon|^{2p} + c\varepsilon^p \Big\| \int_0^T Y(T,s) \mathcal{Z}_\varepsilon d\tilde{\beta}(s) \Big\|_\infty^p. \end{aligned}$$

Taking expectation after supremum on both sides, we obtain

$$\mathbb{E} \sup_{[0,T_0]} \Big\| \int_0^T Y \left\{ \mathcal{Z}_\varepsilon^2 - \frac{\sigma^2}{2} \right\} ds \Big\|_\infty^p \le C \varepsilon^{2p - pr - 2\kappa_0} + C \varepsilon^p \mathbb{E} \sup_{[0,T_0]} \Big\| \int_0^T Y(T,s) \mathcal{Z}_\varepsilon d\tilde{\beta}(s) \Big\|_\infty^p. \tag{5.1}$$

In order to obtain (3.7), let us bound the last term on the right-hand side of (5.1). Using Sobolev embedding from Lemma 2.3 yields

$$\mathbb{E} \sup_{[0,T_0]} \Big\| \int_0^T Y(T,s) \mathcal{Z}_\varepsilon(s) d\tilde{\beta}(s) \Big\|_\infty^p \le \mathbb{E} \sup_{[0,T_0]} \Big\| \int_0^T Y(T,s) \mathcal{Z}_\varepsilon(s) d\tilde{\beta}(s) \Big\|_\alpha^p.$$

By a variant of the Burkholder–Davis–Gundy theorem (see, Theorem 1.2.5 in [14] or the paper of Hausenblas and Seidler [9]), we obtain for $p \geq 2$

$$\begin{aligned}\mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T e^{4(T-s)\partial_X^2} y(s)\mathcal{Z}_\varepsilon(s)d\tilde{\beta}(s)\Big\|_\infty^p &\leq C\mathbb{E}\left(\int_0^{T_0}\|y(s)\mathcal{Z}_\varepsilon(s)\|_\alpha^2\,ds\right)^{\frac{p}{2}}\\ &\leq C\mathbb{E}\left(\int_0^{T_0}|\mathcal{Z}_\varepsilon(s)|^2\,\|y(s)\|_\alpha^2\,ds\right)^{\frac{p}{2}}\\ &\leq C\varepsilon^{-pr-\kappa_0}. \qquad \square\end{aligned}$$

As a final result in this section, we prove an averaging result for a mild formulation of (GL).

LEMMA 5.1. *If $A$ is a solution of* (GL) *with* $\sup_{[0,T_0]}\|A\|_\alpha \leq C$, *then*

$$\int_0^T e^{4(T-s)\partial_X^2} A(s)\left\{\mathcal{Z}_\varepsilon^2(s) - \frac{\sigma^2}{2}\right\}ds = \mathcal{O}(\varepsilon^{1-2\kappa_0}) \tag{5.2}$$

*for any* $\kappa_0 > 0$.

*Proof.* Define for $s \in [0,T]$,

$$Y(T,s) = e^{4(T-s)\partial_X^2}A(s),$$

with

$$dY = (-4\partial_X^2)e^{4(T-s)\partial_X^2}A(s)ds + e^{4(T-s)\partial_X^2}dA.$$

Using (GL), we obtain

$$dY = e^{4(T-s)\partial_X^2}\left[(\nu - \frac{3}{2}\sigma^2)A - 3|A|^2A\right]ds = e^{4(T-s)\partial_X^2}G(s)ds.$$

Using Lemmas, 2.2–2.4, we derive

$$\|G\|_\infty \leq C\,\|G\|_\alpha \leq C\|A\|_\alpha + C\|A\|_\alpha^3.$$

Thus

$$\sup_{[0,T_0]}\|G\|_\infty \leq C.$$

Now applying Lemma 3.3 yields (5.2). $\square$

**6. Main results.** In this section, we give the proof of the main result.

DEFINITION 6.1. *Define the residual $\rho(T)$ as*

$$\rho(T) = w_A(T) - e^{T\mathcal{L}_\varepsilon}w_A(0) - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\left[\nu(w_A+\mathcal{Z}_\varepsilon) - (w_A+\mathcal{Z}_\varepsilon)^3\right]ds, \tag{6.1}$$

*where $w_A$ is defined as in* (3.8).

