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STOCHASTIC MODULATION EQUATIONS ON UNBOUNDED DOMAINS*

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Abstract. We study the impact of small additive space-time white noise on nonlinear stochastic partial differential equations (SPDEs) on unbounded domains close to a bifurcation, where an infinite band of eigenvalues changes stability due to the unboundedness of the underlying domain. Thus we expect not only a slow motion in time, but also a slow spatial modulation of the dominant modes, and we rely on the approximation via modulation or amplitude equations, which acts as a replacement for the lack of random invariant manifolds on extended domains.

One technical problem for establishing error estimates in the stochastic case rises from the spatially translation invariant nature of space-time white noise on unbounded domains, which implies that at any time the error is always very large somewhere far out in space. Thus we have to work in weighted spaces that allow for growth at infinity.

As a first example we study the stochastic one-dimensional Swift-Hohenberg equation on the whole real line [1, 2]. In this setting, because of the weak regularity of solutions, the standard methods for deterministic modulation equations fail, and we need to develop new tools to treat the approximation. Using energy estimates we are only able to show that solutions of the Ginzburg-Landau equation are Hölder continuous in spaces with a very weak weight, which provides just enough regularity to proceed with the error estimates.

Key words. modulation equations, amplitude equations, convolution operator, regularity, Rayleigh-Bénard, Swift-Hohenberg, Ginzburg-Landau

AMS subject classifications. 60H15,60H10

1. Experiments. A celebrated model in pattern formation is the Rayleigh-Bénard convection, an experimental phenomenon where a fluid between two plates is heated from below and kept at a constant temperature from above. Here the full description would be a 3D-Navier-Stokes equation coupled to the heat equation, a mathematical model that is yet too complicated for our analytical tools. In this article we review the results of [1] and [2] and thus we consider the simpler Swift-Hohenberg model [8] that is used as a reduced model for the convective instability.

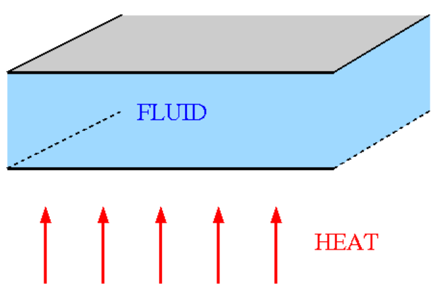

FIGURE 1.1. *Rayleigh-Bénard convection*

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1.1. Convective instability. The convective instability is the first bifurcation in the Rayleigh-Bénard problem. Below a critical temperature T_c the fluid is at rest and no pattern is formed. The heat is just transported by conduction through the system.

Above the critical temperature T_c convection rolls start to form. Hot fluid is going up and cold fluid is going down, and they cannot do that in the same place, so we have areas where the motion is upwards and other areas where it is downwards. In a view from above, a striped pattern starts to show up.

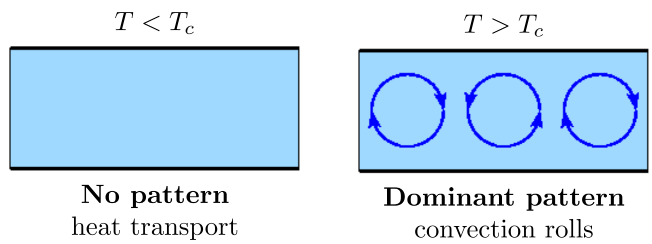


FIGURE 1.2. *Bifurcation at the convective instability, the figure shows a cut through the fluid with the plates above and below.*

1.2. Pattern formation below criticality. Very close to the critical point, stochastic effects were observed first in electro-convection (see Rehberg et al. [18]) and much later in Rayleigh-Bénard convection (see Oh, Ahlers et al. [16, 17]). In both experiments, pattern formation slightly below the critical threshold (i.e., a critical temperature T_c in Rayleigh Bénard) was observed. Nevertheless the distance to bifurcation had to be of the order of the noise's strength, which made it extremely difficult to observe in experiments, as the source of noise in Rayleigh-Bénard are thermal fluctuations. Similar observations in numerical experiments and using formal center manifold approximations were done by Hutt et al. [10, 9].

