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Amplitude Equations – natural slow-fast systems

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Focusing on the stochastic Swift-Hohenberg equation (SH) only, we review results on the rigorous error estimates for amplitude equations. We discuss the impact of various models of the noise, together with open problems.

Introduction. Complicated models near a *change of stability* generate slow-fast systems in a natural way. The dominant pattern (or modes) evolve on a *slow time-scale*, while stable pattern decay and disappear on a *fast time-scale.* The evolution is then given by simplified models for the amplitudes of dominant pattern, the so called *amplitude equations.* There are many examples of formal derivation for such equations. For a review see [7].

For PDEs on bounded domains the theory of invariant or center manifolds is available, where solutions are well approximated by an ODE on the manifold. Unfortunately, invariant manifolds for stochastic PDEs move in time. Moreover, there is a lack of center manifold theory. This is similar to PDE on unbounded domains, where a whole band of eigenvalues changes stability, and amplitude or modulation equations are successfully applied. See [8, 10, 11].

Using amplitude equations, our aim is to understand the impact of noise on the dominant pattern and how noise is transported by the nonlinearity. For simplicity we consider only (SH) on the real line. In pattern formation (SH) is a celebrated toy model for the convective instability in the Rayleigh-Bénard model.

$$\text{(SH)} \qquad \partial_t u = \mathcal{L}u + \nu\varepsilon^2 u - u^3 + \xi_\varepsilon \,,$$

where $\mathcal{L} = -(1+\partial_x^2)^2$. The dominant modes are $\mathcal{N} = \mathrm{span}\{\sin, \cos\}$. Moreover ξ_ε is some small noise process, which may change for different applications.

Slow-Fast System. (SH) naturally generates a system, given by a slow SDE and a fast SPDE. To illustrate this, consider $\xi_\varepsilon = \varepsilon^2 \partial_t W$ in (SH). Split $u(t) = \varepsilon v_c(\varepsilon^2 t) + \varepsilon v_s(\varepsilon^2 t)$ with $v_c \in \mathcal{N}$ and $v_s \perp \mathcal{N}$. Then

$$\text{(SLOW)} \qquad \partial_T v_c = \qquad \nu v_c - P_c(v_c + v_s)^3 + \partial_T \tilde{W}_c \,,$$

$$\text{(FAST)} \qquad \partial_T v_s = \varepsilon^{-2}\mathcal{L}v_s + \nu v_s - P_s(v_c + v_s)^3 + \partial_T \tilde{W}_s \,,$$

where P_c projects onto $\mathcal{N}$, $P_s = I - P_c$, and $\tilde{W}(T) = \varepsilon W(T\varepsilon^{-2})$ is a rescaled version of the driving Wiener process W.

Full Noise. Consider (SH) subject to periodic boundary conditions on $[0, 2\pi]$ with noise $\xi_\varepsilon = \varepsilon^2 \partial_t W$, where W is some suitable Q-Wiener process W.

Theorem 1 (Approximation, [1])**.** *Consider $u(0) = \varepsilon a(0) + \varepsilon^2 \psi(0)$ with $a(0)$ and $\psi(0)$ both $\mathcal{O}(1)$. Let $a(T) \in \mathcal{N}$ solve*

$$\partial_T a = \nu a - P_c a^3 + \partial_T \tilde{W}_c \,,$$

and let $\psi(t) \perp \mathcal{N}$ be an OU-process solving $\partial_t \psi = \mathcal{L}\psi + \partial_t W_s$. Then $u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2 \psi(t) + \mathcal{O}(\varepsilon^{3-})$ for $t \in [0, T_0 \varepsilon^{-2}]$.

The Approximation remains true for invariant measures. Moreover, Attractivity verifies that any solution scales as needed for the Theorem after some time.

Degenerate noise. Additive noise may lead to stabilization (or a shift of bifurcation) of dominant modes (pattern disappears). See [9]. Consider the noise $\xi_\varepsilon = \sigma\varepsilon\partial_t \beta$ for some real-valued Brownian motion β.

$$\text{Ansatz:} \qquad u(t,x) = \varepsilon A(\varepsilon^2 t)\mathrm{e}^{ix} + c.c. + \varepsilon Z(t) + \mathcal{O}(\varepsilon^2)$$

with some fast OU-process $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$ and a complex-valued amplitude A. Using explicit averaging results with error bounds, we obtain the Amplitude equation [5]

$$\text{(A)} \qquad \partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \,.$$

Open Problems. For higher order corrections [5] or quadratic nonlinearities and degenerate noise [3], averaging results with explicit error estimates for integrals of the type $\int_0^T X(\tau) Z(\tau\varepsilon^{-2})^q d\beta(\tau)$ are necessary. These results [3] are based on Levy's representation theorem, which restricts the result to $\dim(\mathcal{N}) = 1$. It remains open, how to obtain error estimates for the limit, if $X(t) \in \mathbb{R}^n$, $n > 1$.

Interesting results arise for (SH) on large [2] (full noise) or unbounded domains [6] (degenerate noise). In both cases the complex amplitude A is slowly modulated in space and given by a stochastic Ginzburg-Landau PDE. Here many questions are still open, due to the lack of regularity and unboundedness of solutions of the stochastic Ginzburg-Landau equation on $\mathbb{R}$ with space-time white noise.

Another interesting question [4] is for non-Gaussian noise without scale invariance. For the driving process L we need limits of $\varepsilon^{\alpha} L(\epsilon^{-2}t)$ with error estimates.

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Nonlinear Dispersive Equations, Solitary Waves and Noise

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(joint work with A. Debussche, R. Fukuizumi, E. Gautier)

Nonlinear dispersive waves in general, and solitons in particular are universal objects in physics. They may as well describe the propagation of certain hydrodynamic waves, as localized waves in plasma physics, signal transmission in fiber optics, or phenomena such as energy transfer in excitable molecular systems. In all those cases, the formation of stable, coherent spatial structures have been experimentally observed, and may be mathematically explained by the theory of nonlinear integrable (or soliton) equations. However, none of those systems is exactly described by soliton equations, and those equations may only be seen as asymptotic models for the description of the physical phenomena. Moreover, as soon as microscopic systems are under consideration, thermal fluctuations may not be negligible. They give rise in general to stochastic fluctuations in the corresponding model, and their interaction with the waves has to be studied. In some other situations, the underlying asymptotic model is not even an integrable equation, even though it is a nonlinear dispersive equation. Solitary waves may still exist in this latter situation, and even if the mathematical theory is then much less