LEMMA 6.2. *If* $\sup_{[0,T_0]}\|A\|_\alpha < \infty$ *for* $\alpha > \frac{1}{2}$, *then for all* $p > 1$ *there is a constant* $C > 0$ *such that*

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\rho(T)\|_\infty^p \leq C\varepsilon^{-3p\kappa_0}\phi_\varepsilon^p, \tag{6.2}$$

*where $\phi_\varepsilon$ is defined as in* (3.10).

*Proof.* From (3.8), we obtain

$$\begin{aligned}
\rho(T) &= A(T)e^{iX\varepsilon^{-1}} - e^{T\mathcal{L}_\varepsilon}A(0)e^{iX\varepsilon^{-1}} - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(\nu A - 3A\mathcal{Z}_\varepsilon^2 - 3|A|^2A)e^{iX\varepsilon^{-1}}ds \\
&\quad + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3iX\varepsilon^{-1}}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds \\
&\quad + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds + c.c. \\
&\quad - \nu\int_0^T e^{-(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon ds + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon^3 ds.
\end{aligned}$$

Using Lemma 4.7, we obtain

$$\begin{aligned}
\rho(T) &= \Big[A(T) - e^{4T\partial_X^2}A(0) - \int_0^T e^{4(T-s)\partial_X^2}(\nu A - 3A\mathcal{Z}_\varepsilon^2 - 3|A|^2A)ds\Big]\cdot e^{iX\varepsilon^{-1}} \\
&\quad + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3iX\varepsilon^{-1}}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds \\
&\quad + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds + c.c. - \nu\int_0^T e^{-\varepsilon^{-2}(T-s)}\mathcal{Z}_\varepsilon ds \\
&\quad + \int_0^T e^{-\varepsilon^{-2}(T-s)}\mathcal{Z}_\varepsilon^3 ds + \mathcal{O}(\varepsilon^{-3\kappa_0}\phi_\varepsilon).
\end{aligned}$$

From (GL) we have

$$\begin{aligned}
\rho(T) &= 3\int_0^T e^{4(T-s)\partial_X^2}A\left(\mathcal{Z}_\varepsilon^2 - \frac{1}{2}\sigma^2\right)ds\cdot e^{iX\varepsilon^{-1}} + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3iX\varepsilon^{-1}}ds \\
&\quad + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2iX\varepsilon^{-1}}ds \\
&\quad + c.c. + \mathcal{O}(\varepsilon^{-3\kappa_0}\phi_\varepsilon).
\end{aligned}$$

We take the norm $\|\cdot\|_\infty^p$ on both sides and use Lemma 4.8 to obtain

$$\begin{aligned}
\|\rho\|_\infty^p &\le C\Big\|\int_0^T e^{4(T-s)\partial_X^2}A\left(\mathcal{Z}_\varepsilon^2 - \frac{1}{2}\sigma^2\right)ds\Big\|_\infty^p \\
&\quad + C\Big(\varepsilon^{2p} + \varepsilon^{p\alpha-\frac{p}{2}}\Big)\cdot\Big[\|A^2\|_\alpha^p + |\mathcal{Z}_\varepsilon|^p\,\|A^2\|_\alpha^p + |\mathcal{Z}_\varepsilon|^p\,\big\||A|^2\big\|_\alpha^p\Big] \\
&\quad + C\varepsilon^{-3p\kappa_0}\phi_\varepsilon^p.
\end{aligned}$$

Taking the expectation value after the supremum and using the bound on $\mathcal{Z}_\varepsilon$ from Lemma 3.2, Lemma 2.4, and the averaging result for mild formulations from Lemma 5.1 yields (6.2). ∎

