CHECKING LINEAR RESPONSE THEORY IN DRIVEN BISTABLE SYSTEMS

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The validity of linear response theory to describe the response of a nonlinear stochastic system driven by an external periodical time dependent force is put to a critical test. A variety of numerical and analytical approximations is used to compare its predictions with numerical solutions over an extended parameter regime of driving amplitudes and frequencies and noise strengths. The relevance of the driving frequency and the noise value for the applicability of linear response theory is explored for single and multi-frequency input signals.

1. Introduction

The response ofdissipative physical systems to small amplitude external perturbations is usually described with the powerful tools of linear response theory (LRT) [1]. This method has found widespread application for the phenomenon of *Stochastic Resonance* which describes the constructive role ofnoise for the transduction and detection ofweak information-carrying signals in systems that possess some sort ofthreshold [2, 3, 4]. In this context, the stochastic dynamics ofa rocked bistable potential has served as an archetype to study many features of this exciting and prominent physical effect [2]. For sufficiently small external forces, one can introduce a smallness parameter proportional to the external amplitude. The evolution equation for the probability density describing the system dynamics can be treated to first order in perturbation theory. Thus, one finds that the effect of the perturbation on the evolution ofthe relevant system variables can be described in terms ofsmall deviations fom their behaviors in the unperturbed system. In particular, for long times and for systems which in the absence of driving reach an equilibrium distribution, LRT provides an approximate expression for the probability distribution where the deviations with respect to the equilibrium distribution in the absence ofdriving are proportional to the driving amplitude. Consequently, LRT predicts that the first moment ofthe probability distribution scales linearly with the driving amplitude.

When the driving force is periodic in time, the linearity of the evolution equation for the probability density allows us to use the Floquet ideas [5]. The main consequence is that the long time probability distribution is periodic in time with the same period as the external driving. This is a general result valid for any value ofthe input amplitude. Consequently, the first moment will also be periodic in time and it can be expanded in a Fourier series containing the fundamental and higher order harmonics. LRT predicts that the only harmonics that should show up in the system response are those already present in the input.

In recent work [6], we have considered the dynamics ofa noisy bistable system subject to the action ofan external driving. We have pointed out the relevance ofparameters other than the amplitude ofthe driving force, for the validity of LRT. We showed that for a periodic single-frequency external force, the validity of LRT depends not just on the amplitude ofthe driving term but also crucially on its frequency. The purpose of this work is to further extend the analysis in two aspects. First, in [6], the 2-mode approximation for the susceptibility described in [7] was used. This analytical expression is not valid when the noise strength is large. In this work, we evaluate the susceptibility using a numerical procedure which is not subject to the limitations ofsmall noise strengths. This allows us to explore the system response in a wider range ofparameter values. Limitations ofthe LRT description can not be ascribed to an invalid estimate ofthe system susceptibility. Second, we consider the response ofthe system when the external driving contains many frequencies [8]. In particular, we analyze an external driving of the form depicted in Fig. 1. Driving forces of this kind have been recently considered by Gingl et al. [9] in their studies ofthe signal-to-noise ratio in double wells.

After the introduction of the model in Sec. 2 we study analytically the system response in Sec. 3 within LRT and check our predictions ofLRT against the numerically obtained behavior for a sinusoidal driving, see Sec. 4, and a multifrequency input in Sec. 5. Our conclusions are presented in Sec. 5.

2. TheModel

Let us consider a system characterized by a single degree offeedom, x, whose time evolution is governed by the nonlinear Langevin equation (in dimensionless form),

$$\dot{x}(t) = x(t) - x^3(t) + F(t) + \eta(t)\,, \tag{1}$$

where $F(t)$ represents an external periodic signal with period T and $\eta(t)$ is a Gaussian white noise with zero average and $\langle \eta(t)\eta(s)\rangle = 2D\delta(t-s)$. The corresponding linear Fokker-Planck equation (FPE) for the probability density $P(x,t)$ reads

