

# COMPLEXITY OF HIERARCHICAL REFINEMENT FOR A CLASS OF ADMISSIBLE MESH CONFIGURATIONS

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## Abstract

An adaptive isogeometric method based on  $d$ -variate hierarchical spline constructions can be derived by considering a refine module that preserves a certain class of admissibility between two consecutive steps of the adaptive loop [6].

In this paper we provide a complexity estimate, i.e., an estimate on how the number of mesh elements grows with respect to the number of elements that are marked for refinement by the adaptive strategy. Our estimate is in the line of the similar ones proved in the finite element context, [3, 24].

**Keywords:** isogeometric analysis; hierarchical splines; THB-splines; adaptivity.

## 1 Introduction

Throughout the last years, Isogeometric Methods have gained widespread interest and are a very active field of research, [10, 2], investigating a wide range of applications and theoretical questions. Due to the tensor-product structure of splines, there exists very stable procedure to perform mesh refinement and degree raising which are known in the literature as  $h$ -refinement,  $p$ -refinement,  $k$ -refinement [10]. While these algorithms are very efficient, the preservation of the tensor-product

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structure at least locally to each patch, produces a dramatic increase of degrees of freedom together with elongated elements. Mainly for this reason, several approaches have been proposed to alleviate these constraints and they all need the definition of B-splines over non-tensor-product meshes. Indeed, there are several strategies and we mention here T-splines [1], hierarchical B-splines [15, 19, 20] and THB-splines [16], but also LR splines [12, 5], hierarchical T-splines [14], modified T-splines [18], PHT-splines [11, 26] amongst others.

Clearly, the development of adaptive strategies exploiting the potential of non-tensor-product splines is an interesting and important step which has been approached in a number of papers, at least from the practical point of view. In fact, despite their performance in experiments [1, 13, 2, 20, 14], the advantages of mesh-adaptive Isogeometric Methods have not been assessed in theory until today. Partial results, in particular on approximation, efficient and reliable error estimates, and convergence of the adaptive procedure, have been proven in preliminary work [6] in the context of (truncated) hierarchical splines, and we aim at continuing this study providing further ingredients that are needed to prove the optimal convergence of the proposed adaptive approach in the spirit of Adaptive Finite Element Methods [3, 24, 9, 8].

In particular, in this paper we address the complexity of the mesh refinement procedure proposed in [6]. The relation between the set of marked elements and the overall number of refined elements introduced by the refine module is not straightforward: additional elements may be refined to create only (strictly) *admissible* meshes. Admissibility is a restriction to suitably graded meshes that allows for the adaptivity analysis of hierarchical isogeometric methods. By starting from an initial mesh configuration, the complexity estimate provides a bound for the ratio of the newly inserted elements and the cumulative number of elements marked for refinement in the subdivision process that leads from the initial to the final mesh. This allows to control the propagation of the refinement beyond the set of elements initially selected by the marking strategy. An analogous complexity analysis is currently available for bivariate and trivariate T-splines [22, 21].

This paper is organized as follows. In Section 2, we recall notation and basic results from [6]. Section 3 is devoted to the announced complexity estimate. Conclusions and an outlook to future work are given in Section 4.

## 2 Hierarchical refinement

In this section, we recall notation and basic results from [6]. Since the complexity analysis of the REFINE module can be directly performed in the parametric setting, we avoid to introduce the two different notations for parametric/physical domains (corresponding to with/without the hat symbol  $\hat{\cdot}$  in [6]).

## 2.1 The truncated hierarchical basis

Let  $V^0 \subset V^1 \subset \dots \subset V^{N-1}$  be a nested sequence of tensor-product  $d$ -variate spline spaces defined on a closed hypercube  $D$  in  $\mathbb{R}^d$ . For each level  $\ell$ , with  $\ell = 0, 1, \dots, N-1$ , we denote by  $\mathcal{B}^\ell$  the normalized tensor-product B-spline basis of the spline space  $V^\ell$  defined on the grid  $G^\ell$ , and we assume that  $G^0$  consists of open hypercubes with side length 1. The Cartesian product of  $d$  open intervals between adjacent (and non-coincident) grid values defines a quadrilateral element  $Q$  of  $G^\ell$ . For all  $Q \in G^k$  we denote by  $h_Q := 2^{-k}$  the length of its side, and by  $\ell(Q)$  its level, i.e.,  $\ell(Q) = k$ . Moreover, we assume a fixed degree  $\mathbf{p} = (p_1, \dots, p_d)$  at any hierarchical level. The analysis could be generalized to the more general case of a non-uniform initial knot configuration by suitably taking into account the corresponding maximum local mesh size and to variable degrees as well.