Definition 6.3. *Define the set $\Omega_0 \subset \Omega$ such that all of the estimates*

$$\sup_{T\in[0,T_0]}|\mathcal{Z}_\varepsilon(T)| < \varepsilon^{-\kappa_0}\,, \tag{6.3}$$

$$\left|\int_0^{T_0}\left\{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\right\}d\tau\right| < \varepsilon^{1-3\kappa_0}\,, \tag{6.4}$$

*and*

$$\sup_{T\in[0,T_0]} \|\rho(T)\|_\infty < \varepsilon^{-4\kappa_0}\phi_\varepsilon$$

*hold on* $\Omega_0$ *for all* $\varepsilon \in (0,1)$.

COROLLARY 6.4. *For all* $p>0$ *there exists a constant* $C_p$ *such that on* $\Omega_0$,

$$\mathbb{P}(\Omega_0) \geq 1 - C_p\varepsilon^p \quad \textit{for all} \quad \varepsilon\in(0,1). \tag{6.5}$$

*Proof.* We note that

$$\mathbb{P}(\Omega_0) \geq 1 - \mathbb{P}\left(\sup_{[0,T_0]} |\mathcal{Z}_\varepsilon(T)| \geq \varepsilon^{-\kappa_0}\right) - \mathbb{P}\left(\int_0^{T_0}\left\{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\right\}d\tau \geq \varepsilon^{1-3\kappa_0}\right)$$
$$- \mathbb{P}\left(\sup_{[0,T_0]}|\rho(T)| \geq \varepsilon^{-4\kappa_0}\phi_\varepsilon\right).$$

Using Chebychev's inequality, we get

$$\mathbb{P}(\Omega_0) \geq 1 - \varepsilon^{q\kappa_0}\mathbb{E}\sup_{[0,T_0]}|\mathcal{Z}_\varepsilon|^q - \varepsilon^{4q\kappa_0}\phi_\varepsilon^q\mathbb{E}\sup_{[0,T_0]}\|\rho\|_\infty^q$$
$$- \varepsilon^{-q+3q\kappa_0}\mathbb{E}\left(\int_0^{T_0}\left\{|\mathcal{Z}_\varepsilon|^2 - \frac{\sigma^2}{2}\right\}d\tau\right)^q.$$

From Lemmas 3.2, 3.3, and 6.2, we obtain

$$\mathbb{P}(\Omega_0) \geq 1 - C_q\varepsilon^{q\kappa_0-\kappa_0} - C_q\varepsilon^{q\kappa_0}.$$

Thus for sufficiently large $q$,

$$\mathbb{P}(\Omega_0) \geq 1 - C_p\varepsilon^p \qquad \text{for all} \quad p>0. \qquad \square$$

Finally, we use the previously obtained results to prove the main assertion of Theorem 3.4 for the approximation of the solution of the SPDE ($\mathrm{SH}_\varepsilon$).

*Proof of Theorem* 3.4. Define

$$R(T) = u(T) - w_A(T) - \mathcal{Z}_\varepsilon(T). \tag{6.6}$$

Considering the mild formulation for ($\mathrm{SH}_\varepsilon$), we obtain

$$u(T) = e^{T\mathcal{L}_\varepsilon}u(0) + \nu\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}u(s)ds - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}u(s)^3ds + \mathcal{Z}_\varepsilon(T). \tag{6.7}$$