OBSERVATION FROM EXPERIMENTS, [18]:

Below threshold (but close)	**Well above threshold**
trivial solution is not stable	convection rolls are stable
pattern is slowly modulated	pattern is almost periodic

2. Introduction. The typical setting in the following presentation of our results shows complicated systems given for example by (stochastic) partial differential equations. Near a change of stability (or bifurcation) of the trivial solution, we have a natural separation of time-scales. The (Fourier) modes similar to the bifurcating pattern move on a slow time-scale given by the distance from bifurcation, while the other modes move and disappear on an order one time-scale.

The typical results we are aiming at are the approximation of the full dynamics by means of the *amplitude* of the bifurcating pattern, which is given by a (stochastic) differential equation. On unbounded domains a full band of eigenfunctions changes stability. In order to take this into account the amplitude of the dominating pattern is slowly *modulated* in space.

This approximation by modulation (or amplitude) equations is well established in the physics literature, but only on a formal level. From a mathematical point of view, the deterministic problems are well studied. Starting from the first publications [4, 11, 14, 13] there is a rich literature, featuring also recent contributions, for example [20, 6], just to name two.

Let us point out that the celebrated center manifold reduction, which works well for deterministic PDEs on bounded domains, is not available for PDEs on unbounded domains. Moreover, it is not useful in the stochastic setting: because of the inherent non-autonomy of the system due to noise, the manifold itself would move through the whole phase space, and thus any reduction to the manifold does not reduce the complexity of the dynamics at all.

We finally give an outline of the paper. In Section 3 we state the setting of the Swift-Hohenberg equation and discuss the spectrum of the linearized operator together with modulated pattern. We then briefly recall the main results on large domains in Section 4, while in Section 5 we state in detail the results available on unbounded domains. In the final two sections we give a remark on pattern formation below criticality and provide an outlook on several possible extensions of the result.

3. Swift-Hohenberg. For our results we consider for simplicity only a toy problem given by the Swift-Hohenberg equation. It can be derived via heuristic reduction from the Rayleigh-Bénard problem close to the convective instability, as was originally shown by Swift & Hohenberg [8]. See also [19] for a more rigorous approach. The equation is given as:

$$\partial_t u = -(1+\partial_x^2)^2 u + \nu\varepsilon^2 u - u^3 + \varepsilon^{3/2}\xi \,, \tag{SH}$$

where we assume

- $u(t,x) \in \mathbb{R}, \quad t>0,\ x\in\mathbb{R}$
- periodic boundary conditions – or – unbounded domain
- $\xi = \partial_t W$ Gaussian space-time white noise.

Thus in the sense of generalized processes the mean of the noise is zero and it is uncorrelated in space and time:

$$\mathbb{E}\,\xi(t,x) = 0\,, \qquad \mathbb{E}\,\xi(t,x)\xi(s,y) = \delta(t-s)\delta(x-y)\,.$$

As a mathematical model, the noise is given as a derivative of a standard cylindrical Wiener process $\{W(t)\}_{t\ge 0}$ in $L^2(\mathbb{R})$, meaning that

$$W(t) = \sum_k \beta_k(t) e_k$$

where $\{e_k\}_k$ is any orthonormal basis in $L^2(\mathbb{R})$ and $\{\beta_k\}_k$ is a sequence of i.i.d. real-valued Brownian motions.

3.1. Eigenvalues – Spectral gap. In our example of the Swift-Hohenberg operator we can calculate all eigenvalues of the linearized operator explicitly:

$$\mathcal{L} = -(1+\partial_x^2)^2 \quad \text{and thus} \quad \mathcal{L}\mathrm{e}^{ikx} \;=\; \lambda(k)\,\mathrm{e}^{ikx}\,,$$

subject to periodic boundary conditions on an interval or on the whole real line. Obviously,

$$\lambda(k) = -(1-k^2)^2\,.$$

In Figure 3.1 we plotted the eigenvalues of $\mathcal{L}$ for those $k\in\mathbb{R}$ which lead to admissible eigenfunctions that satisfy the boundary conditions. We see that the spectral gap between the largest two eigenvalues shrinks as the domain gets larger: on an interval of length $\mathcal{O}(\varepsilon^{-1})$ already many eigenvalues are $\mathcal{O}(\varepsilon^2)$ away from the largest eigenvalue 0.

