ASYMPTOTIC COMPACTNESS OF STOCHASTIC COMPLEX GINZBURG-LANDAU EQUATION ON AN UNBOUNDED DOMAINS

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ABSTRACT. The Ginzburg-Landau-type complex equations are simplified mathematical models for various pattern formation systems in mechanics, physics, and chemistry. In this paper, we consider the complex Ginzburg-Landau (CGL) equations on the whole real line perturbed by an additive space-time white noise. Our main result shows that it generates an asymptotically compact stochastic or random dynamical system. This is a crucial property for the existence of a stochastic attractor for such CGL equations. We rely on suitable spaces with weights, due to the regularity properties of space-time white noise, which gives rise to solutions that are unbounded in space.

1. INTRODUCTION

This paper is concerned with attractors of semilinear stochastic partial differential equations (SPDEs) on unbounded domains. This is the topic of many recent papers. See for example the work of Bates, Lu, and Wang [4, 47, 48] or related for systems on unbounded lattices [5, 3], or the work by Brzezniak and Li [8, 7] or [9].

Usually, results on random attractors on unbounded domains assume some decay condition of solutions at infinity, which rules out the possibility of perturbing the equation with translation invariant noise, like space-time white noise. Here one expects the solution to be unbounded at infinity in space. In [8] estimates in L^2 or L^4 force a decay of solutions at infinity, while for example in [4] explicit far field estimates are used, in order to show that the solution decays in space at infinity.

For simplicity of presentation we focus in this article only on the stochastic complex Ginzburg-Landau (CGL) equation, which is an important model equation in the description of spatial pattern formation and of the onset of instabilities in non-equilibrium fluid dynamical systems (see [6, 14]). For the deterministic CGL, a large amount of work has been devoted to the study of the well-posedness of solutions, the global attractors and the related dynamical issues, see references [2, 11, 12] [16]–[27] [31]–[33] [36]–[38] [49, 50] and therein references.

Recently, stochastic complex Ginzburg-Landau equations are extensively used in physics, and frequently appear in the study of dynamical critical phenomena. For instance, for a statistical-mechanical system they may describe the time evolution of order parameter [40]. On bounded domains the well-posedness of solutions and the existence

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of global attractors for SCGL have been investigated in [28, 51]. The existence of invariant measures for stochastic Ginzburg-Landau equations on an unbounded domain has been studied in [42].

Using weighted L^2-spaces (cf. for example also [10]), with a class of weights that allow for algebraic growth at infinity, we are able to show the asymptotic compactness of the stochastic dynamical system in our weighted spaces. But in order to establish this, we need to rely both on nonlinear stability given by a non-linearity with superlinear growth and the regularizing effect of the dominant linear differential operator.

Although our method of proof is quite general, for simplicity of presentation we consider only additive space-time white noise as an example for translation invariant noise. The main assumptions for the noise are verified in Lemma 3.1. We mainly need the pathwise regularity of the stochastic convolution, which is sometimes a delicate question, especially, if the covariance operator of the underlying Wiener process and the differential operator in the equation do not commute.

Also for simplicity of presentation, we focus on stochastic dynamical systems (SDS) instead of random dynamical systems (RDS). Nevertheless our main results on asymptotic compactness (AC) hold for RDS, too.

We will continue the line of research introduced in [30, 39, 41] or for stochastic systems by [13, 8]. The key of the approach, in order to obtain AC of the SDS related to SCGL equations (1.1) (1.2), are energy type estimates. It is well known that the new ingredient necessary for stochastic PDEs is the transformation of the SCGL equation (1.1) into a partial differential equation with random coefficients such that this random equation is independent of initial time τ, but now initial data depends on τ. Our main new feature is the use of weighted L^2-spaces in order to allow for space-time white noise, which was to our knowledge not treated before.

To be more precise in the following, the stochastic complex Ginzburg-Landau (SCGL) equation is given by

$$du - \{a\partial_y^2 u + \chi_0 u - b|u|^{2\sigma} u\}dt = dW(t), \tag{1.1}$$

$$u(y,t)\big|_{t=\tau} = u_0(y). \tag{1.2}$$

Here u is a complex valued function of $(y,t) \in \mathbb{R}\times\mathbb{R}_+$, complex constants $a = a_1 + ia_2$, real constants $a_1 > 0$, $b > 0$, $\sigma > 0$ and $\chi_0 \in \mathbb{R}$, $W(t)$ is a suitable cylindrical Wiener process, defined later on.

In order to be self-contained, we first establish the global well-posedness of the solution for SCGL given by (1.1) (1.2). Therefore, we use weigthed spaces allowing for the growth of solutions at infinity, which is expected for equations driven by space-time white noise.

Our main result proves AC of the SDS. To be more precise, we establish AC for bounded deterministic sets. This means that any of those sets evolves under the random flow asymptotically towards a compact object, at least under the pull-back convergence, which is typical for RDS or SDS. The AC for tempered (i.e., random) sets, which is crucial for the existence of a random attractor, will be studied in a following work. Finally, we comment on the existence of a global stochastic and thus random attractor $\mathcal{A}$, which will be independent of the weight chosen.

Analogously to [13], we are working in the framework of a general SDS on a separable Banach space. For RDS our results are similar to the results of [8], but they did not treat space-time white noise. On the other hand, using their results we could establish the existence of invariant measures based on random ω-limit sets for RDS. But for the existence of a random attractor we would need AC for tempered sets, as for example stated as a general result in [3], where systems on a lattice are studied.

Throughout this paper, different positive constants are all denoted by the same letter C. If necessary, we denote by $C(\cdot,\cdot)$ a constant depending only on the quantities appearing in parenthesis.

The outline of this paper is as follows. In Section 2, we recall some appropriate concepts and tools from the theory of SDS and introduce some notation. In Section 3, we prove the existence of the stochastic flow (and hence of the corresponding SDS) associated with SCGL equations (1.1), (1.2). In Section 4, we show the asymptotic compactness of the SDS corresponding to SCGL and in the final section 5, we comment on global stochastic or random attractors, which are independent of the weight chosen.

2. Notation and Preliminaries

In order to investigate the long time dynamics of SCGL (1.1) under the influence of white noise, we need some appropriate concepts and tools from the theory of *stochastic dynamical systems* (see [1, 13]). For simplicity of presentation, we restrict the results to SDS and attractors for bounded deterministic sets. Nevertheless, the results also hold for attractors of tempered random sets for a RDS.

2.1. **Stochastic Dynamical Systems and Asymptotic Compactness.** Let H be a complete separable metric space and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. We consider a family of mappings $S(t,\tau;\omega):\ H\longrightarrow H$, $(t,\tau)\in\mathbb{R}_\triangle=\{(t,\tau)\big|-\infty<\tau\le t<\infty\}$, parameterized by $\omega\in\Omega$, satisfying for $\mathbb{P}$-a.e. ω the following properties:

$$\begin{aligned}(i)\ \ & S(t,\tau;\omega)x=S(t,r;\omega)\circ S(r,\tau;\omega)x \quad \text{and}\\ & S(\tau,\tau;\omega)x=x \quad \text{for all} \quad \tau\le r\le t \ \text{ and } \ x\in H,\\ (ii)\ \ & S \ \text{ is } \big(\mathcal{B}(\mathbb{R}_\triangle)\otimes\mathcal{F}\otimes\mathcal{B}(H),\mathcal{B}(H)\big) - \text{measurable.}\end{aligned}$$

A SDS is said to be continuous, if and only if $S(t,\tau;\omega)$ is continuous in H for all $(t,\tau)\in\mathbb{R}_\triangle$ and for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Below we state the concept of a stochastic (global) pullback attractor of bounded deterministic sets for an SDS (see, e.g., [1, 13] and the references therein), which extends the corresponding definition of a global universal attractor in autonomous systems (cf. [29, 43], for example).

Following [13], we define:

Definition 2.1. *Given $t\in\mathbb{R}$ and $\omega\in\Omega$, we say that the bounded set $K(t,\omega)\subset H$ is an* attracting set *if for all bounded sets $B\subset H$,*

$$d\big(S(t,\tau;\omega)B,K(t,\omega)\big)\to 0 \quad \text{as} \quad \tau\to-\infty.$$

Definition 2.2. *Let $A(t,\omega)$ be a nonempty compact set for all $t\in\mathbb{R}$. This set is called a* stochastic pullback attractor *if it attracts all bounded sets from $-\infty$, and it is the minimal closed set with this property. Moreover, it is invariant in the sense that*

$$S(t,\tau;\omega)A(\tau,\omega)=A(t,\omega), \quad \forall\tau\le t.$$

The first result on existence of stochastic attractors is given by the following theorem from [13].