Substituting (6.6) into (6.7), we obtain

$$R(T) = e^{T\mathcal{L}_\varepsilon}R(0) + \nu\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}R\,ds - 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon R^2\,ds$$
$$- 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon^2 R\,ds - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}R^3\,ds - 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}w_A^2R\,ds$$
$$- 6\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}w_A\mathcal{Z}_\varepsilon R\,ds - 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}w_AR^2\,ds + \rho(T),$$

where the residual $\rho(T)$ is defined as in (6.1). Taking the norm $\|\cdot\|_\infty$ on both sides and using Corollary 4.6 yields on $\Omega_0$

$$\begin{aligned}\|R(T)\|_\infty \le C\|R(0)\|_\infty &+ C\int_0^T \|R\|_\infty ds + C\int_0^T |\mathcal{Z}_\varepsilon|\|R\|_\infty^2 ds\\ &+ C\int_0^T \left|\mathcal{Z}_\varepsilon^2\right| \|R\|_\infty ds + C\int_0^T \|R\|_\infty^3 ds + C\int_0^T |\mathcal{Z}_\varepsilon|\, \|R\|_\infty ds\\ &+ C\int_0^T \|R\|_\infty^2 ds + C\varepsilon^{-4\kappa_0}\phi_\varepsilon\ ,\end{aligned}$$

where we used $\|w_A\|_\infty \le C$. Define for some $D$ to be fixed later the stopping time $T_\star$ as the largest time, such that $T_\star \le T_0$ and $\|R(T)\|_\infty \le D\varepsilon^{-4\kappa_0}\phi_\varepsilon$ for all $T \le T_\star$. We obtain for $T \le T_\star$ that

$$\begin{aligned}\|R(T)\|_\infty &\le (C\varepsilon^{\kappa_0}d + C)\varepsilon^{-4\kappa_0}\phi_\varepsilon\\ &\qquad + C\left[1 + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + |\mathcal{Z}_\varepsilon^2| + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2 + |\mathcal{Z}_\varepsilon|\right]\int_0^T \|R\|_\infty ds\\ &\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon + C\left[\frac{3}{2} + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + \frac{1}{2}|\mathcal{Z}_\varepsilon|^2 + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2\right]\int_0^T \|R\|_\infty ds\\ &\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon + \int_0^T \left[C_2 + \frac{1}{2}C|\mathcal{Z}_\varepsilon|^2\right]\|R\|_\infty ds\ ,\end{aligned}$$

where $C_1 = Cd + C$ and

$$C[\frac{3}{2} + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2] \le C\left[2 + \frac{3}{2}D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2\right] \le C_2.$$

Note that by assumption on $\kappa_0$, we can choose $C_2$ independent of $D$, provided $\varepsilon > 0$ is sufficiently small. Using Gronwall's inequality, we obtain

$$\begin{aligned}\|R(T)\|_\infty &\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon\left[1 + \int_0^T \left[C_2 + \frac{1}{2}C\,|\mathcal{Z}_\varepsilon|^2\right]\exp\left\{\int_s^T \left[C_2 + \frac{1}{2}C|\mathcal{Z}_\varepsilon|^2\right]dr\right\}ds\right]\\ &\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon\left[1 + \int_0^{T_0}\left[C_2 + \frac{1}{2}C\,|\mathcal{Z}_\varepsilon|^2\right]\exp\left\{C_2T + \frac{1}{2}C\int_0^{T_0}|\mathcal{Z}_\varepsilon|^2\,dr\right\}ds\right].\end{aligned}$$

Taking the supremum over $[0, T_\star]$ yields

$$\sup_{T\in[0,T_\star]}\|R(T)\|_\infty \le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon[1 + \tilde{C}_2] \quad \text{on} \ \ \Omega_0\ , \tag{6.8}$$

where we used (see (6.4))

$$\int_0^{T_0}|\mathcal{Z}_\varepsilon|^2 d\tau \le \varepsilon^{1-3\kappa_0} + \frac{\sigma^2}{2}T_0 \le \tilde{C} \quad \text{on } \Omega_0 \tag{6.9}$$

and defined

$$\tilde{C}_2 = (C_2T_0 + \frac{1}{2}C\tilde{C})e^{(C_2T_0 + \frac{1}{2}C\tilde{C})}\,.$$