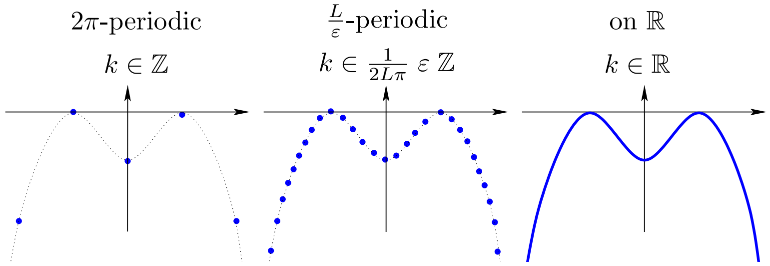
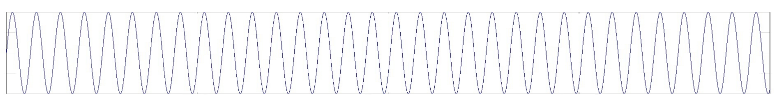


FIGURE 3.1. *Band of Eigenvalues for the example* $\mathcal{L} = -(1+\partial_x^2)^2$ *on bounded, large, and unbounded domains. We plot the wave-number* k *against the eigenvalue* $\lambda(k) = -(1-k^2)^2$ *for the corresponding eigenfunction* $\mathrm{e}^{\mathrm{i}kx}$.

3.2. Modulated pattern. As many eigenvalues are close to the change of stability, we need to understand how many eigenfunctions with wave-number around $k = \pm 1$ influence the pattern.

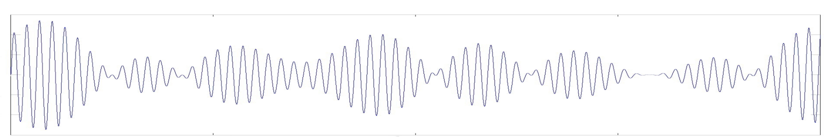
Let us compare a 2π-periodic pattern

$$u(x) = \varepsilon A \mathrm{e}^{ix} + \text{c.c.} \quad \text{with} \quad A \in \mathbb{C}$$



with a modulated pattern

$$u(x) = \varepsilon A(\varepsilon x)\mathrm{e}^{ix} + \text{c.c.} \quad \text{with} \quad A : \mathbb{R} \mapsto \mathbb{C}$$



If we consider the amplitude A in polar coordinates, then its absolute value $|A|$ determines the size of the modulated pattern, while the angle is a phase shift of the pattern. Both move slowly in space here.

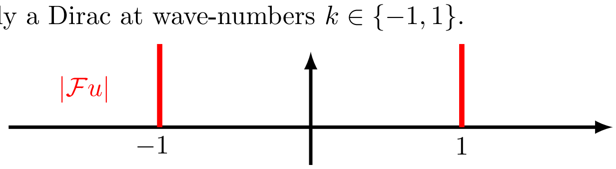
We can calculate that

$$u(x) = \varepsilon A(\varepsilon x)\mathrm{e}^{ix} + \text{c.c.}$$

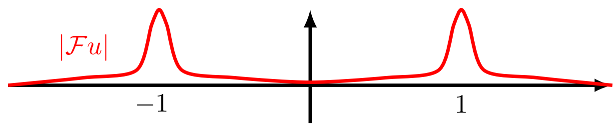
has Fourier transform

$$\mathcal{F}u(k) = \mathcal{F}A\left((k-1)/\varepsilon\right) + \overline{\mathcal{F}A}\left((k+1)/\varepsilon\right)\,.$$

For 2π-periodic pattern the function is in the span of e^{ix} and e^{-ix}. Thus the Fourier transform is only a Dirac at wave-numbers $k \in \{-1, 1\}$.



For the slow modulation of a 2π-periodic pattern, the Fourier transform widens up but it is still concentrated around $k \in \{-1, 1\}$. A whole band of infinitely many Fourier modes defines the structure of the solution.



4. Large domains. Here we present the results of Blömker, Hairer & Pavliotis [3] without stating the technical details.

THEOREM 4.1 (Approximation [3]). *Consider a $2L/\varepsilon$-periodic solution u of (SH) If $u(0,x) = \varepsilon A(0,\varepsilon x)\cdot \mathrm{e}^{ix} + c.c. + \mathcal{O}(\varepsilon^2)$ is a modulated wave with admissible initial condition $A(0,\cdot) = \mathcal{O}(1)$, then*

$$\forall t \in [0, T_0\varepsilon^{-2}] \qquad u(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon x)\cdot \mathrm{e}^{ix} + c.c. + \mathcal{O}(\varepsilon^{2-})\ ,$$

where the amplitude $A(T,X) \in \mathbb{C}$ solves (GL).

