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Finite Element Discretization of Multiscale Elliptic Problems

Daniel Peterseim
(joint work with Axel Målqvist)

Background. The numerical solution of second order elliptic problems with strongly heterogeneous and highly varying (non-periodic) diffusion coefficient is a challenging part within the simulation of modern composite materials. The coefficient may represent different materials or material phases and, hence, heterogeneities and oscillations of the coefficient typically appear on several non-separated scales. Similar difficulties arise in geophysical applications such as ground water flow, oil recovery modeling, or CO_2 sequestration.

The abstract mathematical setup of this note is as follows. Given some bounded polygonal Lipschitz domain Ω in 2 or 3 space dimensions, some uniformly elliptic diffusion matrix $A \in L^\infty\left(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)$, and some force $f \in L^2(\Omega)$, we seek $u \in V := H^1_0(\Omega)$ such that

$$a\,(u,v) := \int_\Omega \langle A\nabla u, \nabla v\rangle \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x =: F(v) \quad \text{for all } v \in V.$$

If A varies rapidly on microscopic scales, classical polynomial based finite element methods are unable to capture neither the microscopic nor the macroscopic behavior of the solution unless the meshwidth is chosen fine enough (i.e., smaller than the smallest scale in the coefficient). To overcome this lack of performance, many methods that are based on general (non-polynomial) ansatz functions have been developed, amongst others [5, 4, 2, 1]. In these methods, the problem is split into coarse and (possibly several) fine scales. The fine scale effect on the coarse scale is either computed numerically or modeled analytically. The resulting modified coarse problem can then be solved numerically and its solution contains crucial information from the fine scales. Although many of these approaches show promising results in practice, their convergence analysis typically relies on strong assumptions such as periodicity and scale separation. Those assumptions, which essentially justify homogenization, appear unrealistic in the applications under consideration.

A New Variational Multiscale Method [8]**.** Without any additional assumptions on the coefficient, we construct for any (possibly coarse) shape regular mesh $\mathcal{T}_H$ of size H an upscaled variational problem with solution u_H^{ms} such that the estimate $\|u - u_H^{\mathrm{ms}}\|_{H^1(\Omega)} \leq C_f H$ holds with a constant C_f that depends on f and the contrast of A but *not* on its variations. The upscaled problem is related to a Galerkin method with respect to a modified coarse space. This coarse space is spanned by one modified nodal basis function per vertex in $\mathcal{T}_H$ and their computation involves only local solves on patches of coarse elements.

We shall briefly summarize our construction. Let V_H denote the space of continuous $\mathcal{T}_H$-piecewise affine finite element functions that matches the homogeneous Dirichlet boundary condition. The key tool in our construction is linear surjective (quasi-)interpolation operator $\mathfrak{I}_{\mathcal{T}} : V \to V_H$ from [3, Section 6]. Its kernel $V^{\mathrm{f}} := \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\}$ represents the microscopic features of V that are not captured by V_H. Since V^{f} is a closed subspace, we have the decomposition

$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}},$$

where V_H^{ms} denotes the orthogonal complement of V^{f} in V for the scalar product a. The space V_H^{ms} is coarse in the sense that $\dim V_H^{\mathrm{ms}} = \dim V_H$. Given the classical nodal basis (tent) function $\lambda_x \in V_H$ for some x in the set of vertices $\mathcal{N}_H$ of $\mathcal{T}_H$, let $\phi_x \in V^{\mathrm{f}}$ solve the corrector problem

$$a(\phi_x, w) = a(\lambda_x, w) \quad \text{for all } w \in V^{\mathrm{f}}. \tag{1}$$

We then have $V_H^{\mathrm{ms}} = \mathrm{span}\{\lambda_x - \phi_x \mid x \in \mathcal{N}_H\}$. Needless to say that the corrections ϕ_x have theoretical purpose only because they are solution of some infinite dimensional problem and because they have global support in general. However, [8] shows that both issues can be handled efficiently. The correction ϕ_x decays exponentially fast (with respect to the number of layers of coarse elements) away from x and that a simple truncation leads to localized basis functions with good approximation properties. This decay is due to the fact that ϕ_x solves a variational problem in the kernel of the interpolation operator where functions are constraint to have vanishing averages in nodal patches. Moreover, this result is stable with respect to perturbations arising from the discretization of the local problems.