Theorem 2.3. *Assume that the SDS has a compact attracting set K. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the SDS exhibits a stochastic pullback attractor.*

Similar results hold true for random attractors of RDS that are AC and exhibit a bounded (or tempered) absorbing set. Let us remark, that absorbing sets are a special type of an attracting set. But as our approach is based on energy estimates, both properties (absorbing and attracting) are established in a quite similar way.

We will discuss the following two types of AC, in contrast to [13] where the existence of a compact absorbing set defines AC.

Definition 2.4. *An SDS is called* AC for bounded (deterministic) sets*, if for all bounded sequences $\{u_n\}_{n\in\mathbb{N}} \subset H$ and all families of times $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \to -\infty$ for $n \to \infty$, the family $\{S(t,\tau_n,\omega)u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.*

Definition 2.5. *An SDS is* AC for (random) tempered sets*, if for all families of times $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \to -\infty$ for $n \to \infty$ and all sequences $\{u_n\}_{n\in\mathbb{N}}$ in H such that $\lim_{n\to\infty} e^{\epsilon\tau_n}u_n = 0$ for all $\epsilon > 0$ one has that the family $\{S(t,\tau_n,\omega)u_n\}_{n\in\mathbb{N}}$ exhibits a convergent subsequence.*

In our main result we verify that SCGL is AC for bounded sets. As our attracting set is not uniformly bounded in time, we would need AC for tempered sets, in order to conclude the existence of a stochastic or random attractor. This is postponed to a future work.

2.2. **Function Spaces.** As the phase space H for our equation, we consider for some fixed $\rho > \frac{1}{2}$ the scale of weighted Hilbert spaces given by

$$L^2_\mu = L^2(\mathbb{R}, d\mu) \quad \text{with respect to the measure } d\mu = m(y)dy, \text{ where } m(y) = (1+|y|^2)^{-\rho}\ .$$

The spaces $L^p_\mu = L^p(\mathbb{R}, d\mu)$ are defined analogously. This will allow for an algebraic growth of solutions in space, but we aim at growth as slow as possible.

The inner product on L^2_μ is denoted by:

$$(f,g)_\mu = \int_{\mathbb{R}} f\bar{g}d\mu,$$

The corresponding Sobolev spaces $H^m(\mathbb{R}, d\mu)$ are denoted by H^m_μ for all $m \geq 1$.

We will see later (cf. Lemma 3.1) that the condition $\rho > \frac{1}{2}$ is necessary, as only then we can expect the solution to be in L^2_μ.

Let us finally remark, that it is easy to check, that L^p_μ is continuously embedded into L^2_μ for $p > 2$, as $\mu(\mathbb{R}) < \infty$ for $\rho > \frac{1}{2}$.

2.3. **Noise.** For simplicity, in this article we consider only space time-white noise for the random perturbation. This is given by a standard cylindrical Wiener process W on $L^2(\mathbb{R})$. To be more precise, consider some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and define

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j\ ,$$

where $\{e_j\}$ is any orthonormal basis of the standard $L^2(\mathbb{R})$ and $\{\beta_j\}_{j\in\mathbb{N}}$ is a sequence of mutually independent real valued two-sided Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us remark, that $\{m^{-1/2}e_j\}$ is an orthonormal basis in L^2_μ, and thus we can consider W as a cylindrical Wiener process in L^2_μ, too. But on that space, we cannot expect the covariance operator to be the identity.

3. A Priori Estimates, Existence and Uniqueness of Solution

For the existence of global solution in time and for results on AC, in this section a priori estimates are established. We introduce the stationary Ornstein-Uhlenbeck (OU) process, in order to transform the SCGL (1.1) into a random partial differential equation. Then a priori estimates can be obtained by standard methods.

3.1. **Stochastic Convolution.** The OU-process η is defined as

$$\begin{aligned}\eta(t,\omega) &:= \Big(\int_{-\infty}^t e^{(t-s)(a\partial_y^2-k)}dW(s)\Big)(\omega)\\ &:= \Big(\int_{-\infty}^t S(t-s)dW(s)\Big)(\omega)\ ,\end{aligned}\tag{3.1}$$

where W is our Wiener process defined in Section 1. The process η is the stationary solution of the following linear stochastic partial differential equation

$$\frac{\partial\eta}{\partial t} = \{a\partial_y^2 - k\}\eta + \dot{W},\tag{3.2}$$

with some *control* parameter $k > 0$. This is important, as otherwise the existence of a stationary solution is not obvious.

In the following, we also use a different representation given by a two-sided Brownian Sheet $\mathcal{B}(t,x)$, $t > 0$, $x \in \mathbb{R}$. From integration with respect to $\mathcal{B}$ (cf. Walsh [46]) we obtain in law

$$\eta(t) = \int_{-\infty}^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathcal{B}(ds,dy)\ ,\tag{3.3}$$

where

$$G_t(x) = \frac{1}{2\sqrt{\pi|a|t}}e^{-\frac{x^2}{4at}-kt}$$

is the Greens-functions corresponding to the semigroup S, such that $[S(t)f](x) = \int_{\mathbb{R}} G_t(x-y)f(y)dy$. Now using Gaussianity of η and Itô-formula (see for example [20, 46]) we obtain:

$$\begin{aligned}\mathbb{E}\|\eta(t)\|^p_{L^p_\mu} &= \int_{\mathbb{R}} \mathbb{E}\Big[\int_{-\infty}^t\int_{\mathbb{R}} G_{t-s}(x-y)\mathcal{B}(ds,dy)\Big]^p m(x)dx\\ &\le \int_{\mathbb{R}} \Big[\mathbb{E}\Big(\int_{-\infty}^t\int_{\mathbb{R}} G_{t-s}(x-y)\mathcal{B}(ds,dy)\Big)^2\Big]^{p/2} m(x)dx\\ &\le \int_{\mathbb{R}} \Big[\int_{-\infty}^t\int_{\mathbb{R}} G_{t-s}(x-y)^2dy\ ds\Big]^{p/2} m(x)dx\\ &= \|m\|_{L^1}\|G\|_{L^2([0,\infty)\times\mathbb{R})}\\ &< \infty\ ,\end{aligned}\tag{3.4}$$

as

$$\begin{aligned}\|G\|^2_{L^2([0,\infty)\times\mathbb{R})} &= \int_0^\infty \int_{\mathbb{R}} \frac{1}{4\pi|a|t} e^{-\frac{a_1 y^2}{2|a|^2 t}-2kt} dy\, dt \\ &= \int_0^\infty \frac{\sqrt{2}}{4\sqrt{\pi a_1 t}} e^{-2kt} dt = C.\end{aligned}$$

Now it is easy to check, that for all a, b, and T

$$\mathbb{E}\int_a^b \|\eta(t)\|^p_{L^p_\mu} dt < \infty \quad \text{and} \quad \mathbb{E}\int_{-\infty}^{T+1} e^{t-T}\|\eta(t)\|^p_{L^p_\mu} dt < \infty\ .$$

Moreover, from Kolmogorov continuity criterion theorem or Theorem 5.16 in [9, p. 134], we find that additionally $\eta(\cdot,\omega) \in C_{\text{loc}}(\mathbb{R}; L^p_\mu)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We proved the following Lemma, which is the main and only regularity assumption on η, that we will use in the following sections.

Lemma 3.1. *For the OU-process η defined in (3.1), for all $p \geq 1$, $T \in \mathbb{R}$, and all bounded $I \subset \mathbb{R}$*

$$\begin{aligned}&\mathbb{P}(\eta \in L^p(I, L^p_\mu) \cap C^0_{loc}(\mathbb{R}, L^p_\mu)) = 1 \\ &\mathbb{P}\Big(\int_{-\infty}^{T+1} e^{t-T}\|\eta(t)\|^p_{L^p_\mu} dt < \infty\Big) = 1\end{aligned} \tag{3.5}$$

3.2. **Energy-type estimates.** We introduce the new variable

$$\tilde{u}(t) := u(t) - \eta(t,\omega), \tag{3.6}$$

where the stationary process η solves the problem (3.2). Now from the standard method relying on the mild formulation, we obtain the following random partial differential equation for the new variable $\tilde{u}$:

$$\partial_t \tilde{u} - a\Delta\tilde{u} - \chi_0\tilde{u} + b|\tilde{u}+\eta|^{2\sigma}(\tilde{u}+\eta) = (k+\chi_0)\eta.$$

We consider the equation path-wise and treat η as a known process. For convenience, we drop the tilde and rewrite the equation above. Thus, we finally get

$$\partial_t u - a\Delta u - \chi_0 u + b|u+\eta|^{2\sigma}(u+\eta) = (k+\chi_0)\eta \tag{3.7}$$

with initial data

$$u(t)\big|_{t=\tau} = u_0 - \eta(\tau,\omega) \in L^2_\mu, \tag{3.8}$$

where η is the stationary solution to (3.2).