The amplitude equation is a stochastic Ginzburg-Landau equation:

$$\partial_T A = (4\partial_X^2 + \nu)A - 3|A|^2 A + \eta \qquad \text{(GL)}$$

with

- $2L$-periodic solutions
- $\mathbb{C}$-valued space-time white noise $\eta = \partial_T \mathcal{W}$

The complex-valued standard cylindrical Wiener-process $\mathcal{W}$ arises from rescaling the discrete Fourier transform of the real-valued Wiener process W for Fourier-modes with wave-number k close to 1. See also Section 5.2.

Let us remark that even in the case of ξ in (SH) colored and regular in space, the amplitude equation (GL) has space-time white noise, due to rescaling in space and time.

The estimates in Theorem 4.1 above are given in C^0-norms and the initial condition $A(0)$ is called *admissible* if it splits into a more regular H^1-part, and a Gaussian part, which we can bound in C^0. This is a quite natural assumption for SPDEs using the standard transformation with the stochastic convolution.

5. Unbounded domains. The key technical problem for deriving an approximation result via amplitude equations for (SH) on unbounded domains is the regularity of solutions. All previous results require too much regularity that we do not have in the stochastic setting. The theory for deterministic PDEs always uses uniform bounds in space on derivatives of the amplitude A. While the pioneering works [4, 11], which needed a uniform bound on the fourth derivative, were much improved since then, all results still need a uniform bound.

The previously stated Theorem 4.1 on large but still bounded domains needs a split condition in space for a more regular H^1-part and a Gaussian part only in C^0. Nevertheless, solutions are always uniformly bounded in space.

In two papers Klepel, Mohammed & Blömker [15, 12] discussed the case of spatially constant noise. Also in this setting they need too much regularity, as the solution of the amplitude equation (GL) has to be $H^{1/2+}$ in space and thus it is uniformly bounded.

We formulate the regularity that we expect for the amplitude A as a theorem:

THEOREM 5.1 (Lack of regularity). *With space-time white noise on the whole real line and with sufficiently smooth initial conditions the amplitude A solving (GL) is*

- *γ-Hölder-continuous in space and time only with $\gamma < 1/2$,*

- *unbounded in space, i.e.* $\|A(T,\cdot)\|_\infty = \infty$ *for all* $T > 0$.

To address the lack of regularity we can on one hand consider mild solutions, that take care of the problems with differentiability. On the other hand, we need weighted Hölder spaces which are defined by the norm (for some small $\kappa > 0$)

$$\|u\|_{C^{0,\alpha}_\kappa} = \sup_{L>1}\{L^{-\kappa}\|u\|_{C^{0,\alpha}([-L,L])}\} \; .$$

5.1. Mild formulation. Recall the Swift-Hohenberg equation:

$$\partial_t u = \underbrace{\mathcal{L}u + \nu\varepsilon^2 u}_{=:\mathcal{L}_\nu u} - u^3 + \varepsilon^{3/2}\partial_t W \tag{SH}$$

Its *mild solution* (see [5]), also called variation of constants formula, is

$$u(t) = \mathrm{e}^{t\mathcal{L}_\nu}u(0) - \int_0^t \mathrm{e}^{(t-s)\mathcal{L}_\nu}u^3(s)\,ds + \varepsilon^{3/2}W_{\mathcal{L}_\nu}(t)$$

with the *stochastic convolution* given by

$$W_{\mathcal{L}_\nu}(t) = \int_0^t \mathrm{e}^{(t-s)\mathcal{L}_\nu}dW(s) \; .$$

Remark 1. *Results for existence and uniqueness of mild solutions are usually straightforward using fixed-point theorems. Unfortunately this is not the case in the weighted spaces we are considering. The nonlinearity is unbounded and the semigroup is only regularizing in terms of differentiability but not in terms of weights. Thus the right-hand-side of the fixed-point equation is not a self-mapping.*

So the existence and uniqueness is first established for weak solutions via an approximation with large but bounded domains, and then one can show that weak solutions are sufficiently regular to be also mild. We will go not into details here, for those see [2].

5.2. Results for the linearized equation. This is the key stochastic result from Bianchi & Blömker [1]. It is one of the essential building blocks to prove a result for the residuum of the nonlinear equation.