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\big\{(-x + x^3 - F(t))P\big\} + D\frac{\partial^2 P}{\partial x^2}\,. \tag{2}$$

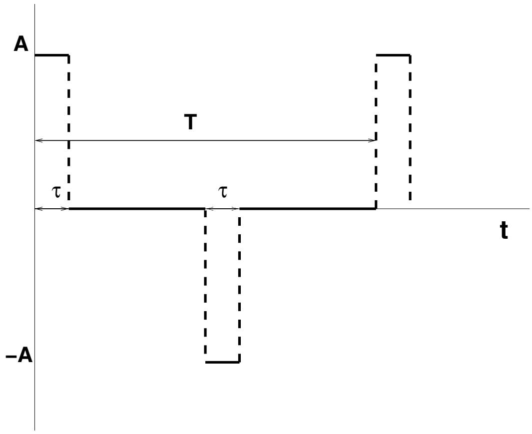

Fig. 1. Sketch of a periodic external force containing many harmonics of the fundamental frequency.

The unperturbed system has an equilibrium distribution ofthe form

$$P_{eq}(x) = N \exp\left(-\frac{U_0(x)}{D}\right), \tag{3}$$

where N is a normalization constant and $U_0(x)$ is the unperturbed potential

$$U_0(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \tag{4}$$

This potential has two minima located at $x_m = \pm 1$ and a maximum at $x_M = 0$, with a barrier height of0 .25.

The analysis ofthe dynamics is simplified by making use oftwo important theorems: the H-theorem, which ensures the existence ofa unique long time distribution function $P_\infty(x,t)$ [10, 11] and the Floquet theorem, which guarantees that $P_\infty(x,t)$ is periodic in time with the same period as the external force [5]. Then, the long time distribution can be expanded in the Fourier series

$$P_\infty(x,t) = \sum_{n=0}^{\infty} [H_n(x)\cos(n\Omega t) + I_n(x)\sin(n\Omega t)], \tag{5}$$

where the quantity $\Omega = 2\pi/T$ denotes the fundamental angular frequency of the external driving signal.

The first moment ofthe probability distribution can be used to characterize the system response to the external driving. In the long time limit, $\langle x(t)\rangle_\infty$ becomes a periodic function of time. It can be written as a Fourier series,

$$\langle x(t)\rangle_\infty = \sum_{n=1}^{\infty} [M_n\cos(n\Omega t) + N_n\sin(n\Omega t)]. \tag{6}$$

Unfortunately, the two powerful theorems cited above equation (5) are not enough for a detailed analysis ofthe system behavior under the action ofthe external driving. The nonlinearity ofthe Langevin dynamics precludes the exact analytical knowledge ofthe probability distribution and, therefore, the coefficients M_n and N_n in the Fourier expansion are unknown. In order to gain quantitative information, one needs to turn to approximations ofanalytical or numerical character.

3. The Linear Response Theory Description of the System Response

The basic quantity ofLRT is the system response function, $\chi(t)$. It is related to the equilibrium time correlation function ofthe system in the absence ofexternal driving, $K(t)$, via the fluctuation-dissipation theorem (FDT) [7], i.e.,

$$\chi(t) = \begin{cases} 0 & : \quad t \leq 0 \\ -\frac{1}{D}\dot{K}(t) & : \quad t > 0\,. \end{cases} \tag{7}$$

In LRT the long time average value $\langle x(t)\rangle_\infty^{LRT}$ is given by

$$\langle x(t)\rangle_\infty^{LRT} = \int_0^\infty \mathrm{d}\tau\, \chi(\tau) F(t-\tau)\,. \tag{8}$$

The external driving $F(t)$ can be expanded in a Fourier series as

$$F(t) = \sum_{n=1}^{\infty} (f_n \cos(nOmegat) + g_n \sin(n\Omega t))\,, \tag{9}$$

with the Fourier coefficients, f_n and g_n, given by

$$\begin{aligned} f_n &= \frac{2}{T}\int_0^T \mathrm{d}t\, F(t)\cos(n\Omega t)\,, \\ g_n &= \frac{2}{T}\int_0^T \mathrm{d}t\, F(t)\sin(n\Omega t)\,. \end{aligned} \tag{10}$$

