[2] O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., **35** (1998), 255–299.
[3] P. Cummings and X. Feng, *Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations*, Math. Models Methods Appl. Sci., **16** (2006), 139–160.
[4] C. J. Gittelson, R. Hiptmair and I. Perugia, *Plane wave discontinuous Galerkin methods: analysis of the h-version*, M2AN Math. Model. Numer. Anal., **43** (2009), 297–332.
[5] U. Hetmaniuk, *Stability estimates for a class of Helmholtz problems*, Communications in Mathematical Sciences, **5** (2007), 665–678.
[6] R. Hiptmair, A. Moiola and I. Perugia, *Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version*, SIAM J. Numer. Anal., **49** (2011), 264-284.
[7] R. Hiptmair, A. Moiola and I. Perugia, in preparation.
[8] T. Huttunen, P. Monk and J. P. Kaipio, *Computational aspects of the ultra-weak variational formulation*, J. Comput. Phys., **182** (2002), 27–46.
[9] J. Melenk, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, USA, 1995.
[10] A. Moiola, R. Hiptmair and I. Perugia, *Plane wave approximation of homogeneous Helmholtz solutions*, Z. Angew. Math. Phys., **62** (2011), 809-837

Comparison of Finite Element Methods for the Poisson Model Problem

Daniel Peterseim
(joint work with C. Carstensen and M. Schedensack)

In the recent preprint [7], the authors establish the equivalence of conforming Courant finite element method (CFEM) and nonconforming Crouzeix-Raviart finite element method (CRFEM) in the sense that the respective energy error norms are equivalent up to generic constants and higher-order data oscillations in a Poisson model problem. The Raviart-Thomas mixed finite element method is better than the previous methods whereas the conjecture of the converse relation is proved to be false. Those results complete the analysis of comparison initiated by Braess [2]. This note extends the comparison to several Discontinuous Galerkin FEM (DGFEM), e.g., symmetric interior penalty method (SIPG) [10, 12, 1], nonsymmetric interior penalty method (NIPG) [14], and local DG (LDG) [9, 8].

Given a bounded polygonal domain Ω in the plane and data $f \in L^2(\Omega)$, the Poisson model problem seeks $u \in V := H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in V.$$

Let $\mathcal{T}$ be some shape regular triangulation of Ω with associated mesh size function $h_\mathcal{T}$. The Courant finite element space of H^1-conforming $\mathcal{T}$-piecewise affine functions is denoted $V_\mathrm{C}(\mathcal{T}) := P_1(\mathcal{T}) \cap V$. The corresponding (unique) Galerkin approximation $u_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})$ satisfies

$$a(u_\mathrm{C}, v_\mathrm{C}) = \textstyle\int_\Omega f v_\mathrm{C} \,\mathrm{d}x \quad \text{for all } v_\mathrm{C} \in V_\mathrm{C}(\mathcal{T}).$$

Abstract DGFEM. Consider the space $V_{\mathrm{DG}}(\mathcal{T}) := P_1(\mathcal{T})$ of $\mathcal{T}$-piecewise affine functions with associated norm $\|\bullet\|_{\mathrm{DG}} := (\|\nabla\bullet\|^2_{L^2(\Omega)} + |\bullet|^2_{\mathrm{J}})^{1/2}$ and jump seminorm

$$|\bullet|^2_{\mathrm{J}} := \sum_{E\in\mathcal{E}} |E|^{-1}\|[\bullet]_E\|^2_{L^2(E)};$$

$[v_{\mathrm{DG}}]_E$ denotes the jump of $v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ across the edge $E \in \mathcal{E}$ as usual.

The bounded and coercive (with respect to $\|\bullet\|_{\mathrm{DG}}$) DG bilinear form $a_{\mathrm{DG}} : V_{\mathrm{DG}}(\mathcal{T}) \times V_{\mathrm{DG}}(\mathcal{T}) \to \mathbb{R}$ extends $a|_{V_{\mathrm{C}}(\mathcal{T})\times V_{\mathrm{C}}(\mathcal{T})}$ to $V_{\mathrm{DG}}(\mathcal{T}) \times V_{\mathrm{DG}}(\mathcal{T})$ and satisfies

$$(1) \qquad |a(v, v_{\mathrm{C}}) - a_{\mathrm{DG}}(v_{\mathrm{DG}}, v_{\mathrm{C}})| \le C_1\|v - v_{\mathrm{DG}}\|_{\mathrm{DG}}\|\nabla v_{\mathrm{C}}\|_{L^2(\Omega)}$$

for all $v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})$, $v \in V$, and $v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ with some universal positive constant C_1 independent of $h_{\mathcal{T}}$. The (unique) DG approximation $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ satisfies

$$a_{\mathrm{DG}}(u_{\mathrm{DG}}, v_{\mathrm{DG}}) = \textstyle\int_\Omega f v_{\mathrm{DG}}\,\mathrm{d}x \quad \text{for all } v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}).$$

