

Numerical upscaling of eigenvalue problems

AXEL MÅLQVIST

(joint work with Daniel Peterseim)

1. INTRODUCTION

We present a numerical upscaling technique for computing eigenpairs of self-adjoint linear elliptic second order differential operators with arbitrary positive bounded coefficients. The precise setting is as follows. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with piecewise flat boundary and let $A \in L^\infty(\Omega, \mathbb{R}_{\text{sym}}^{d \times d})$ be a matrix-valued coefficient with uniform spectral bounds. Consider the self-adjoint eigenvalue problem: find eigenpairs (u, λ) such that

$$(1) \quad -\nabla \cdot (A \nabla u) = \lambda u.$$

A standard finite element approximation of these eigenvalues and eigenfunctions is constructed using a shape regular mesh \mathcal{T}_h of Ω with a corresponding finite element space $V_h \subset V := H_0^1(\Omega)$: find eigenpairs $u_h^{(\ell)} \in V_h$ and $\lambda_h^{(\ell)} \in \mathbb{R}$ such that,

$$(2) \quad a(u_h^{(\ell)}, v) := (A \nabla u_h^{(\ell)}, \nabla v) = \lambda_h^{(\ell)} (u_h^{(\ell)}, v), \quad \forall v \in V_h.$$

We are mainly interested in the small eigenvalues. Popular approaches for the computation of these eigenvalues include e.g. Lanczos/Arnoldi-type iterations or the QR-algorithm applied directly to the N_h -dimensional finite element matrices, where $N_h = \dim(V_h)$.

In our approach we avoid the application of an eigenvalue solver to the large-scale problem (2) directly. Instead, inspired by [3], we compute a low-dimensional approximation space $V_{\text{cs}} \subset V_h$ first, with $N_H = \dim(V_{\text{cs}}) \ll N_h$. This preprocessing step is done by (approximately) inverting the operator for special right hand sides and subject to certain linear constraints. Having performed N_H of those computations, the solution of a low-dimensional $N_H \times N_H$ eigenvalue problem by standard solvers yields approximations of the first N_H eigenpairs. The linear problems needed to be solved on the fine scale are totally independent.

Our method is related to some coarse finite element mesh with maximal width H . The accuracy of the approximate eigenvalues is expressed in terms of H . Without any assumptions on the smoothness of eigenfunctions, we prove that the error scales like H^4 . Note that a standard first-order conforming finite element computation yields H^2 under full $H^2(\Omega)$ regularity, see e.g. [2]. Under such strong assumption the two-grid method [5] allows certain postprocessing (solution of linear problems on the fine scale) of the coarse finite element eigenpairs to increase the accuracy to H^4 . This is also possible to exploit for our proposed method to get even higher order convergence.

2. GALERKIN APPROXIMATION AND MAIN RESULT

We let \mathcal{T}_H denote an underlying coarse regular finite element mesh, with mesh function H defined by $H|_T = \text{diam}(T) := H_T$ for all $T \in \mathcal{T}_H$. We denote the

interior nodes of the mesh \mathcal{N} . We let $V_H = \text{span}(\{\phi_x\}_{x \in \mathcal{N}})$ be a finite element space such that $V_H \subset V_h$.

We recall the Clément type interpolant $\mathcal{I}_H : V \rightarrow V_H$ presented in [1]. Let $\mathcal{I}_H v = \sum_{x \in \mathcal{N}} (\mathcal{I}_H v)(x) \phi_x$, where,

$$(\mathcal{I}_H v)(x) = \frac{(v, \phi_x)}{(1, \phi_x)},$$

for all $x \in \mathcal{N}$. The following approximation and stability property holds,

$$H_T^{-1} \|v - \mathcal{I}_H v\|_{L^2(T)} + \|\nabla(v - \mathcal{I}_H v)\|_{L^2(T)} \leq C \|\nabla v\|_{L^2(\omega_T)}, \quad \forall v \in V,$$

where ω_T is the collection of elements in \mathcal{T} overlapping T .