Here, we are assuming that the cycle average ofthe external driving over its period is zero. Insertion ofthe Fourier expansion into Eq. (8) leads to

$$\langle x(t)\rangle_\infty^{LRT} = \sum_{n=1}^{\infty} [M_n^{LRT}\cos(n\Omega t) + N_n^{LRT}\sin(n\Omega t)]\,, \tag{11}$$

where the coefficients M_n^{LRT} and N_n^{LRT} are given by

$$M_n^{LRT} = f_n\chi_n^r - g_n\chi_n^i\,; \qquad N_n^{LRT} = f_n\chi_n^i + g_n\chi_n^r\,. \tag{12}$$

In these formulas, we have introduced χ_n^r and χ_n^i defined as

$$\chi_n^r = \int_0^\infty \mathrm{d}\tau\, \chi(\tau)\cos(n\Omega\tau)\,, \tag{13}$$

$$\chi_n^i = \int_0^\infty \mathrm{d}\tau\, \chi(\tau)\sin(n\Omega\tau)\,. \tag{14}$$

Use ofthe FDT in the above expressions allows us to write immediately

$$\chi_n^r = \frac{\langle x^2\rangle_{eq} - n\Omega \int_0^\infty \mathrm{d}t\, K(t)\sin(n\Omega t)}{D}, \tag{15}$$

$$\chi_n^i = \frac{n\Omega}{D}\int_0^\infty \mathrm{d}t\, K(t)\cos(n\Omega t)\,. \tag{16}$$

For nonlinear problems, explicit expressions for $K(t)$ are unknown, but useful approximations have been presented in the literature [12]. For the bistable potential in Eq. (4), we can use the two-mode approximation ofJung and H¨anggi [7]. It is based on a large difference in the time scales associated to inter-well and intra-well motions, and it is expected to be valid for small values of the noise strength D. With this model, one gets

$$K(\tau) = g_1 \exp(-\lambda_1\tau) + g_2 \exp(-\alpha\tau)\,, \tag{17}$$

where [13]

$$\lambda_1 \approx \frac{\sqrt{2}}{\pi}\left(1 - \frac{3}{2}D\right)\exp[-1/(4D)] \tag{18}$$

and $\alpha = 2$. The weights, g_1 and g_2, can be obtained from the moments of the equilibrium distribution in the absence ofdriving using the expressions

$$g_2 = \frac{\lambda_1\langle x^2\rangle_{eq}}{\lambda_1 - \alpha} + \frac{\langle x^2\rangle_{eq} - \langle x^4\rangle_{eq}}{\lambda_1 - \alpha}, \tag{19}$$

$$g_1 = \langle x^2\rangle_{eq} - g_2\,. \tag{20}$$

Ifthe noise strength D is small, so that the time scales ofinter and intra-well motions are well separated, the 2-mode approximation to $K(t)$ (Eq. (17)) can be used to evaluate the coefficients χ_n^r, χ_n^i needed to describe the behavior of $\langle x(t)\rangle_\infty^{LRT}$. When the separation between the time scales associated to inter- and intra-well motions is not clear, the 2-mode approximation to $K(t)$ is unreliable. Fortunately, the correlation function in the absence of driving can also be evaluated numerically from the FPE. (See [14] for details). Comparison of the numerical results with those obtained using Eq. (17) indicates that for noise strengths $D < 0.2$ the 2-mode formula gives an adequate approximation.

The LRT leads to explicit predictions about the behavior ofthe system response to an external driving force. Clearly, ifthe amplitude ofthe driver is infinitesimal, i.e., smaller than any other parameter in the problem, there is no doubt about its applicability. But for finite values of the driving amplitude, the reliability of the LRT description for different regions of the parameter values has to be investigated. Differences in the observed response with respect to the LRT predictions in a range ofparameter values indicate a failure ofLRT.

