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Stabilization due to additive noise

Dirk Blömker

Stabilization due to noise is a well-known phenomenon, and there are numerous publications over the last decades. But most examples are for multiplicative noise only. Stabilization can arise somewhat artificially by adding Itô-noise, due to the Itô-Stratonovich correction, as only multiplicative Stratonovich-noise is neutral for the linear stability. In other cases stabilization arises due to averaging over stable and unstable directions. A celebrated example is Kapiza's problem of the inverted pendulum [13]. This averaging is also effective in case of deterministic rotation of the system [10]. But there are very few examples due to additive noise. Very nice is the blow-up through a small tube [15].

We consider two very simple examples of stochastic partial differential equations (SPDEs) close to bifurcation. Using the natural separation of time scales, one derives effective stochastic differential equations (SDEs) for the amplitudes of the dominating pattern. Due to averaging, the noise not acting directly on the dominant pattern may appear as a stabilizing deterministic correction to the SDEs.

Swift-Hohenberg equation. In a series of papers [11, 12], it was numerically and formally (using a center-manifold argument) justified that additive noise is capable of removing patterns in the one-dimensional Swift-Hohenberg equation. See also [9]. The Swift-Hohenberg equation is an SPDE given by

$$\text{(SH)} \qquad \partial_t u = -(1+\partial_x^2)^2 u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta$$

subject to periodic boundary conditions on $[0, 2\pi]$ and β being a real-valued Brownian motion. The constants σ and ν measure the noise strength and the distance from bifurcation, respectively. The eigenfunctions of the differential operator are $\sin(kx)$ and $\cos(kx)$, $k \in \mathbb{Z}$. The kernel span$\{\sin, \cos\}$ is the space of the dominant pattern. Using the Ansatz

$$u(t,x) = \epsilon A(\epsilon^2 t)\mathrm{e}^{ix} + c.c. + \epsilon\sigma Z(t) + \mathcal{O}(\epsilon^2)\ ,$$

with a fast Ornstein-Uhlenbeck (OU) process $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$ and complex-valued amplitude A, we obtain the following amplitude equation [6]:

$$\text{(A)} \qquad \partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2\ .$$

It is interesting that the amplitude equation for the dominant behavior is deterministic, and noise leads to a stabilizing deterministic correction. For a precise statement and proof of the approximation result see [6], which treats a more general situation. Numerical approximation [9] shows that any moment of the uniform in space and time error grows logarithmically with the time-interval, while moments of the error for a fixed time seem to stay small for very long times.

Averaging with error bounds. In formal calculations for the derivation of the amplitude equation, the additional constant terms arise from square of noise in $3A\sigma^2(\epsilon\partial_T\tilde{\beta})^2$, where $\tilde{\beta}(T) = \epsilon\beta(T\epsilon^{-2})$ is a rescaled Brownian motion on the slow time-scale $T = \epsilon^2 t$. In the proofs, using the mild formulation (i.e., variation of constants), we consider the fast OU-process

$$Z(T\epsilon^{-2}) = Z_\epsilon(T) = \epsilon^{-1}\int_0^T e^{-(T-s)\epsilon^{-2}} d\tilde{\beta}(s) \approx \epsilon\partial_T\tilde{\beta}(T)\ .$$

Crucial for the derivation of averaging with explicit error bounds is the following Lemma based on Itô's formula:

Lemma [4, 7] *Let X be a stochastic process with bounded initial condition and differentials, i.e. $dX = \mathcal{O}(\epsilon^{-r})dt + \mathcal{O}(\epsilon^{-r})d\tilde{\beta}$ and $X(0) = \mathcal{O}(\epsilon^{-r})$ for some $r > 0$. Then*

$$\int_0^T X(s)Z_\epsilon(s)^2 ds = \tfrac{1}{2}\int_0^T X(s)ds + \mathcal{O}(\epsilon^{1-2r})\ .$$

Similar results hold true for other even powers of Z_ϵ. For odd powers we have

$$\int_0^T X(s)Z_\epsilon(s)ds = \mathcal{O}(\epsilon^{1-r})\ , \qquad \int_0^T X(s)Z_\epsilon(s)^3 ds = \mathcal{O}(\epsilon^{1-3r})\ , \qquad \cdots$$