In order to define hierarchical spline spaces, we additionally consider a nested sequence of domains  $\Omega^0 \supseteq \Omega^1 \supseteq \dots \supseteq \Omega^{N-1}$ , that are closed subsets of  $D$ . Any  $\Omega^\ell$  is the union of the closure of elements that belong to the tensor-product grid of the previous level. The *hierarchical mesh*  $\mathcal{Q}$  is defined as

$$\mathcal{Q} := \{Q \in \mathcal{G}^\ell, \ell = 0, \dots, N-1\}, \quad (1)$$

where

$$\mathcal{G}^\ell := \{Q \in G^\ell : Q \subset \Omega^\ell \wedge \nexists Q^* \in G^{\ell^*}, \ell^* > \ell : Q^* \subset \Omega^{\ell^*} \wedge Q^* \subset Q\} \quad (2)$$

is the set of active elements of level  $\ell$ . We say that  $\mathcal{Q}^*$  is a refinement of  $\mathcal{Q}$ , and denote  $\mathcal{Q}^* \succeq \mathcal{Q}$ , when  $\mathcal{Q}^*$  is obtained from  $\mathcal{Q}$  by splitting some of its elements via “ $q$ -adic” refinement. Although the hierarchical approach allows us to consider any integer  $q \geq 2$ , we will focus on the case of standard dyadic refinement with  $q = 2$ . Hierarchical B-splines are constructed according to a selection of active basis functions at different levels of detail, see also [19, 25].

**Definition 1.** *The hierarchical B-spline (HB-spline) basis  $\mathcal{H}$  with respect to the mesh  $\mathcal{Q}$  is defined as*

$$\mathcal{H}(\mathcal{Q}) := \{\beta \in \mathcal{B}^\ell : \text{supp } \beta \subseteq \Omega^\ell \wedge \text{supp } \beta \not\subseteq \Omega^{\ell+1}, \ell = 0, \dots, N-1\},$$

where  $\text{supp } \beta$  denotes the intersection of the support of  $\beta$  with  $\Omega^0$ .

The following definition introduces the *truncation* mechanism, the key concept used to define the truncated basis for hierarchical splines [16].

**Definition 2.** *Let*

$$s = \sum_{\beta \in \mathcal{B}^{\ell+1}} c_\beta^{\ell+1}(s)\beta,$$

be the representation of  $s \in V^\ell \subset V^{\ell+1}$  with respect to the finer basis  $\mathcal{B}^{\ell+1}$ . The truncation of  $s$  with respect to  $\mathcal{B}^{\ell+1}$  is defined as

$$\text{trunc}^{\ell+1} s := \sum_{\substack{\beta \in \mathcal{B}^{\ell+1} \\ \text{supp } \beta \not\subseteq \Omega^{\ell+1}}} c_\beta^{\ell+1}(s) \beta.$$

**Definition 3.** The truncated hierarchical B-spline (THB-spline) basis  $\mathcal{T}$  with respect to the mesh  $\mathcal{Q}$  is defined as

$$\mathcal{T}(\mathcal{Q}) := \{ \text{Trunc}^{\ell+1} \beta : \beta \in \mathcal{B}^\ell \cap \mathcal{H}(\mathcal{Q}), \ell = 0, \dots, N-1 \},$$

where  $\text{Trunc}^{\ell+1} \beta := \text{trunc}^{N-1}(\text{trunc}^{N-2}(\dots(\text{trunc}^{\ell+1}(\beta))\dots))$ , for any  $\beta \in \mathcal{B}^\ell \cap \mathcal{H}(\mathcal{Q})$ .

For details on the properties of the truncated basis, we refer to [16, 17].

## 2.2 Admissible meshes

We consider the class of *admissible meshes* introduced in [6].

**Definition 4.** A mesh  $\mathcal{Q}$  is *admissible of class  $m$*  if the truncated basis functions in  $\mathcal{T}(\mathcal{Q})$  which take non-zero values over any element  $Q \in \mathcal{Q}$  belong to at most  $m$  successive levels.