4. TheCaseof Monochromatic Driving

A very large fraction of the papers devoted to the study of *Stochastic Resonance* consider the system response to an external force of the type $F(t) = A\cos\Omega t$, with

$\Omega = 2\pi/T$. In this case, the general formulas obtained within the LRT simplifies considerably. All the coefficients in the expansion in Eq. (9) vanish except one, $f_1 = A$. The average value can be written as

$$\langle x(t)\rangle_\infty^{LRT} = A\int_0^\infty \mathrm{d}\tau\, \chi(\tau)\cos\Omega(t-\tau) = a\cos(\Omega t - \phi)\,, \tag{21}$$

where the response amplitude a is related to the driver amplitude by

$$a = A\sqrt{(\chi_1^r)^2 + (\chi_1^i)^2}\,, \tag{22}$$

and the phase lag ofthe response with respect to the input signal , ϕ, lies in the interval $0 \le \phi \le \pi/2$. It is given by

$$\phi = \arctan\frac{\chi_1^i}{\chi_1^r}\,. \tag{23}$$

Within the 2-mode approximation to $K(t)$, the response amplitude and phase lag are given by

$$a = \frac{A}{D}\left[\frac{g_1^2\lambda_1^2}{\lambda_1^2+\Omega^2} + \frac{g_2^2\alpha^2}{\alpha^2+\Omega^2} + \frac{2g_1g_2\lambda_1\alpha(\lambda_1\alpha+\Omega^2)}{(\lambda_1^2+\Omega^2)(\alpha^2+\Omega^2)}\right]^{\frac{1}{2}}\,, \tag{24}$$

and

$$\phi = \arctan\frac{\frac{g_1\lambda_1\Omega}{\lambda_1^2+\Omega^2} + \frac{g_2\alpha\Omega}{\alpha^2+\Omega^2}}{\frac{g_1\lambda_1^2}{\lambda_1^2+\Omega^2} + \frac{g_2\alpha^2}{\alpha^2+\Omega^2}}\,. \tag{25}$$

The result for the amplitude in Eq. (24) reveals already the limiting validity of linear response theory: Note that for small noise intensity, i.e. $D \ll A$, (weak noise regime) the amplitude a assumes large values. In this latter regime the weak noise theory must be invoked, see in Refs. [15, 16]. Likewise, linear response theory does break down for small angular driving frequency Ω at weak noise obeying $A < D$, see also Ref. [6].

Regardless ofthe way that $K(t)$ is evaluated, it is clear that the LRT description ofthe response ofthe system to a monochromatic driving predicts that: i) the first moment $\langle x(t)\rangle_\infty$ should contain a single harmonics with the frequency of the driving force; ii) the output amplitude should behave linearly with A. In order to check these predictions, we have resorted to numerical solutions ofthe FPE, Eq. (2). The technique is based on the use ofthe split propagator method ofFeit *et al.* [17] and detailed in [18]. From the numerical solution ofthe FPE we can easily obtain the time dependence of $\langle x(t)\rangle_\infty$. As this is a periodic function of time, its Fourier components can be obtained by numerical quadrature.

In Fig. 2 , we show the amplitudes ofthe harmonics appearing in the Fourier analysis of $\langle x(t)\rangle_\infty$ obtained from the numerical solution of the FPE for several sets ofparameter values. In all cases we take for the input amplitude the value $A = 0.35$ that is just slightly below its threshold value ($A_T \approx 0.37$). For the two values of the external frequency, $\Omega = 0.1$ and $\Omega = 0.001$, we consider two illustrative noise strengths: $D = 0.1$ and $D = 1.0$. For both frequencies, increasing the value of D