Theorem 5.2 (Approximation). *Given the Wiener process* W *from (SH), there is a complex-valued Wiener process* $\mathcal{W}$ *for (GL) such that for any* $\kappa > 0$ *with probability almost* 1

$$\sup_{[0,\frac{T_0}{\varepsilon^2}]}\left\|\varepsilon^{\frac{3}{2}}W_{\mathcal{L}_\nu}(t,x) - \left[\varepsilon\mathcal{W}_{4\partial_x^2+\nu}(\varepsilon^2 t,\varepsilon x)\cdot \mathrm{e}^{ix} + c.c.\right]\right\|_{C^0_\kappa} \le C\varepsilon^{\frac{3}{2}-}$$

Definition 5.3. *We say that an* ε*-dependent event* $\mathcal{A}_\varepsilon$ *has* probability almost 1, *if for all* $p \ge 1$ *there is a constant* $C_p > 0$ *such that* $\mathbb{P}(\mathcal{A}_\varepsilon) \ge 1 - C_p\varepsilon^p$.

Let us remark that, in order to control the cubic term in the nonlinear result afterwards, we use the weighted supremum norm and not weaker (and actually much simpler) weighted L^2-norm.

Proof. We provide here only a brief sketch of the proof, for all the technical details see [1]. We rescale the stochastic convolution to the slow time ($T = \varepsilon^2 t$) and large space ($X = \varepsilon x$). Then we split into Fourier-modes larger than 0 and the complex conjugate corresponding to Fourier-modes smaller than 0. This defines the complex valued Wiener process, as there is one canonical process $\mathcal{W}$ such that we can summarize the difference as a single stochastic integral w.r.t. $\mathcal{W}$:

$$\varepsilon^{1/2} W_{\mathcal{L}}(T\varepsilon^{-2}, X\varepsilon^{-1}) - [\mathcal{W}_{4\partial_x^2}(T,X) \cdot \mathrm{e}^{iX/\varepsilon} + \text{c.c.}] = \int_0^T \mathcal{H}_\tau d\mathcal{W}(\tau) \cdot \mathrm{e}^{iX/\varepsilon} + \text{c.c.}$$

with a convolution operator $\mathcal{H}_\tau u = H_\tau \star u$ that mainly contains rescaled differences of the semigroups.

We use a technical estimate that allows to bound $\int_0^T \mathcal{H}_\tau d\mathcal{W}(\tau)$ in weighted Hölder spaces with small exponent and small weight in terms of bounds on the Fourier-transform $\hat{H}_\tau$ in spaces with slightly more regularity than $L^2([0,T_0] \times \mathbb{R})$.

The remaining and lengthy part of the proof shows the bounds for the norm of $\hat{H}_\tau$ in different areas of the Fourier-space. □

5.3. Nonlinear result. The full nonlinear result for (SH) and (GL) was treated in Bianchi, Blömker & Schneider [2]. It contains of two steps: first we bound the residual of the Swift-Hohenberg equation, and then via standard energy-type estimates we establish the approximation result.

5.3.1. Residual. Let A be a solution of (GL) with some conditions on $A(0,\cdot)$. It basically has to be in any $W^{1,p}_\rho$, $p > 1$ for an integrable weight ρ.

Definition 5.4 (Approximation). *For A from above, we define the approximation*

$$u_A(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{ix} + c.c.$$

The key step towards an approximation result is to bound the residual for u_A.

Definition 5.5 (Residual). *For u_A from above we define*

$$Res(t) = u(t) - \mathrm{e}^{t\mathcal{L}_\nu} u_A(0) + \int_0^t \mathrm{e}^{(t-s)\mathcal{L}_\nu} u_A^3(s)\, ds - \varepsilon^{3/2} W_{\mathcal{L}_\nu}(t)$$

We can prove the following result:

Theorem 5.6 (Residual). *For every small $\kappa > 0$ with probability almost* 1

$$\sup_{[0,T_0\varepsilon^{-2}]} \| Res \|_{C^0_\kappa} \le C\varepsilon^{3/2-}.$$

The proof can be found in [2, Theorem 5.9]. Its main strategy is as follows:

- use suitable exchange Lemmas to replace Swift-Hohenberg semigroups by Ginzburg-Landau semigroups,
- take advantage of Theorem 5.2 for stochastic convolution $W_{\mathcal{L}_\nu}$,
- notice that all terms of order $\mathcal{O}(\varepsilon)$ cancel due to (GL).