Since the case  $m = 1$  refers to uniform meshes, we will from now on focus on the case  $m \geq 2$ . The relevance of admissible mesh configurations relies on two properties of THB-splines. First, for each element  $Q$  of an admissible mesh, the number of truncated basis functions of degree  $\mathbf{p} = (p_1, \dots, p_d)$  which are non-zero on  $Q$  is less than  $m \prod_{i=1}^d (p_i + 1)$ . Second, if  $\mathcal{Q}$  is an admissible mesh of class  $m$ , then for all truncated basis functions  $\tau \in \mathcal{T}(\mathcal{Q})$  and elements  $Q \in \mathcal{Q}$  with  $Q \cap \text{supp } \tau \neq \emptyset$ , we have  $|Q| \lesssim |\text{supp } \tau| \lesssim |Q|$ , where the hidden constants in these inequalities depend on  $m$  but not on  $\tau$ ,  $\mathcal{Q}$  and  $N$ . These properties may be suitably exploited in the analysis of adaptive isogeometric methods.

In order to characterize a certain class of admissible meshes, we consider the generalization of the *support extension* usually considered in the tensor-product B-spline case to hierarchical configurations.

**Definition 5.** The *support extension*  $S(Q, k)$  of an element  $Q \in G^\ell$  with respect to level  $k$ , with  $0 \leq k \leq \ell$ , is defined as

$$S(Q, k) := \{ Q' \in G^k : \exists \beta \in \mathcal{B}^k, \text{supp } \beta \cap Q' \neq \emptyset \wedge \text{supp } \beta \cap Q \neq \emptyset \}.$$

By a slight abuse of notation, we will also denote by  $S(Q, k)$  the region occupied by the closure of elements in  $S(Q, k)$ . A relevant subset of admissible meshes can be defined according to the result of Proposition 9 in [6].

**Definition 6.** Let  $\mathcal{Q}$  be the mesh of active elements defined according to (1) and (2) with respect to the domain hierarchy  $\Omega^0 \supseteq \Omega^1 \supseteq \dots \supseteq \Omega^{N-1}$ . A mesh  $\mathcal{Q}$  is strictly admissible of class  $m$  if

$$\Omega^\ell \subseteq \omega^{\ell-m+1} \quad (3)$$

where

$$\omega^{\ell-m+1} := \bigcup \{ \bar{Q} : Q \in G^{\ell-m+1} \wedge S(Q, \ell - m + 1) \subseteq \Omega^{\ell-m+1} \},$$

for  $\ell = m, m + 1, \dots, N - 1$ .

The *overlay*  $\mathcal{Q}_*$  of two meshes  $\mathcal{Q}_1, \mathcal{Q}_2$  is the mesh obtained as the coarsest common refinement of  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ , usually indicated as

$$\mathcal{Q}_* = \mathcal{Q}_1 \otimes \mathcal{Q}_2.$$

Let  $\{\Omega_1^\ell\}_{\ell=0, \dots, N_1-1}$  and  $\{\Omega_2^\ell\}_{\ell=0, \dots, N_2-1}$  be the nested sequence of domains that define the hierarchical meshes  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ , respectively, with  $\Omega_1^0 = \Omega_2^0$ . The domain hierarchy  $\{\Omega_*^\ell\}_{\ell=0, \dots, N_*-1}$ , with  $N_* = \max(N_1, N_2)$ , associated to  $\mathcal{Q}_*$  satisfies

$$\Omega_*^\ell = \Omega_1^\ell \cup \Omega_2^\ell \quad \text{and} \quad \omega_*^\ell \supseteq \omega_1^\ell \cup \omega_2^\ell$$

for  $\ell = 1, \dots, N_* - 1$ , where  $\Omega_i^\ell = \emptyset$  when  $\ell \geq N_i$ , for  $i=1,2$ . Consequently, if  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are strictly admissible, for any level  $\ell$ , we have

$$\Omega_*^\ell \subseteq \omega_*^{\ell-m+1}$$

since any  $Q \in \Omega_*^\ell$  either belongs to  $\Omega_1^\ell$  or  $\Omega_2^\ell$  and the overlay  $\mathcal{Q}_*$  is a refinement of both  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ . The overlay  $\mathcal{Q}_*$  of two strictly admissible meshes is then a strictly admissible mesh. Note that the number of elements of the overlay mesh  $\mathcal{Q}_*$  is bounded as follows, see e.g., [4, 22],

$$\#\mathcal{Q}_* = \#(\mathcal{Q}_1 \otimes \mathcal{Q}_2) \leq \#\mathcal{Q}_1 + \#\mathcal{Q}_2 - \mathcal{Q}_0,$$

where  $\mathcal{Q}_0$  is the initial mesh configuration. The above inequality may be used for discussing the optimality of the resulting adaptive isogeometric method, analogously to adaptive finite element setting.