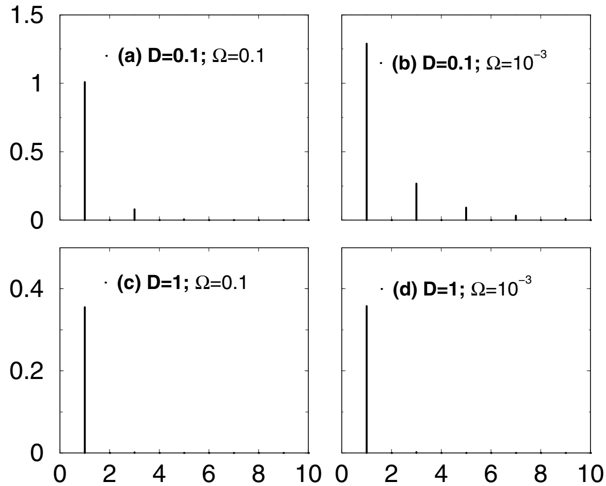

Fig. 2. Amplitudes of the Fourier modes of $\langle x(t)\rangle_\infty$ for a single frequency input signal with amplitude $A = 0.35$ and several values of D and Ω. The order of the harmonics is indicated on the horizontal axis.

reduces the number ofharmonics contributing to the overall behavior of $\langle x(t)\rangle_\infty$. Actually, for the value of A considered, and $D = 1$, the response shows essentially the fundamental harmonics, regardless of the frequency value. (See panels (c) and (d) in Fig. (2)). On the other hand, for $D = 0.1$, the importance ofthe fequency is manifested. From inspection of panels (a) and (b) in Fig. (2) it is clear that as the frequency gets smaller, the system response contains more harmonics than the input. This is an indication ofthe breakdown ofLRT to describe the system response at low external frequencies and finite driving amplitudes, in agreement with the idea that LRT requires that $A/D < 1$.

As mentioned above, a signature ofthe validity ofLRT is that the amplitude ofthe response should scale linearly with that ofthe input. In Fig. 3, we show the behavior ofthe output amplitude, A_{out}, provided by the numerics (circles and the connecting dashed lines), and the behavior ofthe output amplitude obtained within LRT, a (solid line), for a wide range of input amplitudes A, and different values of D and Ω. The results shown here further corroborate the comments in the above paragraph. For large values ofthe driving fequency, $\Omega = 0\ .1$, (panels (a) and (c)) the difference between the numerical amplitude, A_{out}, and the LRT amplitude, a, is very small even for quite large values of A. Large values of D favor the extension ofthe linear behavior to regions ofhigher driving amplitudes. As seen in panel (c), the linear behavior is approximately valid even for input amplitudes with values very close to its threshold value ($A = 0.37$). Panels (b) and (d) show the results for $\Omega = 0.001$. At this small frequency and small noise ($D = 0.1$, panel (b)), deviations ofthe numerical results fom the LRT behavior are clearly detectable at $A \approx 0.05$. On the other hand, as D increases (panel (d)), even for A values higher than its threshold value, the linearity ofthe system amplitude with respect to the driving amplitude is still approximately valid.

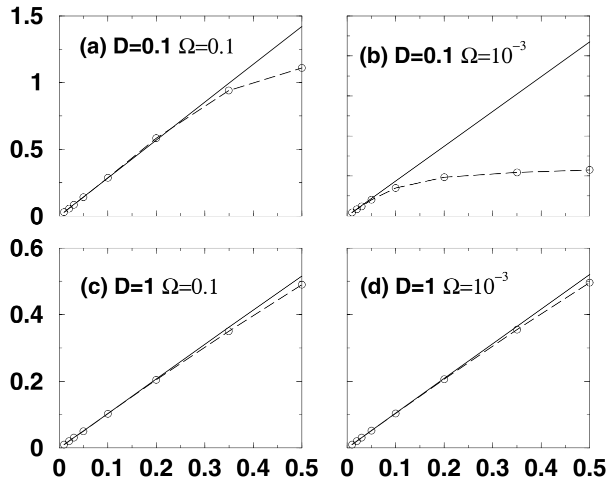

Fig. 3. Behavior of the output amplitude (vertical axes) obtained with the LRT formula Eq. (22) (solid line) and that obtained from the numerics (circles and dashed lines). The input force is sinusoidal with a frequency indicated in the panels and amplitude, A, in the range indicated on the horizontal axes.