Now fix $D > C_1[1 + \tilde{C}_2]$. Hence, (6.8) shows that

$$\sup_{T\in[0,T_\star]} \|R(T)\|_\infty < D\varepsilon^{-4\kappa_0}\phi_\varepsilon\,.$$

Hence, $T_\star = T_0$ and finally,

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|U(t,x) - \varepsilon w_A(\varepsilon^2 t, \varepsilon x) - \varepsilon \mathcal{Z}_\varepsilon(\varepsilon^2 t)\right\|_\infty &\le \varepsilon \sup_{T\in[0,T_0]} \|R(T)\|_\infty \\ &\le C\varepsilon^{1-4\kappa_0}\phi_\varepsilon.\end{aligned}$$

Thus,

$$\mathbb{P}\left\{\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|U(t,x) - \varepsilon w_A(\varepsilon^2 t, \varepsilon x) - \varepsilon \mathcal{Z}_\varepsilon(\varepsilon^2 t)\right\|_\infty > C\varepsilon^{1-4\kappa_0}\phi_\varepsilon\right\} \le 1 - \mathbb{P}(\Omega_0).$$

Using (6.5) yields (3.9). $\square$

## REFERENCES

[1] R. A. Adams *Sobolev Spaces*, Academic Press, New York 1975.
[2] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Springer, Berlin, 1996.
[3] D. Blömker and M. Hairer, *Multiscale expansion of invariant measures for SPDEs*, Comm. Math. Phys., 251 (2004), pp. 515–555.
[4] D. Blömker, M. Hairer, and G. A. Pavliotis, *Modulation equations: Stochastic bifurcation in large domains*, Commun. Math. Phys., 258 (2005), pp. 479–512.
[5] D. Blömker, S. Maier-Paape, and G. Schneider, *The stochastic Landau equation as an amplitude equation*, Discrete Contin. Dyn. Syst. Ser. B, 1 (2001), pp. 527–541.
[6] D. Blömker and W. W. Mohammed, *Amplitude Equations for SPDEs with Cubic Nonlinearities*, Stochastics. An International Journal of Probability and Stochastic Processes iFirst, DOI: 10.1080/17442508.2011.624628, to appear.
[7] P. Collet and J.-P. Eckmann, *The time dependent amplitude equation for the Swift-Hohenberg problem*, Comm. Math. Phys., 132 (1990), pp. 139–153.
[8] M. C. Cross and P. C. Hohenberg, *Pattern formation outside of equilibrium*, Rev. Modern Phys., 65 (1993), pp. 581–1112.
[9] E. Hausenblas and J. Seidler, *A note on maximal inequality for stochastic convolutions*, Czech. Math. J., 51 (2001), pp. 785–790.
[10] P. C. Hohenberg and J. B. Swift, *Effects of additive noise at the onset of Rayleigh-Bénard convection*, Physical Rev. A, 46 (1992), pp. 4773–4785.
[11] A. Hutt, *Additive noise may change the stability of nonlinear systems*, Europhys. Lett., 84 (2008), 34003.
[12] A. Hutt, A. Longtin, and L. Schimansky-Geier, *Additive global noise delays Turing bifurcations*, Physical Rev. Lett., 98 (2007), 230601.
[13] P. Kirrmann, G. Schneider, and A. Mielke, *The validity of modulation equations for extended systems with cubic nonlinearities*, Proc. Roy. Soc. Edinburgh. Sect. A, 122 (1992), pp. 85–91.
[14] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. 135, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[15] A. Mielke and G. Schneider, *Attractors for modulation equations on unbounded domains—existence and comparison*, Nonlinearity, 8 (1995), pp. 743–768.
[16] A. Mielke, G. Schneider, and A. Ziegra, *Comparison of inertial manifolds and application to modulated systems*, Math. Nachr., 214 (2000), pp. 53–69.
[17] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.
[18] G. Schneider, *The validity of generalized Ginzburg-Landau equations*, Math. Methods Appl. Sci., 19 (1996), pp. 717–736.