## 2.3 The REFINE module

In order to define an automatic strategy to steer the adaptive method, we will propagate the refinement procedure over a *suitable neighborhood* of any marked element.

**Definition 7.** *The neighborhood of  $Q \in \mathcal{Q} \cap \mathcal{G}^\ell$  with respect to  $m$  is defined as*

$$\mathcal{N}(\mathcal{Q}, Q, m) := \{Q' \in \mathcal{G}^{\ell-m+1} : \exists Q'' \in S(Q, \ell - m + 2), Q'' \subseteq Q'\},$$

when  $\ell - m + 1 \geq 0$ , and  $\mathcal{N}(\mathcal{Q}, Q, m) = \emptyset$  for  $\ell - m + 1 < 0$ .

A sequence of *strictly* admissible meshes can be recursively defined by suitably extending the refinement of coarser regions beyond the set of marked elements  $\mathcal{M}$  through the algorithms presented in Figure 1.

Note that these algorithms follow the structure of informal high-level descriptions in the spirit of the analogous modules related to the adaptive finite element methods.

$\mathcal{Q}^* = \text{REFINE}(\mathcal{Q}, \mathcal{M}, m)$	$Q = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$
<b>for all</b> $Q \in \mathcal{Q} \cap \mathcal{M}$ <b>do</b> $Q = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$ <b>end for</b> $\mathcal{Q}^* = \mathcal{Q}$	<b>for all</b> $Q' \in \mathcal{N}(\mathcal{Q}, Q, m)$ <b>do</b> $Q = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q', m)$ <b>end for</b> subdivide $Q$ and update $\mathcal{Q}$ by replacing $Q$ with its children

Figure 1: The REFINE and REFINE\_RECURSIVE modules.

By exploiting key properties of the REFINE\_RECURSIVE module, summarized in Lemma 8 and Proposition 9 below, Corollary 10 characterizes the output of the REFINE procedure [6].

**Lemma 8** (Recursive refinement). *Let  $\mathcal{Q}$  be a strictly admissible mesh of class  $m$  and  $Q \in \mathcal{Q}$ . The call to  $\mathcal{Q}^* = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$  terminates and returns a refined mesh  $\mathcal{Q}^*$  with elements that either were already active in  $\mathcal{Q}$  or are obtained by single refinement of an element of  $\mathcal{Q}$ .*

In addition, if  $Q \in \mathcal{G}^\ell$ , the level  $\ell^*$  of all newly created elements  $Q^*$  generated by the call to  $\mathcal{Q}^* = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$  satisfies

$$\ell^* \leq \ell + 1. \tag{4}$$

In order to verify this, we may note that the recursion is applied to elements of level  $< \ell$ , and, in particular, of level  $\leq \ell - m + 1$ . If  $Q^*$  is a child of  $Q$  then  $\ell^* = \ell + 1$ .

Otherwise,  $Q^*$  is obtained by splitting some elements in the chain of neighborhoods generated by set of recursive calls and, consequently,  $\ell^* \leq \ell - m + 2 < \ell + 1$  since  $m \geq 2$ .

**Proposition 9.** *Let  $\mathcal{Q}$  be a strictly admissible mesh of class  $m \geq 2$  and let  $Q \in \mathcal{G}^\ell$ , for some  $0 \leq \ell \leq N - 1$ . Then it follows that the call to  $\mathcal{Q}^* = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$  returns a strictly admissible mesh  $\mathcal{Q}^* \succeq \mathcal{Q}$  of class  $m$ .*

**Corollary 10.** *Let  $\mathcal{Q}$  be a strictly admissible mesh of class  $m \geq 2$  and  $\mathcal{M}$  the set of elements of  $\mathcal{Q}$  marked for refinement. The call to  $\mathcal{Q}^* = \text{REFINE}(\mathcal{Q}, \mathcal{M}, m)$  terminates and returns a strictly admissible mesh  $\mathcal{Q}^* \succeq \mathcal{Q}$  of class  $m$ .*

Note that in each computation of the neighborhood  $\mathcal{N}(Q, Q, m)$ , the choice of level  $\ell - m + 2$  for the support extension yields the smallest neighborhood that is acceptable for preserving the class of admissibility of the mesh when subdividing the given element  $Q$ . Nevertheless, depending on the underlying hierarchical mesh configurations, the basis functions could be also truncated at different intermediate levels.