5. Multiple Frequency Periodic Driving Force

As an example of the response of the system to a periodic driving term containing several harmonics, let us now consider a periodic driver with period T of the form given in Fig. (1). The force vanishes during most of the period except for small time intervals of duration τ where it is kept constant with (alternating) values $\pm A$. The duty cycle of the input is defined as $2\,\tau/T$. The Fourier coefficients in Eq. (9) are given by,

$$f_n = \begin{cases} 0 & : \text{ n even}, \\ \frac{2A}{n\pi}\sin(n\Omega\tau) & : \text{ n odd}, \end{cases} \tag{26}$$

$$g_n = \begin{cases} 0 & : \text{ n even}, \\ \frac{2A}{n\pi}(1-\cos(n\Omega\tau)) & : \text{ n odd}. \end{cases} \tag{27}$$

It then follows from Eqs. (9)–(12) that $\langle x(t)\rangle_\infty^{LRT}$ contains just odd harmonics. This feature is more general, valid even if the LRT description of the system response fails, as it is a consequence of the spatial and temporal symmetries in the problem. The driving force changes sign every half period, i.e., $F(t{+}T/2) = -F(t)$. Then, the symmetry of $U_0(x)$ implies that $P_\infty(-x,t) = P_\infty(x,t+T/2)$. Therefore, the coefficients in the expansion given in Eq. (5) satisfy the relations, $H_n(x) = (-1)^n H_n(-x)$, $I_n(x) = (-1)^n I_n(-x)$ and, consequently, the series expansion in Eq. (6) contains just odd terms.

Figure 4 shows the amplitude of the first 40 harmonics of the fundamental frequency appearing in the output for an input signal with a period of $2\pi/\Omega$ with $\Omega = 0.1$, and a duty cycle of 10%. The solid bars indicate the values of the

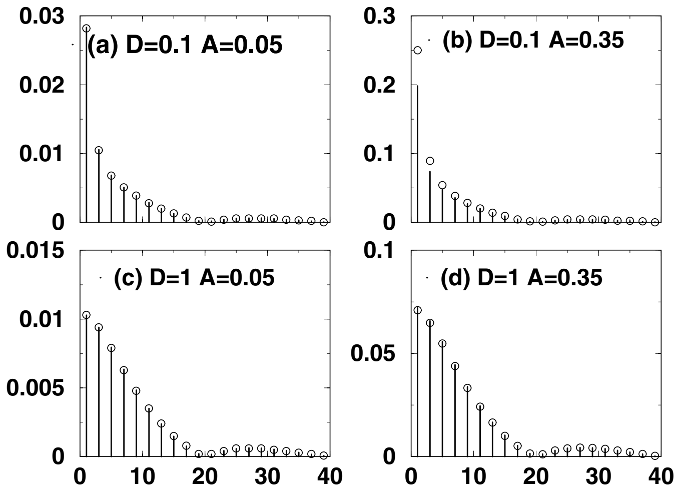

Fig. 4. Amplitude of the harmonics of $\langle x(t)\rangle_\infty^{LRT}$ (bars), and $\langle x(t)\rangle_\infty$ (circles) for an input force of the type sketched in Fig. (1) with a period of $2\pi/0.1$, and a duty cycle of 10%. The order of the harmonics are indicated on the horizontal axes.

amplitudes, a_n, ofeach harmonics of $\langle x(t)\rangle_\infty^{LRT}$ obtained from

$$a_n = \sqrt{(M_n^{LRT})^2 + (N_n^{LRT})^2}\,, \tag{28}$$

where M_n^{LRT} and N_n^{LRT} are given by Eqs. (12)–(16) and $K(t)$ is obtained numerically from the FPE in the absence of driving. The circles indicate the amplitudes ofthe harmonics obtained with the formula $\sqrt{(M_n)^2 + (N_n)^2}$, where M_n and N_n are the Fourier coefficients in Eq. (6) ofthe numerical result, $\langle x(t)\rangle_\infty$. Except for $A = 0.35$ and $D = 0.1$, the LRT predictions and the numerically obtained values coincide.