Assume further that there exists some bounded linear operator $\mathrm{I_C} : V_{\mathrm{DG}}(\mathcal{T}) \to V_{\mathrm{C}}(\mathcal{T})$ and some positive constant C_2 that does not depend on $h_{\mathcal{T}}$ such that

$$(2) \qquad \|v_{\mathrm{DG}} - \mathrm{I_C}\, v_{\mathrm{DG}}\|_{\mathrm{DG}} \le C_2|v_{\mathrm{DG}}|_{\mathrm{J}} \quad \text{holds for all } v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}).$$

It is shown in [11, Section 3.2] that SIPG, NIPG, and LDG fit into this abstract framework with some operator $\mathrm{I_C}$ based on nodal averaging [3, 4, 5, 13].

Main result. The comparison is stated in terms of $A \lesssim B$ which abbreviates the existence of some constant C which only depends on the minimal angle in $\mathcal{T}$, but *not* on the domain Ω and *not* on the mesh-size $h_{\mathcal{T}}$, such that $A \le CB$. The comparison includes data oscillations $\mathrm{osc}(f,\mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(\Omega)}$, where Π_0 denotes the L^2 orthogonal projection onto the piecewise constants.

The comparison result for CFEM and DGFEM reads

$$(3) \qquad \|\nabla u - \nabla u_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T}).$$

Needless to say that (3), by transitivity, establishes the equivalence of SIPG, NIPG, and LDG as well as CRFEM. It is remarkable that those results do *not* rely on any regularity assumption and hold for arbitrary coarse triangulations and not just in an asymptotic regime.

Sketch of proof. The inclusion $V_{\mathrm{C}}(\mathcal{T}) \subset V_{\mathrm{DG}}(\mathcal{T})$ and the triangle inequality yield

$$\|\nabla(u - u_{\mathrm{C}})\|_{L^2(\Omega)} = \|u - u_{\mathrm{C}}\|_{\mathrm{DG}} \le \|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} + \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}}.$$

Coercivity of a_{DG} (with respect to $\|\bullet\|_{\mathrm{DG}}$), Galerkin orthogonality, boundedness of a_{DG}, and the property (2) of the averaging operator $\mathrm{I_C}$ lead to

$$\begin{aligned}\|u_{\mathrm{DG}} - u_{\mathrm{C}}\|^2_{\mathrm{DG}} &\lesssim a_{\mathrm{DG}}(u_{\mathrm{DG}} - u_{\mathrm{C}}, u_{\mathrm{DG}} - u_{\mathrm{C}}) = a_{\mathrm{DG}}(u_{\mathrm{DG}} - u_{\mathrm{C}}, u_{\mathrm{DG}} - \mathrm{I_C}\, u_{\mathrm{DG}})\\ &\lesssim \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}}|u_{\mathrm{DG}}|_{\mathrm{J}} \lesssim \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}}|u - u_{\mathrm{DG}}|_{\mathrm{J}}.\end{aligned}$$

The combination of the previous estimates proves the first inequality in (3). The proof of the second inequality follows directly from [11, Section 3.2], which requires the condition (1). $\square$

Various generalizations. The equivalence of CFEM and DGFEM immediately generalizes to its higher-order variants. Let $V_{\mathrm{C}}^k(\mathcal{T}) := P_k(\mathcal{T}) \cap V$ be the conforming subspace of $\mathcal{T}$-piecewise polynomials of degree at most $k \in \mathbb{N}$; $V_{\mathrm{DG}}^k(\mathcal{T}) := P_k(\mathcal{T})$ denotes the corresponding DG space of the same order. Then

$$\|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}^k\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} + \mathrm{osc}_k(f, \mathcal{T})$$

holds with $\mathrm{osc}_k(f, \mathcal{T}) := \|h_{\mathcal{T}}^k(f - \Pi_{k-1} f)\|$ where Π_{k-1} denotes the L^2 orthogonal projection onto $P_{k-1}(\mathcal{T})$. The hidden generic constants may depend on the polynomial degree k but *not* on the mesh size $h_{\mathcal{T}}$.