### 3 Linear Complexity

This section is devoted to a complexity estimate in the style proposed by Binev, Dahmen and DeVore [3] and, in an alternative version, by Stevenson [24], for adaptive finite element methods.

#### 3.1 Auxilliary results

For every pair of mesh elements  $(Q, Q')$ , let  $\text{dist}(Q, Q')$  be the Euclidean distance of their midpoints. Given a  $Q \in \mathcal{G}^\ell$ , all  $Q' \in \mathcal{N}(Q, Q, m)$  satisfy

$$\text{dist}(Q, Q') \leq \frac{\sqrt{d}}{2} \text{diam}(S(Q, \ell - m + 2)),$$

where  $\ell = \ell(Q)$  and

$$\text{diam}(S(Q, \ell - m + 2)) := 2^{-\ell+m-2} (2p + 1) = 2^{-\ell} C_s,$$

with  $C_s = C_s(p, m) := 2^{m-2}(2p + 1)$ ,  $p := \max_{i=1, \dots, d} p_i$ . Hence,

$$\text{dist}(Q, Q') \leq 2^{-\ell-1} C_d, \quad C_d = C_d(d, p, m) := \sqrt{d} C_s. \quad (5)$$

**Lemma 11.** *Let  $\mathcal{Q}$  be a strictly admissible mesh of class  $m \geq 2$ ,  $\mathcal{M}$  the set of elements of  $\mathcal{Q}$  marked for refinement, and  $Q' \in \mathcal{Q} \cap \mathcal{M}$ . Any newly created  $Q \in \mathcal{Q}^* \setminus \mathcal{Q}$  obtained by the call to  $\mathcal{Q}^* = \text{REFINE\_RECURSIVE}(\mathcal{Q}, Q', m)$  satisfies*

$$\text{dist}(Q, Q') \leq 2^{-\ell(Q)} C \quad \text{with} \quad C := \sqrt{d} \tilde{C}, \quad \tilde{C} := \left( 2^{-1} + \frac{2}{1 - 2^{1-m}} C_s \right), \quad (6)$$

where then  $C$  depends on  $d, p$  and  $m$ .

*Proof.* The existence of  $Q \in \mathcal{Q}^* \setminus \mathcal{Q}$  means that  $\text{REFINE\_RECURSIVE}$  is called over a sequence of elements  $Q' = Q_J, Q_{J-1}, \dots, Q_0$  and corresponding meshes  $\mathcal{Q}_J, \dots, \mathcal{Q}_0$  so that  $Q_{j-1} \in \mathcal{N}(\mathcal{Q}_j, Q_j, m)$ , with  $Q' \in \mathcal{M}$  and  $Q$  being a child of  $Q_0$ , namely  $\ell(Q) = \ell(Q_0) + 1$ . Since  $\ell(Q_{j-1}) = \ell(Q_j) - m + 1$ , it follows

$$\ell(Q_j) = \ell(Q_0) + j(m - 1). \quad (7)$$

We have

$$\text{dist}(Q, Q') \leq \text{dist}(Q, Q_0) + \text{dist}(Q_0, Q')$$

and

$$\text{dist}(Q, Q_0) = 2^{-\ell(Q)} 2^{-1} \sqrt{d}, \quad \text{dist}(Q_0, Q') \leq \sum_{j=1}^J \text{dist}(Q_j, Q_{j-1}).$$

According to (5) and (7), we obtain

$$\begin{aligned} \sum_{j=1}^J \text{dist}(Q_j, Q_{j-1}) &\leq \sum_{j=1}^J 2^{-\ell(Q_j)-1} C_d = \sum_{j=1}^J 2^{-\ell(Q_0)-1-j(m-1)} C_d \\ &< 2^{-\ell(Q_0)} C_d \sum_{j=0}^{\infty} 2^{-j(m-1)} = \frac{2^{-\ell(Q_0)}}{1 - 2^{1-m}} C_d = \frac{2^{-\ell(Q)+1}}{1 - 2^{1-m}} C_d. \end{aligned}$$

Hence,  $\text{dist}(Q, Q') \leq 2^{-\ell(Q)} C$ , where  $C$  is the constant defined in (6).  $\square$

### 3.2 Main result

The main result of this paper states the existence of a generic constant  $\Lambda = \Lambda(d, p, m) < \infty$  that bounds, for any sequence of successive refinements, the ratio between the number of new elements in the final mesh  $\mathcal{Q}_J$  and the number of all marked elements encountered in the refinement process from  $\mathcal{Q}_0$  to  $\mathcal{Q}_J$ .