Notice that, within LRT, even though the system response contains the same harmonics as the driving term, the degree ofamplification ofeach harmonics depends on its frequency. Therefore, the shape of the output $\langle x(t)\rangle_\infty^{LRT}$ will, in general, be different from the shape of the input $F(t)$. Nonetheless, the maximum value of $\langle x(t)\rangle_\infty^{LRT}$ should scale linearly with the maximum of $F(t)$.

In Fig. 5, we show the time evolution of $\langle x(t)\rangle_\infty^{LRT}$ in the long-time regime given by Eqs. (11)–(16), and the entire time evolution of $\langle x(t)\rangle$ including the transient from some initial condition, obtained by numerically solving the corresponding FPE. We show the results for two driving amplitudes: a small value, $A = 0.05$, and $A = 0.35$, that is slightly below its threshold value. The noise values considered are $D = 0.1$ and $D = 1.0$, while the period ofthe driving term is $T = 2\pi/\Omega$, with $\Omega = 0.1$. The duty cycle ofthe input is set at 10%. It is clear fom inspection of panels (c) and (d) that for large noise intensities ($D = 1$), the LRT gives an excellent description of the long time dynamics, even for large driving amplitudes. After a very short transient, the numerically obtained $\langle x(t)\rangle_\infty$ coincides with $\langle x(t)\rangle_\infty^{LRT}$. On the other hand, for $D = 0.1$ and $A = 0.35$, the deviations between the two long time results are noticeable. The peaks ofthe numerical solution are slightly larger

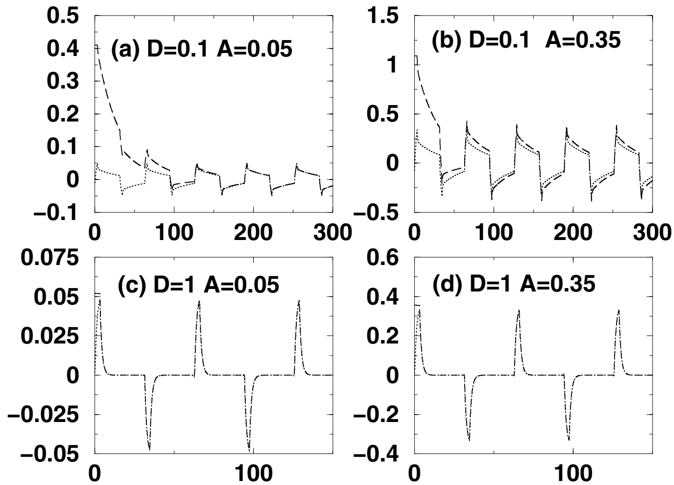

Fig. 5. Time evolution of $\langle x(t)\rangle_\infty^{LRT}$ (dotted lines), and $\langle x(t)\rangle$ (dashed lines) for an input force of the type sketched in Fig. (1) with a period of $2\pi/0.1$, and a duty cycle of 10%.

than the peaks obtained with the LRT formulas. These plots are in total agreement with the Fourier spectra depicted in Fig. 4. The deviations ofLRT predictions with respect to the numerical results are noticeable only for parameter values such that $A/D > 1$ [15, 16].