Now we work on this random partial differential equations.

Lemma 3.2 (Energy Estimates)**.** *Let $u_0 \in L^2_\mu$ and $\sigma > 0$. Then for all sufficiently smooth solutions u of (3.7) and* (3.8)*, the following estimate is valid:*

$$\begin{aligned}&\|u(T)\|^2_{L^2_\mu} + \int_T^{T+1}\Big(\|\nabla u(t)\|^2_{L^2_\mu} + \|u(t)\|^{\sigma_1}_{L^{\sigma_1}_\mu}\Big)dt \\ \leq &Ce^{\tau-T}\|u_0-\eta(\tau)\|^2_{L^2_\mu} + C\int_\tau^{T+1} e^{t-T}\{1+\|\eta(t)\|^2_{L^2_\mu} + \|\eta(t)\|^{\sigma_1}_{L^{\sigma_1}_\mu}\}dt, \quad \forall\, T \geq \tau,\end{aligned} \tag{3.9}$$

$$\|\partial_t u(t)\|_{H'(\tau,T)} \leq r_0(T,\tau,u_0,\eta), \qquad \forall\, T \geq \tau, \tag{3.10}$$

where the constant r_0 can be calculated explicitly, and the constant C is independent of T and τ, $\sigma_1 = 2\sigma+2$, and $H'(\tau,T)$ is the dual space of $H(\tau,T) = L^2(\tau,T;H^1_\mu)\cap L^{2\sigma+2}(\tau,T;L^{2\sigma+2}_\mu)$.

Proof. Multiplying (3.7) by $(1+|y|^2)^{-\rho}\bar u$, integrating by parts and considering the real part in the resulting identity, we get that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_\mu} + a_1\|\nabla u(t)\|^2_{L^2_\mu} - \chi_0\|u(t)\|^2_{L^2_\mu} + b\|u+\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\\ =&\mathrm{Re}\int_{\mathbb{R}}\big(2a\rho(1+|y|^2)^{-1}\bar u y\cdot\nabla u + b|u+\eta|^{2\sigma}(u+\eta)\bar\eta + (k+\chi_0)\eta\bar u\big)d\mu.\end{aligned}\tag{3.11}$$

By using Hölder's inequality, we easily prove the following inequalities

$$\big|(k+\chi_0)\int \eta\bar u d\mu\big| \le \frac{1}{2}\|u\|^2_{L^2_\mu} + C\|\eta\|^2_{L^2_\mu},\tag{3.12}$$

$$\big|b\int |u+\eta|^{2\sigma}(u+\eta)\bar\eta d\mu\big| \le \frac{1}{4}b\|u+\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu} + C\|\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu},\tag{3.13}$$

$$\big|\int 2a\rho(1+|y|^2)^{-1}\bar u y\cdot\nabla u d\mu\big| \le \frac{1}{2}a_1\|\nabla u\|^2_{L^2_\mu} + \frac{2|a\rho|^2}{a_1}\|u\|^2_{L^2_\mu}.\tag{3.14}$$

Taking $M \gg 1$ sufficiently big, we obtain for $|u|\ge M$ that

$$(\chi_0 + 1 + 2|a\rho|^2/a_1)|u|^2 \le (|\chi_0| + 1 + 2|a\rho|^2/a_1)M^{-2\sigma}|u|^{2\sigma+2}\ .$$

Since $\rho > 1/2$, we thus derive

$$\begin{aligned}&(\chi_0 + 1 + 2|a\rho|^2/a_1)\|u\|^2_{L^2_\mu}\\ &\le(|\chi_0| + 1 + 2|a\rho|^2/a_1)\Big\{\int_{|u|\ge M}|u|^2 d\mu + \int_{|u|\le M}|u|^2 d\mu\Big\}\\ &\le\frac{1}{4}b\|u+\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu} + C\|\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu} + C.\end{aligned}\tag{3.15}$$

Putting together (3.11–3.15), we have

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|^2_{L^2_\mu} + \|u(t)\|^2_{L^2_\mu} + a_1\|\nabla u(t)\|^2_{L^2_\mu} + b\|u+\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\\ &\qquad\le C(1+\|\eta\|^2_{L^2_\mu} + \|\eta\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}).\end{aligned}\tag{3.16}$$

Applying Gronwall inequality to (3.16), we obtain for all $T\ge\tau$

$$\begin{aligned}&\|u(T)\|^2_{L^2_\mu} + \int_\tau^T e^{t-T}\|\nabla u(t)\|^2_{L^2_\mu} + \|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}dt\\ &\le Ce^{\tau-T}\|u_0-\eta(\tau)\|^2_{L^2_\mu} + C\int_\tau^T e^{t-T}\{1+\|\eta(t)\|^2_{L^2_\mu} + \|\eta(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\}dt.\end{aligned}\tag{3.17}$$

By (3.16) and (3.17), we obtain

$$\begin{aligned}&\int_T^{T+1}\Big(\|\nabla u(t)\|^2_{L^2_\mu} + \|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\Big)dt\\ &\le Ce^{\tau-T}\|u_0-\eta(\tau)\|^2_{L^2_\mu} + C\int_\tau^{T+1} e^{t-T}\{1+\|\eta(t)\|^2_{L^2_\mu} + \|\eta(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\}dt.\end{aligned}\tag{3.18}$$

Now (3.9)is obtained from (3.17) and (3.18) and the first part of the Lemma is proved.

For any $T \geq \tau$ and $v(y,t) \in L^2(T,T+1;H^1_\mu) \cap L^{2\sigma+2}(T,T+1;L^{2\sigma+2}_\mu)$, multiplying (3.7) by $(1+|y|^2)^{-\rho}\bar{v}$ and integrating by parts, we get that

$$\begin{aligned}
&\int_T^{T+1} \big(\partial_t u(\cdot,t), v(\cdot,t)\big)_\mu dt + a\int_T^{T+1} \big(\nabla u(\cdot,t), \nabla v(\cdot,t)\big)_\mu dt \\
&\quad =\chi_0 \int_T^{T+1} \big(u+\eta, v(\cdot,t)\big)_\mu dt - b\int_T^{T+1} \big(|u+\eta|^{2\sigma}(u+\eta), v(\cdot,t)\big)_\mu dt \\
&\qquad + 2a\rho\int_T^{T+1} \big((1+|y|^2)^{-1} y\cdot\nabla u, v(\cdot,t)\big)_\mu dt + k\int_T^{T+1} \big(\eta, v(\cdot,t)\big)_\mu dt.
\end{aligned} \tag{3.19}$$

Using Hölder inequality, we have

$$\Big|a\int_T^{T+1} \big(\nabla u(\cdot,t), \nabla v(\cdot,t)\big)_\mu dt\Big| \leq C\|\nabla u\|_{L^2(T,T+1;L^2_\mu)}\|\nabla v\|_{L^2(T,T+1;L^2_\mu)}, \tag{3.20}$$

$$\Big|\chi_0\int_T^{T+1} \big(u+\eta, v(\cdot,t)\big)_\mu dt\Big| \leq C\|u+\eta\|_{L^2(T,T+1;L^2_\mu)}\|v\|_{L^2(T,T+1;L^2_\mu)}, \tag{3.21}$$

$$\begin{aligned}
&\Big|b\int_T^{T+1} \big(|u+\eta|^{2\sigma}(u+\eta), v(\cdot,t)\big)_\mu dt\Big| \\
&\leq C\|u+\eta\|^{2\sigma+1}_{L^{2\sigma+2}(T,T+1;L^{2\sigma+2}_\mu)}\|v\|_{L^{2\sigma+2}(T,T+1;L^{2\sigma+2}_\mu)},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\Big|2a\rho\int_T^{T+1} \big((1+|y|^2)^{-1} y\cdot\nabla u, v(\cdot,t)\big)_\mu dt\Big| \\
&\leq C\|\nabla u\|_{L^2(T,T+1;L^2_\mu)}\|v\|_{L^2(T,T+1;L^2_\mu)},
\end{aligned} \tag{3.23}$$

$$\Big|k\int_T^{T+1} \big(\eta, v(\cdot,t)\big)_\mu dt\Big| \leq C\|\eta\|_{L^2(T,T+1;L^2_\mu)}\|v\|_{L^2(T,T+1;L^2_\mu)}. \tag{3.24}$$