Denote nodal patches of k-th order about $x \in \mathcal{N}_H$ by ω_x^k. Given some finescale finite element space $V_h \supset V_H$ that captures microscopic scales sufficiently well, define discrete and localized finescale spaces $V_h^{\mathrm{f}}(\omega_{x,k}) := \{v \in V^{\mathrm{f}} \cap V_h \mid v|_{\Omega \setminus \omega_{x,k}} = 0\}$, $x \in \mathcal{N}_H$. Then the solutions $\phi_{x,k}^h \in V_h^{\mathrm{f}}(\omega_{x,k})$ of

$$a(\phi_{x,k}^h, w) = a(\lambda_x, w) \quad \text{for all } w \in V_h^{\mathrm{f}}(\omega_{x,k}), \tag{2}$$

are discrete approximations of ϕ_x from (1) with local support. Note that these corrector problems are completely decoupled and can be computed in parallel without any communication.

The proposed (variational) multiscale finite element method then seeks an approximation $u_{H,k}^{\mathrm{ms},h}$ of u in the coarse multiscale space

$$V_{H,k}^{\mathrm{ms},h} = \mathrm{span}\{\lambda_x - \phi_{x,k}^h \mid x \in \mathcal{N}_H\} \subset V. \tag{3.a}$$

The approximation $u_{H,k}^{\mathrm{ms},h}$ satisfies the upscaled system of equations

$$a(u_{H,k}^{\mathrm{ms},h}, v) = F(v) \quad \text{for all } v \in V_{H,k}^{\mathrm{ms},h}. \tag{3.b}$$

This method is a new variant of the variational multiscale methods introduced in [6]. Note that $\dim V_{H,k}^{\mathrm{ms},h} = |\mathcal{N}_H| = \dim V_H$, i.e., the number of degrees of freedom of the proposed method (3) is the same as for the classical finite element method related to the space V_H. The basis functions of the multiscale method have local support. The overlap is proportional to the parameter k.

Review of A Priori Error Analysis. The error analysis in [8] shows that the error $u - u_{H,k}^{\mathrm{ms},h}$ for $k \approx \log \frac{1}{H}$ satisfies the following a priori estimate:

$$\|A^{1/2}\nabla(u - u_{H,k}^{\mathrm{ms},h})\| \leq C_f H + \inf_{v_h \in V_h} \|A^{1/2}\nabla(u - v_h)\|; \tag{4}$$

H being the mesh size of the underlying coarse finite element mesh, h being the fine mesh size for the local (parallel) computations. The desired accuracy TOL, e.g., $TOL \approx 0.01$ is achieved by choosing $H \approx TOL$ independent of any scales in the problem and by ensuring that the local problems are solved sufficiently accurate. For example, if $A \in W^{1,\infty}$ (bounded with bounded weak derivative) and ε is the smallest present scale, i.e., $\|\nabla A\|_{L^\infty(\Omega)} \lesssim \varepsilon^{-1}$, the second term in the right-hand side of (4) may be replaced by the worst case bound $Ch\varepsilon^{-1}$ for a first-order ansatz space V_h (see [9]). In this case, $C_f(H + \frac{h}{\varepsilon})$ bounds the error of our multiscale approximation (3).

The proof in [8] does not rely on regularity of the solution and gives a very explicit expression for the rate of convergence. The analysis confirms previous numerical results in [6, 7] and gives the (variational) multiscale method the solid theoretical foundation that has previously been missing. We further stress that our result is not asymptotic but holds for arbitrary coarse mesh parameter H.

The author is supported by the DFG Research Center Matheon Berlin.

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