Note that $X = \mathcal{O}(f_\epsilon)$, if for all $p > 1$ and $T > 0$ there is a $C > 0$ such that

$$\mathbb{E}\sup_{s\in[0,T]}|X(s)|^p \le C{f_\epsilon}^p\ .$$

Burgers type equation. Stabilization effects were observed numerically in [1, 14] for an equation of the following type;

$$\text{(B)} \qquad \partial_t u = -(1+\partial_x^2)u + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma\epsilon\partial_t\beta\sin(2\cdot)$$

subject to Dirichlet boundary conditions on $[0,\pi]$ with dominant space span$\{\sin\}$. The highly degenerate noise acts only on the 2^{nd} mode by β. Consider:

$$u(t,x) = \epsilon a(\epsilon^2 t)\sin(x) + \epsilon\sigma Z(t)\sin(2x) + \mathcal{O}(\epsilon^2)$$

with fast OU-process $Z(t) = \int_0^t e^{-3(t-\tau)} d\beta(\tau)$. This is rigorously justified by a-priori estimates. In [4] we obtain the following amplitude equation:

$$da = (\nu - \tfrac{\sigma^2}{88})a\,dT - \tfrac{1}{12}a^3 dT + \tfrac{\sigma}{6} a \circ d\tilde{\beta} \tag{A2}$$

in Stratonovich sense, with rescaled Brownian motion $\tilde{\beta}(T) = \epsilon\beta(\epsilon^{-2}T)$. Obviously, 0 is stabilized for $\nu \in (0, \sigma^2/88)$. For a precise statement of the approximation result and its proof in a significantly generalized situation see [4]. Numerical justification in [9] verified the validity of the approximation for large times and moderate or even large ϵ.

Outlook – Open problems. We comment on a series of related results, generalizations and open problems. Interesting questions in regularity and scaling arise for example for Levy noise [5].

Averaging of martingals of the type $\int_0^T XZ_\epsilon^q d\beta$ is necessary for (B) with high-dimensional noise or for higher order corrections for (SH). The averaging is well known, but for error estimates in [4] we are based on Levy's characterization theorem, restricting the result to one-dimensional dominant modes.

Modulated patterns arise if the underlying domain is large or unbounded. Here we need to approximate by a modulated wave of the type $A(\epsilon^2 t, \epsilon x)e^{ix} + c.c.$, where A solves a SPDE of Ginzburg-Landau type. See [4, 8]. The truly unbounded space with space-time white noise is still open. Solutions seem to be both spatially unbounded and not sufficiently regular for the tools available.

The results presented are limited to long transient time-scales. For the approximation of long-time behavior in terms of invariant measures for (SH) see [2].

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Homogenization of random parabolic operators. Diffusion approximation

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(joint work with Marina Kleptsyna and Alexandre Popier)

The talk focuses on homogenization problem for divergence form second order parabolic operators whose coefficients are rapidly oscillating functions of both spatial and temporal variables. The corresponding Cauchy problem takes the form

$$\begin{gathered}\frac{\partial u^\varepsilon}{\partial t} = \mathrm{div}\Big(a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\Big)\nabla u^\varepsilon)\big), \qquad (x,t) \in \mathbb{R}^d \times (0,T) \\ u^\varepsilon(x,0) = g(x), \qquad u^\varepsilon\big|_{\partial G} = 0,\end{gathered} \tag{1}$$

with $g(x) \in L^2(\mathbb{R}^d)$. We assume that the coefficients of $a(z,s)$ are periodic functions of spatial variables while their dependence of time is random stationary ergodic, $\alpha > 0$. Moreover, the matrix $a(z,s)$ is real symmetric, uniformly bounded and positive definite.

It was proved in [1], [2] that the solutions of the original problem converges almost surely to a deterministic limit, the limit function being a solution of homogenized equation with constant coefficients:

$$\frac{\partial u^0}{\partial t} = \mathrm{div}(a^{\mathrm{eff}}\nabla u^0), \qquad u^0(x,0) = g(x), \tag{2}$$

The question of interest is the asymptotic behaviour of the normalized difference of the original and homogenized solutions.

It turns out that the limits behaviour of the said normalize difference depends crucially on whether $\alpha < 2$, or $\alpha = 2$, or $\alpha > 2$. In the talk we mostly dwell on the the self-similar case $\alpha = 2$.

In order to formulate the diffusion approximation result we need an auxiliary function, so-called corrector.

Lemma (see [3]). *The equation*

$$\partial_s \chi(z,s) = \mathrm{div}_z\big(a(z,s)[\nabla_z \chi(z,s) + \mathbf{I}]\big)$$

has a stationary in s and periodic in z solution. The solution is unique up to an additive (random) constant.