**Theorem 12** (Complexity of REFINE). *Let  $\mathcal{M} := \bigcup_{j=0}^{J-1} \mathcal{M}_j$  be the set of marked elements used to generate the sequence of strictly admissible meshes  $\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_J$  starting from  $\mathcal{Q}_0 = G^0$ , namely*

$$\mathcal{Q}_j = \text{REFINE}(\mathcal{Q}_{j-1}, \mathcal{M}_{j-1}, m), \quad \mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1} \quad \text{for } j \in \{1, \dots, J\}.$$

*Then, there exists a constant  $\Lambda > 0$  so that*

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j,$$

*with  $\Lambda = \Lambda(d, p, m) := 4(4\tilde{C} + 1)^d$ , where  $\tilde{C}$  is defined in (6).*

*Proof.* We denote by  $\mathbb{G} := \bigcup_j G^j$  the set of the initial mesh elements and all elements that can be generated from their successive dyadic subdivision. Let  $Q \in \mathbb{G}$ ,  $Q_{\mathcal{M}} \in \mathcal{M}$ , let

$$\lambda(Q, Q_{\mathcal{M}}) := \begin{cases} 2^{\ell(Q) - \ell(Q_{\mathcal{M}})} & \text{if } \ell(Q) \leq \ell(Q_{\mathcal{M}}) + 1 \text{ and } \text{dist}(Q, Q_{\mathcal{M}}) < 2^{1 - \ell(Q)} C, \\ 0 & \text{otherwise.} \end{cases}$$

The proof consists of two main steps devoted to identify

- (i) a lower bound for the sum of the  $\lambda$  function as  $Q_{\mathcal{M}}$  varies in  $\mathcal{M}$  so that each  $Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0$  satisfies

$$\sum_{Q_{\mathcal{M}} \in \mathcal{M}} \lambda(Q, Q_{\mathcal{M}}) \geq 1; \tag{8}$$

- (ii) an upper bound for the sum of the  $\lambda$  function as the refined element  $Q$  varies in  $\mathcal{Q}_J \setminus \mathcal{Q}_0$  so that, for any  $j = 0, \dots, J - 1$ , each  $Q_{\mathcal{M}} \in \mathcal{M}_j$  satisfies

$$\sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q_{\mathcal{M}}) \leq \Lambda. \tag{9}$$

If inequalities (8) and (9) hold for a certain constant  $\Lambda$ , we have

$$\begin{aligned} \#\mathcal{Q}_J - \#\mathcal{Q}_0 &= \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} 1 \leq \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \sum_{Q_{\mathcal{M}} \in \mathcal{M}} \lambda(Q, Q_{\mathcal{M}}) \\ &\leq \sum_{Q_{\mathcal{M}} \in \mathcal{M}} \Lambda = \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j, \end{aligned}$$

and the proof of the theorem is complete. We detail below the analysis of (i) and (ii).

(i) Let  $Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0$  be an element generated in the refinement process from  $\mathcal{Q}_0$  to  $\mathcal{Q}_J$ , and let  $j_1 < J$  be the index so that  $Q \in \mathcal{Q}_{j_1+1} \setminus \mathcal{Q}_{j_1}$ . Lemma 11 together with (4) state the existence of  $Q_1 \in \mathcal{M}_{j_1}$  with

$$\text{dist}(Q, Q_1) \leq 2^{-\ell(Q)} C \quad \text{and} \quad \ell(Q) \leq \ell(Q_1) + 1,$$

and, consequently  $\lambda(Q, Q_1) = 2^{\ell(Q)-\ell(Q_1)} > 0$ . The repeated use of Lemma 11 yields a sequence  $\{Q_2, Q_3, \dots\}$  with  $Q_{i-1} \in \mathcal{Q}_{j_i+1} \setminus \mathcal{Q}_{j_i}$ , for  $j_1 > j_2 > j_3 > \dots$ , and  $Q_i \in \mathcal{M}_{j_i}$  such that

$$\text{dist}(Q_{i-1}, Q_i) \leq 2^{-\ell(Q_{i-1})} C \quad \text{and} \quad \ell(Q_{i-1}) \leq \ell(Q_i) + 1. \quad (10)$$

We iteratively apply Lemma 11 as long as

$$\lambda(Q, Q_i) > 0 \quad \text{and} \quad \ell(Q_i) > 0,$$

until we reach the first index  $L$  with  $\lambda(Q, Q_L) = 0$  or  $\ell(Q_L) = 0$ . By considering the three possible cases below, inequality (8) may be derived as follows.