We are ready to present the decomposition of the space V_h into a coarse and a fine part. We let the fine scale space be defined by,

$$V_{\text{fs}} := \text{kernel } \mathcal{I}_H = \{v \in V_h : \mathcal{I}_H v = 0\},$$

and the coarse scale space by,

$$V_{\text{cs}} = \{v \in V_h : a(v, w) = 0, \text{ for all } w \in V_{\text{fs}}\}.$$

This yields an a -orthogonal split of the space $V_h = V_{\text{cs}} \oplus V_{\text{fs}}$. We note that $\dim(V_{\text{cs}}) = \dim(V_H) = |\mathcal{N}|$. In order to compute a basis for V_{cs} we solve $|\mathcal{N}|$ corrector problems: find $\psi_x \in V_{\text{fs}}$ such that,

$$a(\psi_x, v) = a(\phi_x, v), \quad \text{for all } v \in V_{\text{fs}},$$

and let $V_{\text{cs}} = \text{span}(\{\phi_x - \psi_x\}_{x \in \mathcal{N}})$. The Galerkin approximation of equation (2) in the space V_{cs} reads: find $u_{\text{cs}}^{(\ell)} \in V_{\text{cs}}$ and $\lambda_{\text{cs}}^{(\ell)} \in \mathbb{R}$ such that,

$$a(u_{\text{cs}}^{(\ell)}, v) = \lambda_{\text{cs}}^{(\ell)} (u_{\text{cs}}^{(\ell)}, v), \quad v \in V_{\text{cs}}.$$

We now present an error bound for the approximate eigenvalues.

Theorem 1. *Let H be sufficiently small so that $H \leq C \ell^{-1/4} \sqrt{\frac{\alpha}{\lambda_h^{(0)}}}$. Then it holds*

$$(3) \quad \frac{\lambda_{\text{cs}}^{(\ell)} - \lambda_h^{(\ell)}}{\lambda_h^{(\ell)}} \leq C \sqrt{\ell} \left(H \sqrt{\frac{\lambda_h^{(\ell)}}{\alpha}} \right)^4 \quad \text{for all } \ell = 1, \dots, |\mathcal{N}|,$$

for some constant C only depending on Ω and the shape regularity constant and with α being the lower spectral bound of A .

Remark 1. In [3] it is shown that $\phi_x - \psi_x$ decays exponentially (in the number of coarse elements) away from node x . This allows the use of truncated patches of size $H \log(H^{-1})$, with Dirichlet boundary conditions, rather than solving for ψ_x on the entire domain Ω . Theorem 1 holds also when using truncated domains.

3. NUMERICAL EXAMPLE

Let $\Omega := (-1, 1)^2 \setminus [0, 1]^2$ be the L-shaped domain. Consider the constant scalar coefficient $A = 1$ and consider uniform coarse meshes with maximal mesh widths $\sqrt{2}H = 2^{-1}, \dots, 2^{-4}$ of Ω . The reference mesh \mathcal{T}_h has maximal mesh width $h = 2^{-7}/\sqrt{2}$. We consider a $P1$ conforming finite element approximation of the eigenvalues on the reference mesh \mathcal{T}_h and compare these discrete eigenvalues $\lambda_h^{(\ell)}$ with coarse scale approximations depending on the coarse mesh size H .

Table 5 shows results for the case without truncation, i.e., all linear problems have been solved on the whole of Ω . For fixed ℓ , the rate of convergence of the

ℓ	$\lambda_h^{(\ell)}$	$e^{(\ell)}(1/2\sqrt{2})$	$e^{(\ell)}(1/4\sqrt{2})$	$e^{(\ell)}(1/8\sqrt{2})$	$e^{(\ell)}(1/16\sqrt{2})$
1	9.6436869	0.003494567	0.000034466	0.000000546	0.000000010
2	15.1989274	0.009621397	0.000079887	0.000000845	0.000000010
3	19.7421815	0.023813222	0.000213097	0.000002073	0.000000023
4	29.5281571	0.096910157	0.000724615	0.000006574	0.000000076
5	31.9265496	0.094454625	0.000874659	0.000009627	0.000000138
6	41.4922250	-	0.002395227	0.000019934	0.000000254
7	44.9604884	-	0.002443271	0.000019683	0.000000223
8	49.3631826	-	0.003651870	0.000028869	0.000000308
9	49.3655623	-	0.004266472	0.000032835	0.000000355
10	56.7389993	-	0.006863742	0.000055219	0.000000618

TABLE 5. Errors $e^{(\ell)}(H) =: \frac{\lambda_H^{(\ell)} - \lambda_h^{(\ell)}}{\lambda_h^{(\ell)}}$ for $\ell = 1, \dots, 10$, constant coefficient $A = 1$, and various choices of the coarse mesh size H .

eigenvalue error $\lambda_H^{(\ell)} - \lambda_h^{(\ell)}$ in terms of H observed in Table 5 is between 6 and 7 which is even better than predicted in Theorem 1. For more elaborate numerical results we refer to [4].

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