Plugging (3.17), (3.18) and (3.20)– (3.24) into (3.19) yields

$$\Big|\int_T^{T+1} \big(\partial_t u(\cdot,t), v(\cdot,t)\big)_\mu dt\Big| \leq C\|v\|_{L^2(T,T+1;H^1_{\mu_1})} + C\|v\|_{L^{2\sigma+2}(T,T+1;L^{2\sigma+2}_\mu)}. \tag{3.25}$$

Finally, (3.10) is obtained from (3.25) and this lemma is proved. □

3.3. **Existence and Uniqueness.** The following Theorem is verified by standard methods, but as some of the estimates are used later, we state its proof for completeness.

Theorem 3.3. (Existence and Uniqueness) *Let $u_0 \in L^2_\mu$ and $\sigma > 0$. Then equations* (3.7) *and* (3.8) *possesses a unique solution $u(t) = u\big(t,\tau,\omega,u_0-\eta(\tau,\omega)\big)$ such that $u\big(\tau,\tau,\omega,u_0-\eta(\tau,\omega)\big) = u_0 - \eta(\tau,\omega)$, $u(t) \in C_{loc}([\tau,\infty);L^2_\mu) \cap L^2_{loc}([\tau,\infty);H^1_\mu) \cap L^{2\sigma+2}_{loc}([\tau,\infty);L^{2\sigma+2}_\mu)$ and $\partial_t u \in H'(\tau,T)$ for any $T \geq \tau$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ and the energy estimates (3.9) and (3.10) hold.*

Moreover, for every two solution $u_1(t)$ and $u_2(t)$ of equations (3.7) *and* (3.8)*, the following estimate holds:*

$$\|u_1(t)-u_2(t)\|^2_{L^2_\mu} \leq e^{K_0 t}\|u_1(\tau)-u_2(\tau)\|^2_{L^2_\mu}, \quad \forall\, t \geq \tau, \tag{3.26}$$

where $K_0 = 2|\chi_0| + |a|^2\rho^2/a_1$.

Proof. (I) *Existence of Solution.* For the proof we rely on the Galerkin method and a compactness argument, which extracts a convergent subsequence from the Galerkin approximations. This is used to prove path-wise the existence of solution of equations (3.7) and (3.8). Let us remark that, as we rely on subsequences, the measurability w.r.t. ω of the solutions follow from the uniqueness of solutions.

Let $\{e_j\}_{j=1}^{\infty} \subset H^2_\mu \cap L^2_\mu$ be the orthonormal basis of L^2_μ, fixed in the definition of W. We denote the Galerkin approximation by

$$u_N = \sum_{j=1}^{N} \alpha_j(t) e_j, \qquad u_{0N}(\tau) := P_N(u_0 - \eta(t)) = \sum_{j=1}^{N} (u_0 - \eta(\tau), e_j)\, e_j \quad (\forall N \geq 1)$$

where the orthogonal projection onto the Galerkin space is denoted by

$$P_N : \ L^2_\mu \longrightarrow \ \mathrm{span}\{e_1, \cdots, e_N\}.$$

It is obvious that $\lim_{N\to\infty} \|u_{0N} - \{u_0 - \eta(\tau)\}\|_{L^2_\mu} = 0$ and $P_N u_N = u_N$. Moreover, $\|P_N v\|_{L^2_\mu} \leq \|v\|_{L^2_\mu}$ for all v.

Define now for all $N \geq 1$ the Galerkin approximation as the solution of the following problem:

$$\partial_t u_N - P_N\{a\Delta u_N + \chi_0 u_N - b|u_N + \eta|^{2\sigma}(u_N + \eta) + (k + \chi_0)\eta\} = 0, \tag{3.27}$$

$$u_N\big|_{t=\tau} = u_{0N}(\tau)\ . \tag{3.28}$$

As this is basically an ODE in $\mathbb{R}^N$ with polynomial nonlinearity, it is straightforward to establish the existence and uniqueness of solutions.

Following the proof of Lemma 3.2, we derive in a completely analogous way

(3.29)
$$\begin{aligned}
&\|u_N(T)\|^2_{L^2_\mu} + \int_\tau^T \Big(\|u_N(t)\|^2_{H^1_\mu} + \|u_N(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\Big)dt \\
\leq& CTe^{\tau-T}\|u_0 - \eta(\tau)\|^2_{L^2_\mu} + CT\int_\tau^T e^{t-T}\big(1 + \|\eta(t)\|^2_{L^2_\mu} + \|\eta(t)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}\big)dt, \ \ \forall T \geq \tau,
\end{aligned}$$

and

$$\|\partial_t u_N(t)\|_{H'(\tau,T)} \leq r_0(T, \tau, u_0, \eta), \quad \forall T \geq \tau, \tag{3.30}$$

where C and r_0 are independent of N and given in Lemma 3.2. Define the space

$$\mathcal{W} = C([\tau, T]; L^2_\mu) \cap L^2(\tau, T; H^1_\mu) \cap L^{2\sigma+2}(\tau, T; L^{2\sigma+2}_\mu).$$

Thus there exists a subsequence of the sequence u_N, which we also denote by u_N for simplicity, such that u_N converges to u in $\mathcal{W}$ star-weakly and $\partial_t u_N$ converges to $\partial_t u$ in $H'(\tau, T)$ weakly (cf. [44]).

Let us now restrict the convergence to bounded intervals in space, in order to use compact embeddings. To be more precise, for any $M \in \mathcal{N}$, let $I_M = (-M, M)$ and define for $T \geq \tau$ the space

$$\begin{aligned}
\mathcal{W}_M = \Big\{v \Big| \ &\partial_t v \in H'(\tau, T), \\
&v \in C([\tau, T]; L^2(I_M)) \cap L^2(\tau, T; H^1(I_M)) \cap L^{2\sigma+2}(\tau, T; L^{2\sigma+2}(I_M))\Big\}\ .
\end{aligned}$$

The energy estimates (3.29) and (3.30) imply immediately that u_N is bounded in $\mathcal{W}_M$ uniformly with respect to N, as both estimates hold obviously with all norms restricted to I_M on the left hand side of the equations.

Due to the Aubin-Lions lemma (see [34] or [45]), we obtain that (passing to a further subsequence if necessary) u_N converges to u strongly in $L^2(\tau, T; L^2(I_M))$. By using a diagonalization process, we can assume that (passing to a further subsequence if necessary) u_N converges to u strongly in $L^2_{loc}([\tau,\infty); L^2_{loc}(\mathbb{R}))$ and $u_N(y,t)$ converges to $u(y,t)$ for a.e. $(y,t) \in \mathbb{R} \times [\tau,\infty)$.

These assertions on the weak convergence imply immediately that the limit u also satisfies the energy estimates (3.9) and (3.10). Furthermore, one has enough regularity to pass to the limit in equation (3.27) and (3.28). Thus u is the solution of equations (3.7) and (3.8).