- If  $\ell(Q_L) = 0$  and  $\lambda(Q, Q_L) > 0$ , then

$$\sum_{Q_M \in \mathcal{M}} \lambda(Q, Q_M) \geq \lambda(Q, Q_L) = 2^{\ell(Q)-\ell(Q_L)} > 1,$$

since  $\ell(Q) > \ell(Q_L) = 0$ .

- If  $\lambda(Q, Q_L) = 0$  because  $\ell(Q) > \ell(Q_L) + 1$ , then (10) yields  $\ell(Q_{L-1}) \leq \ell(Q_L) + 1 < \ell(Q)$  and hence

$$\sum_{Q_M \in \mathcal{M}} \lambda(Q, Q_M) \geq \lambda(Q, Q_{L-1}) = 2^{\ell(Q)-\ell(Q_{L-1})} > 1.$$

- If  $\lambda(Q, Q_L) = 0$  because  $\text{dist}(Q, Q_L) \geq 2^{1-\ell(Q)} C$ , then a triangle inequality combined with Lemma 11 leads to

$$2^{1-\ell(Q)} C \leq \text{dist}(Q, Q_1) + \sum_{i=1}^{L-1} \text{dist}(Q_i, Q_{i+1}) \leq 2^{-\ell(Q)} C + \sum_{i=1}^{L-1} 2^{-\ell(Q_i)} C.$$

Consequently,  $2^{-\ell(Q)} \leq \sum_{i=1}^{L-1} 2^{-\ell(Q_i)}$ , and we obtain

$$1 \leq \sum_{i=1}^{L-1} 2^{\ell(Q)-\ell(Q_i)} = \sum_{i=1}^{L-1} \lambda(Q, Q_i) \leq \sum_{Q_M \in \mathcal{M}} \lambda(Q, Q_M).$$

(ii) Inequality (9) can be derived as follows. For any  $0 \leq j \leq J-1$ , we consider the set of elements of level  $j$  whose distance from  $Q_M$  is less than  $2^{1-j}C$  defined as

$$B(Q_M, j) := \{Q \in G^j : \text{dist}(Q, Q_M) < 2^{1-j}C\}.$$

According to the definition of  $\lambda$ , the set  $B(Q_M, j)$  collects the elements at level  $j$  so that  $\lambda(Q, Q_M) > 0$ . We then have

$$\sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q_M) \leq \sum_{Q \in G \setminus \mathcal{Q}_0} \lambda(Q, Q_M) = \sum_{j=1}^{\ell(Q_M)+1} 2^{j-\ell(Q_M)} \#B(Q_M, j). \quad (11)$$

Since the diagonal of an element  $Q$  of level  $j$  is  $2^{-j}\sqrt{d}$ , the diagonal of the hypercube composed by the union of the closure of all elements in  $B(Q_M, j)$  is less or equal to

$$2 \cdot 2^{1-j}C + 2^{-j}\sqrt{d} = 2^{-j}\sqrt{d}(4\tilde{C} + 1),$$

where  $\tilde{C}$  is defined in (6). Hence,

$$\#B(Q_M, j) \leq (4\tilde{C} + 1)^d,$$

and the index substitution  $k := 1 - j + \ell(Q_M)$  reduces (11) to

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q_M) &\leq \sum_{j=1}^{\ell(Q_M)+1} 2^{j-\ell(Q_M)} \#B(Q_M, j) = \sum_{k=0}^{\ell(Q_M)} 2^{1-k} \#B(Q_M, j) \\ &< 2 \sum_{k=0}^{\infty} 2^{-k} \#B(Q_M, j) = 4 \#B(Q_M, j) \leq \Lambda, \end{aligned}$$

with  $\Lambda = \Lambda(d, p, m) = 4(4\tilde{C} + 1)^d$ . □

## 4 Conclusions

We developed a complexity estimate which says that the ratio between the refined elements and the marked elements along the refinement history stays bounded, when refinement is performed as proposed in [6]. In particular, this estimate guarantees that the refinement routine ensuring the (strict) admissibility of the refined mesh remains local at least when looking at the overall refinement process. Note that for a single refinement step, it may be impossible to prove such an estimate [23].

Our work paves the way to the analysis of optimal convergence of the adaptive strategy proposed in [6] that will be addressed in further studies [7].