The key problem is that for the exchange Lemmas some regularity (or Gaussianity) is needed to estimate:

$$\mathrm{e}^{t\mathcal{L}}[D(\varepsilon x)\mathrm{e}^{ix}] \approx [\mathrm{e}^{4T\partial_X^2} D](\varepsilon x) \cdot \mathrm{e}^{ix} \quad \text{and} \quad \mathrm{e}^{t\mathcal{L}}[D(\varepsilon x)\mathrm{e}^{3ix}] \approx 0$$

If D is very smooth the proofs are straightforward, but here $D \in \{A^3, A|A|^2\}$ thus we only have Hölder-regularity.

5.3.2. Approximation. For a solution u of (SH) and the approximation u_A we define

$$R = u - u_A - Res$$

which solves

$$\partial_t R = \mathcal{L}_\nu R - (R + u_A + Res)^3 - {u_A}^3.$$

Use standard energy estimates in a weighted L^2-norm (for $\rho > 1$)

$$\|R\|^2_{L^2_{\rho,\varepsilon}} = \int_{\mathbb{R}} (1 + \varepsilon^2 x^2)^{-\rho/2} |R(x)|^2 dx$$

we obtain the following result.

THEOREM 5.7. *With probability almost* 1

$$\sup_{[0,T_0\varepsilon^{-2}]} \|u - u_A\|_{L^2_{\rho,\varepsilon}} \leq C\|u(0) - u_A(0)\|_{L^2_{\rho,\varepsilon}} + C\varepsilon^{1-}.$$

Details of the proof can be found in [2, Theorem 6.3].

6. A comment on pattern formation below criticality. Using amplitude equations, the question of pattern formation has a simple answer. Let us consider (SH) below the bifurcation, but sufficiently close. To be more precise, if σ is the noise-strength, then the distance from bifurcation should be $\mathcal{O}(\sigma^{4/3})$. In such scaling the effective dynamic is described by the amplitude equation, which is independent of σ. Thus the amplitude A is always $\mathcal{O}(1)$ and hence the pattern is visible.

7. Outlook. Let us conclude by commenting on some possible extensions of the results above.

7.1. Other types of noise in (SH). In the result presented here we only treat space-time white noise in both equations. But we could try more regular noise to overcome regularity barriers.

For colored, spatially smooth and translation invariant noise, it seems straightforward that in the approximation result (GL) still has space-time white noise, due to the rescaling both in space and time. Thus it does not help with the regularity.

If we consider trace class noise in $L^2(\mathbb{R})$ then we impose a decay-condition at infinity for (SH). But in that case, due to the spatial rescaling, we expect point-forcing in (GL).

In order to have noise that does not change under the space-time rescaling, one could try to consider algebraic decay of correlations. Here we expect a similar algebraic decay of correlations also for the noise in (GL). However, these types of noise seem to yield poor regularity of solutions, too.

7.2. Quadratic non-linearities. A more accurate Swift-Hohenberg model of the real Rayleigh-Bénard convection has a quadratic nonlinearity. In that setting the analysis is much more involved, as one has much more complicated interaction of Fourier-modes. But it is known from the deterministic results that even in the Rayleigh-Bénard phenomenon the amplitude equation is of Ginzburg-Landau type. Consequently, we expect a similar result to hold in the stochastic case, too.

7.3. Higher-dimensional models. Considering higher dimensional models is a difficult problem, as already in 2D the Ginzburg-Landau equation is no longer well-defined. See Hairer, Ryser & Weber [7] for a result on Allen-Cahn, which should generalize to (GL).

Consider for example Swift-Hohenberg in $\mathbb{R}^2$

$$\partial_t u = -(1+\Delta)^2 u + 4\partial_y^2 u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W \tag{2D-SH}$$

subject to space-time white noise or even smoother spatially colored noise. This formally has the amplitude equation

$$\partial_T A = -4\Delta A + \nu A - 3A|A|^2 + \partial_T \mathcal{W} \tag{2D-GL}$$

also with space-time white noise, which is no longer well-defined, as noted in the aforementioned [7]. Nevertheless, results like these are used in the applied literature.

Here in the spirit of [7], we can consider a smaller strength of the noise to obtain a meaningful limit. In that case the amplitude equation has no longer an additive noise, but additional deterministic terms should appear due to the presence of noise in (SH-2D) and averaging effects in the nonlinearity.

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