Abstracts

Stabilization due to additive noise

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Amplitude Equations describe essential dynamics of a complicated (stochastic) partial differential equation near a change of stability. The approximations are derived using the natural separation of time-scales near a bifurcation for a multi-scale analysis. Here we focus on results for equations with locally quadratic nonlinearity and show various applications (stabilization, random invariant manifolds, modulated pattern)

1. General setting - examples

Consider an Equation of the type

$$du = (Lu + B(u,u) + \epsilon^2 Au + B(u,u))dt + \epsilon^2 dW \ ,$$

where L is a non-positive operator with non-empty kernel $\mathcal{N}$, B is a bilinear operator and Au a linear perturbation. The noise $\partial_t W$ is Gaussian and white in time. Typical examples are

- Burgers: $\partial_t u = \partial_x^2 u + \nu u + u\partial_x u + \sigma\xi$
- Kuramoto Shivashinsky: $\partial_t u = -\Delta^2 u - \nu\Delta u + |\nabla u|^2 + \sigma\xi$
- Surface Growth: $\partial_t u = -\Delta^2 u - \nu\Delta u - \Delta|\nabla u|^2 + \sigma\xi$

or the Rayleigh Benard system (3D-Navier-Stokes coupled to a heat equation).

2. The general approximation result

One aim of amplitude equations is to show on large time scales:

$$u(t) = \epsilon a(\epsilon^2 t) + \mathcal{O}(\epsilon^2)$$

where on the slow time-scale $a \in \mathcal{N}$ solves an equation of the type

$$da = (\nu a + \mathcal{F}(a))dT + d\beta \ ,$$

where $\mathcal{F}$ is a cubic, β Brownian motion in $\mathcal{N}$ given by the projection $\beta = P_c W$ of W onto $\mathcal{N}$. Note that in the case $P_c W = 0$, the dynamics is essentially deterministic, as the dominant modes are only driven by noise acting directly on the dominant modes. We could consider larger noise then.

3. Stabilization by degenerate additive noise

In a special case [10], rigorously verified in [6], for highly degenerate noise the amplitude equation for $u(t) \approx a(\epsilon^2 t)\sin$ (in Stratonovic sense) is

$$da = (\nu - \tfrac{\sigma^2}{88})a\ dT - \tfrac{1}{12}a^3 dT + \tfrac{\sigma}{6} a \circ d\beta, \tag{A}$$

In the SPDE $\sigma\epsilon$ is strength of the noise and ν distance from bifurcation. For sufficiently large additive noise in the SPDE the dominating mode is stabilized (i.e. the bifurcating pattern destabilized).

As numerical example consider the following Burgers-type SPDE

$$\text{(B)} \qquad \partial_t u = (\partial_x^2 + 1)u + \epsilon^2 u + u\partial_x u + \sigma\epsilon\xi \; ,$$

where $u(t,x) \in \mathbb{R}$ for $t > 0$, $x \in [0,\pi]$ subject to Dirichlet b.c. Set $\epsilon = 0.1$ and use highly degenerate noise $\xi(t,x) = \partial_t\beta(t)\sin(2x)$, which acts only on the second Fourier mode, where $\beta(t)$ is a standard one-dimensional Brownian motion.

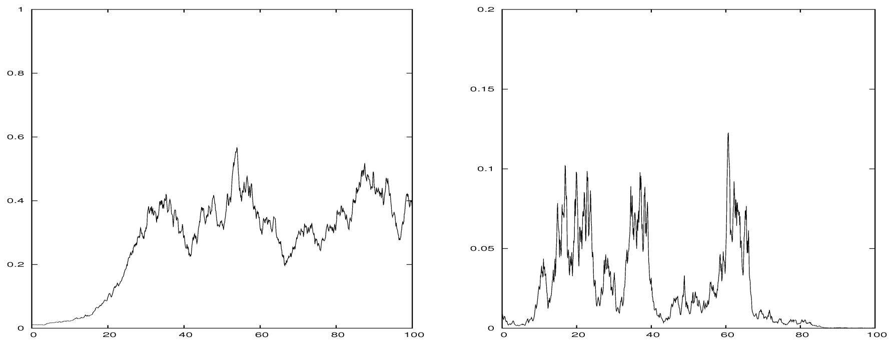

Figure 1. Time evolution of the first Fourier mode of the solution of a Galerkin truncation of (B) for $\sigma = 2$ (left) and for $\sigma = 10$ (right) for a single typical realization. Stabilization by additive noise on that mode is clearly seen (cf. [8]).

4. Large domains - modulated pattern

Amplitude or modulation equations are a standard tool for spatially extended deterministic PDEs, which help to overcome the lack of center manifold theory in such a setting. Despite long use in physics (cf. [9]), starting from [12, 11] (see also [13]) the rigorous theory of modulation equations has only been developed in the last two decades.

Little is known about modulation equations for extended SPDEs. There is some work in this direction, for the example the study of the motion of solitons in the stochastic Korteweg-de-Vries [1] and the derivation of the stochastic Ginzburg-Landau equation as the amplitude equation for the Swift-Hohenberg equation [7].

Consider as an example an equation of Kuramoto Sivashinsky type:

$$\partial_t u = -(\partial_x^2 + 1)^2 u - \nu\epsilon^2\partial_x^2 u + |\partial_x u|^2 + \epsilon^{3/2}\xi$$

with $u = u(t,x)$, $t > 0$, $x \in \mathbb{R}$, and ξ is space-time white noise.

The expected result is a modulated pattern

$$u(t,x) \approx \epsilon A(\epsilon^2 t, \epsilon x)e^{ix} + \text{compl.conj.} + \mathcal{O}(\epsilon^2) \; ,$$

where $A \in \mathbb{C}$ solves a complex Ginzburg-Landau equation of the type

$$\partial_T A = \partial_X^2 A + \nu A - cA|A^2 + \eta \; ,$$

and η is complex valued space-time white noise. See [7, 14].

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Random attractors for stochastic Navier-Stokes equations in some unbounded domains

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In this talk I will present new developments in the theory of infinite dimensional random dynamical systems. The starting point is my recent paper with Y. Li [4]. In that paper we constructed a RDS for the stochastic Navier-Stokes equations in some unbounded domains $\mathcal{O} \subset \mathbb{R}^2$. We proved that that RDS is asymptotically compact, what roughly means that if a sequence x_n of initial data is bounded in the energy Hilbert space H and the sequence of initial times $(-t_n)$ converges to $-\infty$, then the sequence $u(0, -t_n, x_n)$, where $u(t, s, x)$, $t \geq s$ is a solution of the SNSEs such that $u(s, s, x) = x$, is relatively compact in H. A RDS satisfying this condition is called an asymptotically compact one. We also proved that for any asymptotically compact RDS on a separable Banach space, the Ω-limit set of any bounded *deterministic* set B is non-empty, compact and forward invariant with respect to the RDS (and attracting the set B). We were not able to show existence of a random attractor, as such a proof would require that there exists a family of closed and bounded random sets such that each element of it is absorbed