Often, the large number of degrees of freedom in DGFEM compared to CFEM is justified by the possibility of using non-conforming meshes that may contain some finite number of hanging nodes per edge. Define $V_{\mathrm{DG}}^k(\mathcal{T}) := P_k(\mathcal{T})$ for some non-conforming triangular mesh $\mathcal{T}$. It is shown in [13] that also for such meshes there exists an averaging operator $\mathrm{I}_{\mathrm{C}} : V_{\mathrm{DG}}^k(\mathcal{T}) \to V$ that satisfies (2) with suitably redefined jump seminorm. The image $\mathrm{I}_{\mathrm{C}}(V_{\mathrm{DG}}^k(\mathcal{T})) = V_{\mathrm{DG}}^k(\mathcal{T}) \cap V$ defines some conforming space $V_{\mathrm{C}}^k(\mathcal{T})$. One might not want to use $V_{\mathrm{C}}^k(\mathcal{T})$ for actual computations but the corresponding Galerkin solution u_{C}^k serves for a comparison. The proof of (3) remains valid in this setting and establishes the comparison

$$\|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}^k\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} + \mathrm{osc}_k(f, \mathcal{T})$$

for non-conforming meshes. Hence, SIPG, NIPG, and LDG are equivalent also on non-conforming meshes and their accuracy is limited by the accuracy that is provided by its largest conforming subspace.

These new results on DGFEM will be included in an upcoming revised version of [7]. Similar comparison results can be achieved for 3-dimensional domains, non-simplicial meshes, or other DG schemes (e.g., WOPSIP [6]). Applications of comparison results include least-squares finite element methods and equality of approximation classes for concepts of optimality for adaptive finite element methods.

The author is supported by the DFG Research Center Matheon Berlin.

References

[1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[2] D. Braess. An a posteriori error estimate and a comparison theorem for the nonconforming P_1 element. *Calcolo*, 46(2):149–155, 2009.

[3] S. C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65(215):897–921, 1996.

[4] S. C. Brenner. Convergence of nonconforming multigrid methods without full elliptic regularity. *Math. Comp.*, 68(225):25–53, 1999.

[5] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, 41(1):306–324 (electronic), 2003.

[6] S. C. Brenner, L. Owens, and L.-Y. Sung. A weakly over-penalized symmetric interior penalty method. *E*lectron. Trans. Numer. Anal., 30:107–127, 2008.

[7] C. Carstensen, D. Peterseim, and M. Schedensack. Comparison Results of Three First-order Finite Element Methods for Poisson Model Problem. *Matheon Preprint*, 831, 2011.

[8] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38(5):1676–1706 (electronic), 2000.
[9] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35(6):2440–2463 (electronic), 1998.
[10] J. Douglas, Jr. and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing methods in applied sciences (Second Internat. Sympos., Versailles, 1975)*, pages 207–216. Lecture Notes in Phys., Vol. 58. Springer, Berlin, 1976.
[11] T. Gudi. A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, 79(272):2169–2189, 2010.
[12] P. Houston, D. Schötzau, and T. P. Wihler. Energy norm a posteriori error estimation of *hp*-adaptive discontinuous Galerkin methods for elliptic problems. *Math. Models Methods Appl. Sci.*, 17(1):33–62, 2007.
[13] O. A. Karakashian and F. Pascal. A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. *SIAM J. Numer. Anal.*, 41(6):2374–2399 (electronic), 2003.
[14] B. Rivière, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM J. Numer. Anal.*, 39(3):902–931, 2001.

Exponential Convergence of *hp*-Version Discontinuous Galerkin Methods for Elliptic Problems in Polyhedral Domains

Dominik Schötzau
(joint work with Christoph Schwab, Thomas Wihler)

In a series of land mark papers in the mid eighties, Babuška and Guo proved that using *hp*-version finite element methods for the numerical approximation of elliptic problems with piecewise analytic data in polygonal domains leads to exponential rates of convergence in the number of degrees of freedom. The convergence bounds are typically of the form

$$\|u - u_N\|_E \leq C \exp(-b\sqrt[3]{N}),$$

where u is the solution of the boundary-value problem, u_N its *hp*-version finite element approximation, $\|\cdot\|_E$ a suitable (energy) norm to measure the error, N the dimension of the *hp*-version finite element space, and C and b are constants independent of N; see [2, 3, 4] and the references therein.

Starting in the nineties, steps were undertaken to extend these results to polyhedral domains in $\mathbb{R}^3$; see [1] and the references therein. It was asserted and confirmed numerically that the errors decay exponentially as $C\exp(-b\sqrt[5]{N})$, i.e., with an exponent that contains the fifth root of N.

In this talk, we prove this convergence rate for *hp*-version discontinuous Galerkin (DG) discretizations of the model problem

$$\begin{aligned} -\nabla\cdot(\boldsymbol{A}\nabla u) &= f \qquad \text{in } \Omega \subset \mathbb{R}^3, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$