(II) *Uniqueness of Solution.* Our main task now is to prove estimate (3.26), which immediately implies the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions of equations (3.7) and (3.8) with initial conditions $u_1(0)$ and $u_2(0)$, respectively, instead of u_0. Then the energy estimate (3.9) is also valid for $u_1(t)$ and $u_2(t)$. Define $v(t) = u_1(t) - u_2(t)$, which satisfies the equation

$$\partial_t v - a\Delta v - \chi_0 v + b|u_1+\eta|^{2\sigma}(u_1+\eta) - b|u_2+\eta|^{2\sigma}(u_2+\eta) = 0, \tag{3.31}$$

$$v(\tau) = u_1(\tau) - u_2(\tau)\ . \tag{3.32}$$

As v is sufficiently regular, we can multiplying equation (3.31) by $\bar{v}(t)$ and integrate over $x \in \mathbb{R}$. Using integration by parts and taking the real part, we have

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2_{L^2_\mu} + 2a_1\|\nabla v(t)\|^2_{L^2_\mu} - \mathrm{Re}\int_{\mathbb{R}} \big(4a\rho(1+|y|^2)^{-1} vy\cdot\nabla v\big)d\mu \\ + 2b\,\mathrm{Re}\Big(|u_1+\eta|^{2\sigma}(u_1+\eta) - |u_2+\eta|^{2\sigma}(u_2+\eta), v\Big)_\mu \\ = 2|\chi_0|\|v(t)\|^2_{L^2_\mu}.\end{aligned} \tag{3.33}$$

First note the fact that

$$Re\,\big(|u_1+\eta|^{2\sigma}(u_1+\eta) - |u_2+\eta|^{2\sigma}(u_2+\eta), v\big)_\mu \geq 0\ .$$

Using Hölder inequality yields

$$\left|\int_{\mathbb{R}} \big(4a\rho(1+|y|^2)^{-1}\bar{v}y\cdot\nabla v\big)d\mu\right| \leq a_1\|\nabla v(t)\|^2_{L^2_\mu} + \frac{|a|^2\rho^2}{a_1}\|v(t)\|^2_{L^2_\mu}. \tag{3.34}$$

Inserting (3.34) into (3.33), and using Gronwall inequality, we obtain that

$$\|v(T)\|^2 + a_1\int_\tau^T \|\nabla v(t)\|^2 dt \leq \|u_1(\tau) - u_2(\tau)\|^2 e^{K_0 t}, \tag{3.35}$$

where the positive constant is $K_0 = 2|\chi_0| + |a|^2\rho^2/a_1$. Finally, (3.35) implies estimate (3.26)and the Theorem 3.3 is proved. □

Let us finally state a simple but yet usefull fact. recall that $u = u\big(t,\tau,\omega,u_0 - \eta(\tau)\big)$ is a solution of equations (3.7) and (3.8) given by theorem 3.3. By the proof of Lemma 3.1.2 [44], we have

$$\frac{d}{dt}\|u\|^2 = 2Re(\partial_t u, u)_\mu$$

in the distributional sense on $\mathbb{R}^+$. Thus we obtain the following lemma.

Lemma 3.4 (Energy Equality)**.** *Let $u = u\big(t, \tau, \omega, u_0 - \eta(\tau)\big)$ be a solution of equation* (3.7) *given by theorem 3.3. Then u satisfies the energy equality* (3.11) *in the distribution sense for $t \geq 0$.*

4. Asymptotic compactness

In this section, we construct the SDS corresponding to the SCGL (1.1) and (1.2), and establish AC for bounded sets.

Definition 4.1. *We define a map*

$$S:\ \mathbb{R}_\Delta \times \Omega \times L^2_\mu \longrightarrow L^2_\mu$$

by

$$(t, \tau, \omega, u_0) \mapsto S(t, \tau; \omega)u_0 = u\big(t, \tau, \omega, u_0 - \eta(\tau, \omega)\big) + \eta(t, \omega), \tag{4.1}$$

where $u\big(t, \tau, \omega, u_0 - \eta(\tau, \omega)\big)$, which is defined in Theorem 3.3, is the solution of the transformed equation (3.7)*.*

By the estimate (3.26), the map $S(t, \tau; \omega)$ is obviously continuous in L^2_μ, for all $t \geq \tau$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$. We obtain the following Lemma.

Lemma 4.2. *The map S defined in Definition 4.1 defines a SDS.*

Proof. All properties of a SDS with the exception of the cocycle follow from Theorem 3.3. Hence we only need to show that for any $u_0 \in L^2_\mu$ that

$$S(t, \tau; \omega)u_0 = S(t, r; \omega)S(r, \tau; \omega)u_0, \qquad \forall \tau \leq r \leq t. \tag{4.2}$$

From the definition of S, we have for all $\tau \leq r \leq t$,

$$\begin{aligned} S(t, \tau; \omega)u_0 &= u\big(t, \tau, \omega, u_0 - \eta(\tau, \omega)\big) + \eta(t, \omega), \\ &S(t, r; \omega)S(r, \tau; \omega)u_0 \\ &= u\big(t, r, \omega, S(r, \tau; \omega)u_0 - \eta(r, \omega)\big) + \eta(t, \omega) \\ &= u\Big(t, r, \omega, u\big(r, \tau, \omega, u_0 - \eta(\tau, \omega)\big)\Big) + \eta(t, \omega). \end{aligned}$$

Therefore, in order to prove (4.2), we only need to prove

$$u\big(t, \tau, \omega, u_0 - \eta(\tau, \omega)\big) = u\Big(t, r, \omega, u\big(r, \tau, \omega, u_0 - \eta(\tau, \omega)\big)\Big), \qquad \forall \tau \leq r \leq t. \tag{4.3}$$

Let us fix τ and define two functions u_1 and u_2 by

$$\begin{aligned} u_1(t) =& u\big(t, \tau, \omega, u_0 - \eta(\tau, \omega)\big), \\ u_2(t) =& u\Big(t, r, \omega, u\big(r, \tau, \omega, u_0 - \eta(\tau, \omega)\big)\Big), \qquad \forall \tau \leq r \leq t. \end{aligned}$$

Because $u\big(\tau, \tau, \omega, v_0\big) = v_0$, we infer that

$$\begin{aligned} u_2(\tau) &= u\Big(\tau, \tau, \omega, u\big(\tau, \tau, \omega, u_0 - \eta(\tau, \omega)\big)\Big) \\ &= u\big(\tau, \tau, \omega, u_0 - \eta(\tau, \omega)\big) = u_1(\tau). \end{aligned}$$

Since $u(t, r, \omega, U)$ is a solution of equations (3.7) with initial data $u\big|_{t=r} = U$, we infer that $u_1(t)$ and $u_2(t)$ solve the problem (3.7) and (3.8). Therefore, by the uniqueness of solutions to problem (3.7) and (3.8), we have $u_1(t) = u_2(t)$. Finally, since τ is arbitrary, (4.3) is proved. $\square$

4.1. **Weak and Strong Continuity of the SDS.** Let us now prove continuous dependence in the weak and strong topology of our SDS $S(t,\tau;\omega)$ defined in Definition 4.1. This will be used later in the proof of AC.

Lemma 4.3 (Weak Continuity)**.** *Fix an initial condition $u_0 \in L^2_\mu$ and a sequence $\{u_{0n}\}_n$ converging to u_0 weakly in L^2_μ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\begin{cases} S(t,\tau;\omega)u_{0n} \rightharpoonup S(t,\tau;\omega)u_0 & \text{weakly in } L^2_\mu, \ \ \forall t \ge \tau, \\ S(\cdot,\tau;\omega)u_{0n} \rightharpoonup S(\cdot,\tau;\omega)u_0 & \text{weakly in } H(\tau,T), \ \ \forall T \ge \tau, \end{cases} \tag{4.4}$$

$$S(\cdot,\tau;\omega)u_{0n} \to S(\cdot,\tau;\omega)u_0 \ \ \text{strongly in } L^2(\tau,T;L^2_{loc}(\mathbb{R})), \ \ \forall T \ge \tau\ , \tag{4.5}$$

where again $H(\tau,T) = L^2(\tau,T;H^1_\mu) \cap L^{2\sigma+2}(\tau,T;L^{2\sigma+2}_\mu)$.

Proof. Let us fix ω such that $\eta(\cdot,\omega)$ is finite in $C^0_{\rm loc}(\mathbb{R};L^p_\mu)$ for any $p \ge 2$ and fix $T \ge \tau$. Denote the solution $u\big(t,\tau,\omega,u_{0n}-\eta(\tau,\omega)\big)$ of equation (3.7) by $u_n(t)$. From the a priori estimates (3.9) and (3.10), it follows that

$$\begin{cases} \{u_n(t)\}_n & \text{is bounded in } L^\infty(\tau,T;L^2_\mu) \cap H(\tau,T), \\ \{\partial_t u_n(t)\}_n & \text{is bounded in } H'(\tau,T). \end{cases}$$

As before by a diagonalization process, we can extract a subsequence $\{u_{n'}(t)\}_{n'}$ converging to $\hat u(t)$ weakly star in $L^\infty(\tau,T;L^2_\mu)$ and weakly in $H(\tau,T)$. Then $\hat u(t) \in L^\infty(\tau,T;L^2_\mu) \cap H(\tau,T)$. Thanks to Aubin-Lions lemma (see [26, 33]) and a diagonalization process, we can also assume that (passing to a further subsequence if necessary) $\{u_{n'}(t)\}_{n'}$ converges to $\hat u(t)$ strongly in $L^2(\tau,T;L^2_{loc}(\mathbb{R}))$. Moreover, $\hat u$ satisfies the energy estimates (3.9), where the initial condition is replaced by a constant, depending on a bound on the weakly convergent subsequence u_{0n}.