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## References

- [1] Y. Bazilevs, V. M. Calo, J. A. Cottrell, J. Evans, T. J. R. Hughes, S. Lipton, M. A. Scott, and T. W. Sederberg. Isogeometric analysis using T-Splines. *Comput. Methods Appl. Mech. Engrg.*, 199:229–263, 2010.
- [2] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. Mathematical analysis of variational isogeometric methods. *Acta Numer.*, 23:157–287, 2014.
- [3] P. Binev, W. Dahmen, and R. DeVore. Adaptive Finite Element Methods with convergence rates. *Numer. Math.*, 97:219–268, 2004.
- [4] A. Bonito and R. H. Nochetto. Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method. *SIAM J. Numer. Anal.*, 48:734–771, 2010.
- [5] A. Bressan. Some properties of LR-splines. *Comput. Aided Geom. Design*, 30:778–794, 2013.
- [6] A. Buffa and C. Giannelli. Adaptive isogeometric methods with hierarchical splines: error estimator and convergence. *Math. Models Methods Appl. Sci.*, to appear, 2015.
- [7] A. Buffa and C. Giannelli. Adaptive isogeometric methods with hierarchical splines: optimality and convergence rates. In preparation, 2015.
- [8] C. Carstensen, M. Feischl, M. Page, and D. Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.
- [9] J. M. Cascón, C. Kreuzer, R. H. Nochetto, and K. G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46:2524–2550, 2008.

- [10] J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. John Wiley & Sons, 2009.
- [11] J. Deng, F. Chen, X. Li, Ch. Hu, W. Tong, Z. Yang, and Y. Feng. Polynomial splines over hierarchical T-meshes. *Graph. Models*, 70:76–86, 2008.
- [12] T. Dokken, T. Lyche, and K. F. Pettersen. Polynomial splines over locally refined box-partitions. *Comput. Aided Geom. Design*, 30:331–356, 2013.
- [13] M. R. Dörfel, B. Jüttler, and B. Simeon. Adaptive isogeometric analysis by local h-refinement with T-splines. *Comput. Methods Appl. Mech. Engrg.*, 199:264–275, 2010.
- [14] E. J. Evans, M. A. Scott, X. Li, and D. C. Thomas. Hierarchical T-splines: Analysis-suitability, Bézier extraction, and application as an adaptive basis for isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 284:1–20, 2015.
- [15] D. R. Forsey and R. H. Bartels. Hierarchical B-spline refinement. *Comput. Graphics*, 22:205–212, 1988.
- [16] C. Giannelli, B. Jüttler, and H. Speleers. THB-splines: the truncated basis for hierarchical splines. *Comput. Aided Geom. Design*, 29:485–498, 2012.
- [17] C. Giannelli, B. Jüttler, and H. Speleers. Strongly stable bases for adaptively refined multilevel spline spaces. *Adv. Comp. Math.*, 40:459–490, 2014.
- [18] H. Kang, F. Chen, and J. Deng. Modified T-splines. *Computer Aided Geometric Design*, 30(9):827–843, 2013.
- [19] R. Kraft. Adaptive and linearly independent multilevel B-splines. In A. Le Méhauté, C. Rabut, and L. L. Schumaker, editors, *Surface Fitting and Multiresolution Methods*, pages 209–218. Vanderbilt University Press, Nashville, 1997.
- [20] G. Kuru, C. V. Verhoosel, K. G. van der Zeeb, and E. H. van Brummelen. Goal-adaptive isogeometric analysis with hierarchical splines. *Comput. Methods Appl. Mech. Engrg.*, 270:270–292, 2014.
- [21] P. Morgenstern. 3D Analysis-suitable T-splines: definition, linear independence and  $m$ -graded local refinement. arXiv:1505.05392, submitted for publication.
- [22] P. Morgenstern and D. Peterseim. Analysis-suitable adaptive T-mesh refinement with linear complexity. *Comput. Aided Geom. Design*, 34:50–66, 2015.

- [23] R. H. Nochetto and A. Veerer. Primer of adaptive finite element methods. In *Multiscale and adaptivity: modeling, numerics and applications*, volume 2040 of *Lecture Notes in Math.*, pages 125–225. Springer, Heidelberg, 2012.
- [24] R. Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7:245–269, 2007.
- [25] A.-V. Vuong, C. Giannelli, B. Jüttler, and B. Simeon. A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 200:3554–3567, 2011.
- [26] P. Wang, J. Xu, J. Deng, and F. Chen. Adaptive isogeometric analysis using rational PHT-splines. *Comput. Aided Design*, 43(11):1438–1448, 2011.