The peaks ofthe output correspond to the impulses ofthe driver. It is interesting to observe the relevance ofthe noise strength in the time evolution ofthe output after each peak. When D is large, after each impulse of the driving force, very quickly the system adapts to a situation corresponding to no forcing and the output decays to zero as expected for a bistable system in the absence of driving. On the other hand, when D is small, the decay time is much longer. Diffusion is not fast enough to make the amplitude decays to zero before the next impulse of the driving term appears. This influence ofthe noise strength on the decay time after each impulse is clearly observed in Fig. 6, where we show the time evolution during several periods of $\langle x(t)\rangle_\infty$ for an input signal with $A = 0.35$ and the same duty cycle as in Figs. 4 and 5, for a sequence of noise strengths. From top to bottom, the D values are: $D = 0.05$ (a), $D = 0.1$ (b), $D = 0.5$ (c) and $D = 1.0$ (d). We also note that the heights ofthe peaks do not depict a monotonic behavior versus the noise strength D. The amplitude first increases as D increases (panels (a) to (c)), and then, it decreases when D is further augmented going to panel (d). This non-monotonic behavior ofthe maximum amplitude ofthe output is typical ofthe phenomenon of *Stochastic Resonance* [2, 3, 4].

6. Conclusions

In this work, we have further extended the analysis carried out in [6] to check the validity ofthe LRT description ofthe dynamics ofdriven nonlinear stochastic systems. We have compared the predictions ofLRT with the results ofthe numerical solution ofthe FPE for a wide range ofvalues ofthe noise strength D. As it is well

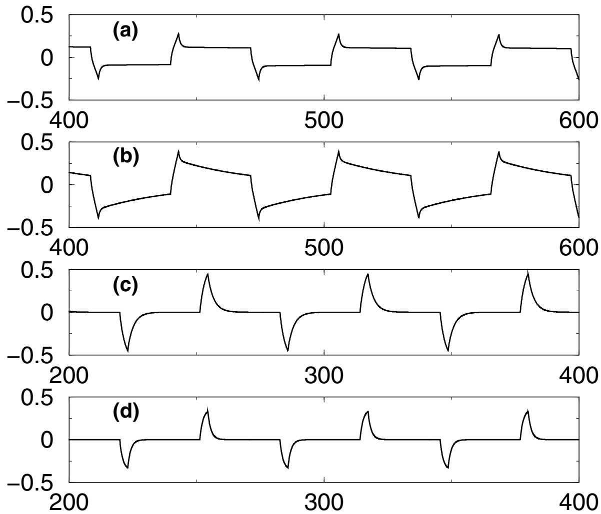

Fig. 6. Time evolution of $\langle x(t)\rangle_\infty$ for an input signal with a duty cycle of 10% and $A = 0.35$. The noise values are: $D = 0.05$ (panel a), $D = 0.1$ (panel b), $D = 0.5$ (panel c) and $D = 1.0$ (panel d).

known, LRT formulas are based on the knowledge of the linear susceptibility, which in turns, depends on the knowledge ofthe equilibrium time correlation fnction in the absence ofdriving. In [6], we restricted ourselves to cases with small values of D so that the 2-mode approximation to the linear susceptibility could be used. Here, we present results without this restriction. The equilibrium time correlation fnction in the absence ofdriving is obtained fom the numerical solution ofthe FP equation for the conditional probability density. Then, the linear susceptibility is obtained by numerical quadrature.

As a firther extension ofthe analysis, we have considered the case ofperiodic input signals with a more complicated structure than a pure sinusoidal signal. In particular, we have studied the behavior under the influence ofa multiple fequency input signal ofthe type considered recently in [9]. The interest ofthis extension lies in the fact that according to the authors of [9], gains in the signal-to-noise ratio larger than unity can be obtained for subthreshold input signals of this kind.

We have tested the predictions ofLRT by concentrating on the analysis ofthe Fourier spectrum ofthe output and the scaling ofthe output amplitude with the input amplitude. Our numerical calculations indicate that, in general terms, LRT provides an adequate approximation to the dynamics even for input amplitudes which are a substantial faction ofits threshold value. For fixed values ofthe input parameters (amplitude and frequency), the quality of LRT predictions improves as the noise strength increases. On the other hand, for small values of the input frequency, and noise values D such that A/D is large, the deviations ofthe numerical findings with respect to the LRT predictions are manifested. Thus, in order to decide on the validity ofLRT when dealing with input signals with finite amplitude, it is also necessary to take into account the input frequency. These general conclusions extend to the case ofmultifequency input signals as well.

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