Passing to the limit in the weak form of the equations (3.7) and (3.8) for $u_n(t)$, we find that $\hat u(t)$ solves the equations (3.7) and (3.8). By the uniqueness of solution, we have $\hat u(t) = u(t) = u\big(t,\tau,\omega,u_0-\eta(\tau,\omega)\big)$. Then, by a contradiction argument, we can deduce from the uniqueness of solutions that in fact the whole sequence $\{u_n(t)\}_n$ converges to $u(t)$ such that (4.5) holds and moreover

$$\begin{cases} u_n(t) \to u(t) & \text{weakly star in } L^\infty(\tau,T;L^2_\mu), \ \ \forall T \ge \tau, \\ & \text{strongly in } L^2(\tau,T;L^2_{loc}(\mathbb{R})), \ \ \forall T \ge \tau, \\ u_n(t) \rightharpoonup u(t) & \text{weakly in } H(\tau,T), \ \ \forall T \ge \tau. \end{cases} \tag{4.6}$$

Hence, for all $v \in H^1_\mu \cap L^{2\sigma+2}_\mu$,

$$\big(u_n(t),v\big)_\mu \to \big(u(t),v\big)_\mu, \quad \text{for a.e. } t \in [\tau,\infty). \tag{4.7}$$

For all $\tau \le t \le t+a \le T$,

$$\begin{aligned} \big(u_n(t+a)-u_n(t),v\big)_\mu &= \int_t^{t+a} \big(\partial_t u_n(s),v\big)_\mu ds \\ &\le \{\|v\|_{H^1_\mu}\ a^{1/2} + \|v\|_{L^{2\sigma+2}_\mu}\ a^{1/(2\sigma+2)}\}\|\partial_t u_n\|_{H'} \\ &\le C_T\{\|v\|_{H^1_\mu}\ a^{1/2} + \|v\|_{L^{2\sigma+2}_\mu}\ a^{1/(2\sigma+2)}\}, \end{aligned} \tag{4.8}$$

where C_T is independent of n. Thus $\{\big(u_n(t),v\big)_\mu\}_n$ is equibounded and equicontinuous on $[\tau,T]$ for all $T>0$, and by uniqueness of the Limit together with Arcela-Ascoli we derive

$$\big(u_n(t),v\big)_\mu \to \big(u(t),v\big)_\mu, \quad \forall t \in [\tau,\infty), \ \ \forall v \in H^1_\mu \cap L^{2\sigma+2}_\mu. \tag{4.9}$$

(4.4) follows from (4.6) and (4.9) by taking into account the facts that $H^1_\mu \cap L^{2\sigma+2}_\mu$ is dense in L^2_μ and u_n bounded in L^2_μ. $\square$

The following Lemma exploits bounds in the energy estimate given by the nonlinear terms. It is crucial, in order to pass from convergence in L^2_μ to convergence in some L^p_μ.

Lemma 4.4. (Strong Convergence) *For any given $T \geq \tau$, suppose that*

$$u_n \to u \ \ \text{strongly in } L^2(\tau, T; L^2_{loc}(\mathbb{R})), \tag{4.10}$$

and that the sequence $\{u_n\}_{n=1}^\infty$ is uniform bounded in $L^{2\sigma+2}(\tau, T; L^{2\sigma+2}_\mu)$, i.e.

$$\|u_n\|_{L^{2\sigma+2}(\tau,T;L^{2\sigma+2}_\mu)} \leq C, \qquad \forall n \geq 1. \tag{4.11}$$

Then

$$u_n \to u \ \ \text{strongly in } L^p(\tau, T; L^p_\mu(\mathbb{R})), \qquad \forall p \in [1, 2\sigma+2). \tag{4.12}$$

Proof. Recall that $d\mu = m(y)dy$, $m(y) = (1+|y|^2)^{-\rho}$, $\rho > \frac{1}{2}$ and $\mu(\mathbb{R}) < \infty$. Let us first prove the result for $p = 2$. Using (4.10), we derive for any $r > 0$

$$\begin{aligned}
&\lim_{n\to\infty} \|u_n - u\|^2_{L^2(\tau,T;L^2_\mu)} \\
&= \lim_{n\to\infty} \int_\tau^T \int_{|y|\leq r} |u_n(s) - u(s)|^2 d\mu ds + \lim_{n\to\infty} \int_\tau^T \int_{|y|\geq r} |u_n(s) - u(s)|^2 d\mu ds \\
&=: I_1 + I_2
\end{aligned}$$

The convergence in L^2_{loc} immediately implies $I_1 = 0$. For I_2 we obtain due to Hölder inequality

$$I_2 \leq (T-\tau)^{\sigma/(\sigma+1)} \Big\{ \int_{|y|\geq r} d\mu \Big\}^{\sigma/(\sigma+1)} \sup_n \|u_n - u\|^{1/(\sigma+1)}_{L^{2\sigma+2}(\tau,T;L^{2\sigma+2}_\mu)}.$$

Together with (4.11) and letting $r \to \infty$ yields

$$\lim_{n\to\infty} \|u_n - u\|^2_{L^2(\tau,T;L^2_\mu)} = 0. \tag{4.13}$$

Let us now turn to $p \in [1, 2)$. Here the claim follows immediately, as

$$\|u_n - u\|^p_{L^p(\tau,T;L^p_\mu)} \leq \{(T-\tau)\mu(\mathbb{R})\}^{1-p/2} \|u_n - u\|^{p/2}_{L^2(\tau,T;L^2_\mu)} \overset{n\to\infty}{\to} 0. \tag{4.14}$$

For any $p \in (2, 2\sigma+2)$, by Hölder interpolation we have

$$\lim_{n\to\infty} \|u_n - u\|^p_{L^p(\tau,T;L^p_\mu)} \leq \lim_{n\to\infty} \|u_n - u\|^{\frac{2\sigma+2-p}{p\sigma}}_{L^2(\tau,T;L^2_\mu)} \|u_n - u\|^{1-\frac{2\sigma+2-p}{p\sigma}}_{L^{2\sigma+2}(\tau,T;L^{2\sigma+2}_\mu)} = 0. \tag{4.15}$$

Thus finally, the convergence in (4.12) is proved. $\square$

4.2. **Asymptotic Compactness.** We now turn to the main result of the paper. The following Theorem verifies AC for bounded sets, as for example defined in [8] or [3] for RDS.

Theorem 4.5. (Asymptotic Compactness) *Let $\{u_j\}_j$ be bounded in L^2_μ and $\{\tau_j\}_j \subset \mathbb{R}$ with $\tau_j \to -\infty$. Then there exists $w \in L^2_\mu$ and a subsequence $\{j'\}$ such that $S(t, \tau_{j'}; \omega)u_{j'} \to w$ strongly in L^2_μ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Let us fix ω such that $\eta(T,\omega)$ is sufficiently regular, as asserted by Lemma 3.1.

From a priori estimates (3.9) and (3.10) and from the fact that $\{u_j\}_j$ is bounded in L^2_μ, there exist constants $J>0$ and $R_0>0$ such that

$$\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\|_{L^2_\mu}\le R_0,\quad \forall j'\ge J, \tag{4.16}$$

where $u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)$ is the solution of equations (3.7) with initial data

$$u\Big|_{t=\tau_{j'}}=u_{j'}-\eta(\tau_{j'}). \tag{4.17}$$

Thus $\{u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\}_{j'}$ is weakly precompact in L^2_μ, and we can assume that

$$u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\rightharpoonup v \text{ weakly in } L^2_\mu \tag{4.18}$$

by passing to a further subsequence if necessary.

Similarly for each $T>0$, we also have that there exist constants $J=J(T)>0$ such that $t-T\ge\tau_{j'}$ and

$$\|u\big(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\|_{L^2_\mu}\le R_0,\quad \forall j'\ge J. \tag{4.19}$$

Thus $\{S(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'}))\}_{j'}$ is weakly precompact in L^2_μ, and by using a diagonal process and passing to a further subsequence if necessary we can assume that

$$u(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'}))\rightharpoonup v_T \text{ weakly in } L^2_\mu \tag{4.20}$$

for any rational number $T>0$.

