Computable A-Posteriori Bounds for SPDEs

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The main question in this work is *"How to numerically compute something that might not exist?"*. Based on A-posteriori error estimates we present a method to obtain existence of globally unique solutions together with error estimates, for equations where this is analytically not known.

Motivation. Our result is motivated by equations where the global existence of solutions is not known, and thus global a-priori estimates are not available for numerical analysis. The final goal of this approach is the three-dimensional Navier-Stokes equation. But for the first deterministic results we focused on a somewhat simpler equation [3] from surface growth with similar properties.

For the latter in [2, 10] a-posteriori analysis was used for the deterministic surface growth PDE to prove numerically the regularity of solutions and thus the global existence and uniqueness. The residual was evaluated using numerical data and analytically an error estimate in terms of the residual was given.

Stochastic Example. In [4] we consider the following simpler SPDE on the Hilbert-space $H = L^2([0,\pi])$

$$du = [Au + F(u)]dt + dW \qquad u(0) = u_0 \ . \tag{1}$$

subject to Dirichlet boundary conditions for the Laplacian A. The perturbation W is some cylindrical Q-Wiener process with bounded covariance operator. Finally, F is the locally-Lipschitz nonlinearity $F(u) = -u^3$.

For (1) the asymptotic convergence results of numerical schemes are well known See for example [9, 8, 6] or [1] for a truncated scheme. Moreover, there is no problem with existence and uniqueness of solutions. See for example [5]. Nevertheless, we study it as a starting point for stochastic results.

Discretization. For the spatial discretization we use the spectral Galerkin-scheme, where P_N is the projection onto the first N Fourier modes. Moreover, for the time-discretization we use an accelerated exponential scheme introduced in [7, 6]. Unfortunately, no sharp asymptotic rate is known for our scheme.

We use a fixed step-size $h > 0$ and for a fixed realization ω we obtain by a random number generator in principle exact values of $\{P_N Z(hk)\}_{k\in\mathbb{N}}$, defined by

$$Z_0 = 0 \ , \qquad Z_{k+1} = e^{hA} Z_k + X_{k+1} = \sum_{j=1}^{k+1} e^{h(k+1-j)A} P_N X_j$$

with i.i.d. $\mathbb{R}^N$-valued Gaussian random variables

$$X_{k+1} = P_N \int_{hk}^{h(k+1)} e^{(h(k+1)-s)A} dW(s) \sim \mathcal{N}(0, P_N \int_0^h e^{2sA} ds P_N) \ .$$

Given the Z_k, the numerical method provides a realization of the approximation $\{u_k\}_{k=1,\dots,M} \subset P_N H$, which is defined recursively as

$$u_n = e^{Ah} P_N u_{n-1} + \int_0^h e^{A(h-s)} ds F_N(u_{n-1}) + X_n \ .$$

Interpolation linearly in time yields the approximation φ. The main result is a bound on the conditional mean-square error given the numerical data:

$$\mathbb{E}[\|u - \varphi\|^2 \mid \{X_k\}_{k\in\mathbb{N}}] = \mathbb{E}[\|u - \varphi\|^2 \mid \{Z_k\}_{k\in\mathbb{N}}] = \text{ small} \ ,$$

which is not an asymptotic result, but one that holds for the given approximation. The term "small" depends on the the numerical data, and we evaluate this part numerically. The general philosophy is to evaluate as much as possible of the error bounds using the numerical data, and only rely on analytic estimates if no numerical evaluation is possible.

Residual. The residual measures the quality of an arbitrary numerical approximation φ. For $t \in (0, T)$ it is defined as

$$\text{Res}(t) = \varphi(t) - e^{At}\varphi(0) - \int_0^t e^{A(t-s)} F(\varphi(s)) ds - Z(t). \tag{2}$$

This contains terms depending on ...

- ... the data u_k and Z_k, which we evaluate only numerically
- ... infinite dimensional parts from the stochastic convolution, which is independent of the data and expectations are evaluated analytically.
- ... an Ornstein-Uhlenbeck bridge between discretization times, which is also independent of the data and Gaussian.

Approximation. The numerical data mainly comes into play via the residual. We additionally need to quantify the continuous dependence of solutions on additive perturbations like the residual. By putting $d(t) = u(t) - \varphi(t)$ we have

$$d(t) = u(t) - \varphi(t) = e^{At} d(0) + \int_0^t e^{A(t-s)} (F(u(s)) - F(\varphi(s))) ds + \text{Res}(t)$$

with $d(0) = (I - P_N)u_0$. Let $r = d - \text{Res}$ which is the solution of the following differential equation

$$\partial_t r = Ar + F(r + \varphi + \text{Res}) - F(\varphi).$$

Recall that $\text{Res}(0) = 0$, so $r(0) = d(0) = Q_N u_0$. Now we use standard a-priori estimates for the equation for r.

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