Then by the proof of Lemma 4.3 and Lemma 4.2 we deduce that

$$\begin{aligned}
v&=w\!-\!\lim_{j'\to\infty}u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\\
&=w\!-\!\lim_{j'\to\infty}u\Big(t,t-T,\omega,u(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'}))\Big)\\
&=u\Big(t,t-T,\omega,w\!-\!\lim_{j'\to\infty}u(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'}))\Big)=u(t,t-T,\omega,v_T),
\end{aligned}$$

where $w\!-\!\lim$ denotes the limit taken in the weak topology of L^2_μ. Thus

$$v=u(t,t-T,\omega,v_T),\qquad \forall T\in\mathbb{N}. \tag{4.21}$$

Now, from (4.18), we find

$$\|v\|_{L^2_\mu}\le\liminf_{j'\to\infty}\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\|_{L^2_\mu}, \tag{4.22}$$

and, in order to show strong convergence, we will show that

$$\limsup_{j'\to\infty}\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'})\big)\|_{L^2_\mu}\le\|v\|_{L^2_\mu}. \tag{4.23}$$

For $T\in\mathbb{N}$ and $t-T\ge\tau_j$ we obtain by integrating the energy equation (3.11) from $t-T$ to t that

$$\begin{aligned}
&\|u\big(t,\tau_j,\omega,u_j-\eta(\tau_j,\omega)\big)\|^2_{L^2_\mu}+2b\int_{t-T}^t e^{\tau-t}\|U_j+\eta(\tau,\omega)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}d\tau\\
&+\int_{t-T}^t e^{\tau-t}\{2a_1\|\nabla U_{j'}\|^2_{L^2_\mu}-(2\chi_0+1)\|U_{j'}\|^2_{L^2_\mu}\}d\tau\\
=&\|u\big(t-T,\tau_j,\omega,u_j-\eta(\tau_j,\omega)\big)\|^2_{L^2_\mu}e^{-T}+2Re\int_{t-T}^t e^{\tau-t}\int_{\mathbb{R}}(k+\chi_0)U_j\bar\eta(\tau,\omega)d\mu\, d\tau
\end{aligned}$$

$$+2Re\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}} b\Big|U_j+\eta(\tau,\omega)\Big|^{2\sigma}\{U_j+\eta(\tau,\omega)\}\bar{\eta}(\tau,\omega)d\mu\,d\tau, \tag{4.24}$$

where

$$U_j=u\big(\tau,\tau_j,\omega,u_j-\eta(\tau_j,\omega)\big)=u\Big(\tau,t-T,\omega,u(t-T,\tau_j,\omega,u_j-\eta(\tau_j,\omega))\Big).$$

From (4.19) we find

$$\limsup_{j'}\|u\big(t-T,\tau_j,\omega,u_j-\eta(\tau_j,\omega)\big)\|^2e^{-T}\le R_0^2e^{-T}. \tag{4.25}$$

Also, by the proof of (4.4) and (4.5) we deduce from (4.20) that

$$U_{j'}=u\Big(\cdot,t-T,\omega,u\big(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big)\Big)\rightharpoonup u(\cdot,t-T,\omega,v_T) \tag{4.26}$$

weakly in $L^2(t-T,t;H^1_\mu)\cap L^{2\sigma+2}(t-T,t;L^{2\sigma+2}_\mu)$,

$$U_{j'}=u\Big(\cdot,t-T,\omega,u\big(t-T,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big)\Big)\to u(\cdot,t-T,\omega,v_T) \tag{4.27}$$

strongly in $L^2(t-T,t;L^2_{loc}(\mathbb{R}))$ and $U_{j'}(y,s)$ converges to $u(s,t-T,\omega,v_T)$ for a.e. $(y,s)\in\mathbb{R}\times(t-T,t)$. By using Lemma 4.4, we get that $U_{j'}$ converges to $u(s,t-T,\omega,v_T)$ strongly in $L^p(t-T,t;L^p_\mu)$ for any $p\in[1,2\sigma+2)$. Since $\mu(\mathbb{R})<\infty$ and

$$\tau\mapsto e^{\tau-t}\bar{\eta}(\tau,\omega)\in L^2(t-T,t;L^2_\mu),$$

we find that

$$\begin{aligned}&\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}} U_{j'}\bar{\eta}(\tau,\omega)d\mu\,d\tau\\ &\longrightarrow\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}} u(\tau,t-T,\omega,v_T)\bar{\eta}(\tau,\omega)d\mu\,d\tau,\end{aligned} \tag{4.28}$$

Next, as exp is L^∞, the strong convergence in $L^2(L^2_\mu))$ implies

$$\int_{t-T}^{t} e^{\tau-t}\|U_{j'}\|^2_{L^2_\mu}d\tau\longrightarrow\int_{t-T}^{t} e^{\tau-t}\|u(\tau,t-T,\omega,v_T)\|^2_{L^2_\mu}d\tau, \tag{4.29}$$

From strong convergence in $L^p(L^p_\mu)$ with $p<2\sigma+2$ we deduce

$$\begin{aligned}&\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}}\Big|U_{j'}+\eta(\tau,\omega)\Big|^{2\sigma}\{U_{j'}+\eta(\tau,\omega)\}\bar{\eta}(\tau,\omega)d\mu\,d\tau\\ &\longrightarrow\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}}\Big|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\Big|^{2\sigma}\cdot\\ &\qquad\qquad\{u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\}\bar{\eta}(\tau,\omega)d\mu\,d\tau,\end{aligned} \tag{4.30}$$

Finally, the weak convergence bounds

$$\begin{aligned}&\int_{t-T}^{t} e^{\tau-t}\|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}d\tau\\ &\le\liminf_{j'\to\infty}\int_{t-T}^{t} e^{\tau-t}\|U_{j'}+\eta(\tau,\omega)\|^{2\sigma+2}_{L^{2\sigma+2}_\mu}d\tau,\end{aligned} \tag{4.31}$$

and

$$\int_{t-T}^{t} e^{\tau-t}\|\nabla u(\tau,t-T,\omega,v_T)\|^2_{L^2_\mu}d\tau\le\liminf_{j'\to\infty}\int_{t-T}^{t} e^{\tau-t}\|\nabla U_{j'}\|^2_{L^2_\mu}d\tau. \tag{4.32}$$

Therefore

$$
\begin{aligned}
&\limsup_{j'\to\infty} -2a_1\int_{t-T}^{t} e^{\tau-t}\|\nabla U_{j'}\|_{L^2_\mu}^2 d\tau = -2a_1 \liminf_{j'\to\infty}\int_{t-T}^{t} e^{\tau-t}\|\nabla U_{j'}\|_{L^2_\mu}^2 d\tau \\
&\le -2a_1\int_{t-T}^{t} e^{\tau-t}\|\nabla u(\tau,t-T,\omega,v_T)\|_{L^2_\mu}^2 d\tau.
\end{aligned}
\tag{4.33}
$$

Now passing to the $\limsup$ as j goes to infinity in (4.24) and using (4.25), (4.28) – (4.31) and (4.33), we get that

$$
\begin{aligned}
&\limsup_{j'\to\infty}\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big)\|_{L^2_\mu}^2 \\
\le& R_0^2 e^{-T} - 2b\int_{t-T}^{t} e^{\tau-t}\|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\|_{L^{2\sigma+2}_\mu}^{2\sigma+2} d\tau \\
&-\int_{t-T}^{t} e^{\tau-t}\{2a_1\|\nabla u(\tau,t-T,\omega,v_T)\|_{L^2_\mu}^2 - (2\chi_0+1)\|u\|_{L^2_\mu}^2\} d\tau \\
&+2Re\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}} b\Big|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\Big|^{2\sigma}\{u+\eta\}\bar{\eta}d\mu\, d\tau \\
&+2Re\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}}(k+\chi_0)u(\tau,t-T,\omega,v_T)\bar{\eta}(\tau,\omega)d\mu\, d\tau.
\end{aligned}
\tag{4.34}
$$

On the other hand, we obtain from the energy equality (3.11) applied to $v = u(t,t-T,\omega,v_T)$ that

$$
\begin{aligned}
\|v\|_{L^2_\mu}^2 &= \|u(t,t-T,\omega,v_T)\|_{L^2_\mu}^2 \\
&= e^{-T}\|v_T\|^2 - 2b\int_{t-T}^{t} e^{\tau-t}\|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\|_{L^{2\sigma+2}_\mu}^{2\sigma+2} d\tau \\
&\quad -\int_{t-T}^{t} e^{\tau-t}\{2a_1\|\nabla u(\tau,t-T,\omega,v_T)\|^2 - (2\chi_0+1)\|u\|_{L^2_\mu}^2\} d\tau \\
&\quad +2Re\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}} b\Big|u(\tau,t-T,\omega,v_T)+\eta(\tau,\omega)\Big|^{2\sigma}\{u+\eta\}\bar{\eta}d\mu\, d\tau \\
&\quad +2Re\int_{t-T}^{t} e^{\tau-t}\int_{\mathbb{R}}(k+\chi_0)u(\tau,t-T,\omega,v_T)\bar{\eta}(\tau,\omega)d\mu\, d\tau.
\end{aligned}
\tag{4.35}
$$

From (4.34) and (4.35) we deduce that

$$
\begin{aligned}
\limsup_{j'\to\infty}\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big)\|_{L^2_\mu}^2 &\le \|v\|_{L^2_\mu}^2 + (R_0^2 - \|v_T\|_{L^2_\mu}^2)e^{-T} \\
&\le \|v\|_{L^2_\mu}^2 + R_0^2 e^{-T}, \qquad \forall T\in\mathbb{N}.
\end{aligned}
\tag{4.36}
$$

Letting $T\to\infty$ in (4.36), we have

$$
\limsup_{j'\to\infty}\|u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big)\|_{L^2_\mu}^2 \le \|v\|_{L^2_\mu}^2,
\tag{4.37}
$$

as claimed. Since L^2_μ is a Hilbert space, the bound (4.37) on the $\limsup$ together with the weak convergence from (4.18) imply

$$
u\big(t,\tau_{j'},\omega,u_{j'}-\eta(\tau_{j'},\omega)\big) \overset{j'\to\infty}{\longrightarrow} v \text{ strongly in } L^2_\mu.
\tag{4.38}
$$

□

5. Comments on the attractor

Let us again by Lemma 3.1 fix ω such that η is sufficiently regular. According to Lemma 3.2, it is easy to show that for all $t > \tau$

$$\|S(t,\tau,\omega)u_0\|_{L^2_\mu} \leq Ce^{\tau-t}\|u_0-\eta\|_{L^2_\mu} + r(t,\omega)$$

where the random constant

$$r(t,\omega) = C\int_{-\infty}^{t+1} e^{s-t}\Big(1+\|\eta(s,\omega)\|^2_{L^2_\mu} + \|\eta(s,\omega)\|^{\sigma_1}_{L^{\sigma_1}_\mu}\Big)dt$$

is $\mathbb{P}$-almost sure finite due to Lemma 3.1.

Thus the SDS $S(t,\tau;\omega)$ possesses an attracting set

$$K(t,\omega) = B_{r(t,\omega)} \subset L^2_\mu\,.$$

It is obvious that by enlarging the radius, one obtains a bounded absorbing set $\mathcal{B}(t,\omega)$ in L^2_μ. Moreover in the setting of RDS, it is easy to show that the set K and thus $\mathcal{B}$ is a tempered set, which means that $K(t,\omega)$ grows at most subexponential for $t \to -\infty$.

Let us remark that for more regular forcing, where η is uniformly bounded in time, using Theorem 4.5, it would be easy to establish the existence of a random pull-back attractor for bounded sets for the RDS, as the absorbing set is then uniformly bounded in time.

Let us also remark, that using Theorem 4.5 and following the proof of Proposition 3.1–Theorem 3.4 in [8] for RDS, one could show that there is a compact Ω-Limit set of the RDS, which supports an invariant measure. This would recover results of [42].

It remains an open problem, whether one can use the concept of ω-limit sets for SDS, too. Moreover, as the absorbing set is tempered, we would need AC for tempered sets, in order to show that the ω-limit set of the absorbing set is a random attractor. In that case it will also attract tempered sets.

Nevertheless, if we suppose that our stochastic forcing is more regular in time, such that $\eta(t)$ is uniformly bounded in t, then it is easy to see, that in this case there is a deterministic bounded absorbing set. Using our results on AC for bounded sets, it is easy to show (cf. e.g. [3]) that the RDS has a bounded random attractor for bounded deterministic sets.

Let us conclude the paper by showing that the possible attractor is actually independent of the weight chosen. This is not obvious, as illustrated by the following example. For the deterministic PDE $\partial_t u = \partial_x^2 u$ the stationary solutions in L^2_μ are $\{0\}$ for $\rho < \frac{1}{2}$, $\mathrm{span}\{1\}$ for $\rho \in (\frac{1}{2}, 1)$, and $\mathrm{span}\{1, x\}$ for $\rho > 1$.

For any $\rho > \frac{1}{2}$ denote the space L^2_μ now by L^2_ρ, in order to allow for different weights.

Theorem 5.1. *Suppose that for all $\rho > \frac{1}{2}$, the SDS $S(t,\tau;\omega)$ associated to SCGL* (1.1) *and* (1.2) *possesses a tempered pull-back global attractor $\mathcal{A}_\rho(t,\omega)$ for tempered sets, i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$, and for all $t \in \mathbb{R}$, the set $\mathcal{A}_\rho(t,\omega)$ is a nonempty compact subset in L^2_ρ which attracts all tempered sets from $-\infty$, it is invariant in the sense that*

$$S(t,\tau;\omega)\mathcal{A}_\rho(\tau,\omega) = \mathcal{A}_\rho(t,\omega), \quad \forall \tau \leq t,$$

and is the minimal closed set with this property.

Then $\mathcal{A}_\rho$ is independent of $\rho > \frac{1}{2}$.

Let us remark, that for more regular noise, similar results hold true for the stochastic or random attractor of bounded sets.

Proof. Let us first show more regularity for $\mathcal{A}_\rho$. For t_0 and ω fixed, take $u_0 \in \mathcal{A}_\rho(t_0,\omega) \subset L^2_\rho$, and define $u(t) = S(t,t_0,\omega)u_0$. From energy estimates $u \in L^{2\sigma+2}(t_0,t,L^{2\sigma+2}_\rho)$, and thus there is a $t_1(\omega) \in (t_0,t)$ such that $u(t_1) \in L^{2\sigma+2}_\rho$.

Now we can use the following fact that for $0 < \theta < \sigma(2\rho-1)/(2\sigma+2)$ Hölder inequality yields

$$\begin{aligned}
\|u\|^2_{L^2_{\rho-\theta}} &= \int_{\mathbb{R}} (1+|x|^2)^{-(\rho-\theta)}|u(x)|^2 dx \\
&\leq \Big(\int_{\mathbb{R}} (1+|x|^2)^{-(\rho-\theta-(\frac{\rho}{\sigma+1}))\frac{\sigma+1}{\sigma}} dx\Big)^{\sigma/(\sigma+1)} \cdot \Big(\int_{\mathbb{R}} m_\rho(x)|u(x)|^{2\sigma+2} dx\Big)^{1/(\sigma+1)} \\
&= C\|u\|^2_{L^{2\sigma+2}_\rho} .
\end{aligned}$$

Thus $u(t_1) \in L^2_{\rho-\theta}$ and again from energy estimates it is straightforward to obtain $u \in C^0([t_1,t], L^2_{\rho-\theta})$ and thus $u(t) \in L^2_{\rho-\theta}$.

Finally, as u_0 was arbitrary, this yields

$$\mathcal{A}_\rho(t,\omega) = S(t,t_0,\omega)\mathcal{A}_\rho(t_0,\omega) \subset L^2_{\rho-\theta}.$$

Repeating the previous argument, this is actually true for all $\theta \in (0,\rho)$.

On one hand $\mathcal{A}_\rho$ is an invariant set in $L^2_{\rho-\theta}$, which has to be attracted by $\mathcal{A}_{\rho-\theta}$. Thus $\mathcal{A}_\rho \subset \mathcal{A}_{\rho-\theta}$.

On the other hand $\mathcal{A}_{\rho-\theta} \subset L^2_{\rho-\theta} \subset L^2_\rho$ is also invariant, and thus $\mathcal{A}_{\rho-\theta} \subset \mathcal{A}_\rho$. □

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