Wave Scattering by a Circular Elastic Plate in Water of Finite Depth: A Closed Form Solution

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ABSTRACT

We present a solution for a circular thin plate of shallow draft on water of finite depth subject to linear wave forcing of a single frequency. The solution, which is given in a closed form, is based on decomposing the solution into angular eigenfunctions. The coefficients in the expansion are then found by matching the potential and its derivative at the plate edge and imposing the free edge conditions for the plate. The matching is accomplished by taking the inner product with respect to the vertical eigenfunctions for the free surface. The equations that are derived are transformed so that the final system of equations involves only the unknowns under the plate. Solutions are presented and compared to the results of Meylan (2002), who presented a solution for a plate of arbitrary geometry.

INTRODUCTION

The problem of a linear, floating thin plate of shallow draft subject to wave forcing is a standard problem in hydroelasticity which can be used to model a range of physical systems. In 2 dimensions, many different solution methods exist. For shallow water, a solution was presented by Stoker (1958); for finite depth, solutions have been presented by Meylan and Squire (1994) and Newman (1994), amongst others. In 3 dimensions, a solution for a circular plate on shallow water was presented by Zilman and Miloh (2000), and general solution methods have been presented by Meylan and Squire (1994), Meylan (2002) and Kashiwagi (1998), amongst others. However, with the exception of Zilman and Miloh (2000), these solutions were based on the free-surface Green’s function and were highly numerical. Even the solution presented by Meylan and Squire (1996) for a circular plate only exploited the circular geometry to calculate the modes of vibration of the free plate.

The only 3-dimensional solution for a thin plate of shallow draft that is not based on a highly numerical method was presented by Zilman and Miloh (2000). The exact same solution method was also independently derived by Tsubo (2001). Their solution was for the case of shallow water and a circular plate. The circular geometry allows separation of variables in the angular direction, so that the solution may be found by decoupling the solutions for each angular eigenfunction. Once this has been accomplished, the solution for each angular direction can be found by solving a linear system of 4 equations. These were derived by matching the potential and its derivative, and by imposing the 2 boundary conditions at the edge of the plate.

We present here an extension of the method of Zilman and Miloh (2000) to the case where the water depth is finite. In this case we can still solve for each angular eigenfunction separately, and we match the potential and its derivative and impose the boundary conditions at the plate edge. However, we must match the potential not at a point but throughout the water depth. This matching is accomplished by taking the inner product with respect to the vertical eigenfunctions which satisfy the free surface condition. We present results for the method, which are compared to the results of Meylan (2002).

GOVERNING EQUATIONS

We begin with the equations for the plate-water system in nondimensional form as the problem is so well known. The derivation and nondimensionalisation are discussed in detail in Meylan (2002). We nondimensionalise the spatial variables with respect to a length parameter \( L \) (for example, \( L \) may be derived from the area of the plate, or \( L \) may be the characteristic length \((D/\rho g)^{1/4}\), where \( D \) is the rigidity constant of the plate, \( \rho \) the density of the water, and \( g \) the gravitational constant) and the time variables with respect to \( \sqrt{L/g} \). We assume that all motions are time-harmonic with radian frequency \( \omega = \sqrt{\alpha} \), so that the velocity potential of the water, \( \phi(x, t) \), can be expressed as the real part of a complex quantity \( \Phi \):

\[
\Phi(x, t) = \text{Re}\{\phi(x)e^{-i\sqrt{\alpha}t}\} \tag{1}
\]

We will use a cylindrical coordinate system, \( x = (r, \theta, z) \), assumed to have its origin at the centre of the circular plate with radius \( a \). The water is assumed to have constant finite depth \( H \), and the \( z \)-direction points vertically upward with the water surface at \( z = 0 \) and the seafloor at \( z = -H \). The boundary value problem can thus
be expressed as:

\[
\Delta \phi = 0, \quad -H < z < 0, \\
\phi_z = 0, \quad z = -H, \\
\phi_r = \alpha \phi, \quad z = 0, r > a, \\
(\beta \Delta^2 + 1 - \alpha \gamma) \phi_z = \alpha \phi, \quad z = 0, r < a
\]

(2)

where the constants \( \beta \) and \( \gamma \) are given by:

\[
\beta = \frac{D}{\rho L^3 g}, \quad \gamma = \frac{\rho h}{\rho L}
\]

(3)

and \( \rho_i \) is the density of the plate. We must also apply the edge conditions for the plate and the radiation condition as \( r \to \infty \). The subscript \( z \) denotes the derivative in the \( z \)-direction.

**SOLUTION METHOD**

**Separation of Variables**

We now separate variables, noting that, since the problem has circular symmetry, we can write the potential as:

\[
\phi(r, \theta, z) = \zeta(z) \sum_{n=-\infty}^{\infty} \rho_n(r)e^{i\alpha \theta}
\]

(4)

Applying Laplace’s equation we obtain:

\[
\xi_{zz} + \mu^2 \xi = 0
\]

(5)

so that:

\[
\zeta = \cos \mu (z + H)
\]

(6)

where the separation constant \( \mu^2 \) must satisfy the standard dispersion equations:

\[
k \tan(kH) = -\alpha, \quad r > a
\]

(7)

\[
\kappa \tan(\kappa H) = \frac{-\alpha}{\beta k^2 + 1 - \alpha \gamma}, \quad r < a
\]

(8)

Note that we have set \( \mu = k \) under the free surface and \( \mu = \kappa \) under the plate. The dispersion equations are discussed in detail in Fox and Squire (1994). We denote the positive imaginary solution of Eq. 7 by \( k_0 \) and the positive real solutions by \( k_m, m \geq 1 \). The solutions of Eq. 8 will be denoted by \( \kappa_m, m \geq -2 \). The fully complex solutions with a positive imaginary part are \( \kappa_{-2} \) and \( \kappa_{-1} \) (where \( \kappa_{-1} = \kappa_{-2} \)), the negative imaginary solution is \( \kappa_0 \), and the positive real solutions are \( \kappa_m, m \geq 1 \). We define:

\[
\phi_m(z) = \frac{\cos k_m(z + H)}{\cos k_m H}, \quad m \geq 0
\]

(9)

as the vertical eigenfunction of the potential in the open water region and:

\[
\psi_m(z) = \frac{\cos \kappa_m(z + H)}{\cos \kappa_m H}, \quad m \geq -2
\]

(10)

as the vertical eigenfunction of the potential in the plate-covered region. For later reference, we note that:

\[
\int_{-H}^{0} \phi_m(z) \phi_m(z) \, dz = A_m \delta_{mn}
\]

(11)

where \( \delta_{mn} \) is the Kronecker delta and:

\[
A_m = \frac{1}{2} \left( \frac{\cos k_m H \sin k_m H + k_m H}{k_m \cos^2 k_m H} \right)
\]

(12)

and:

\[
\int_{-H}^{0} \phi_m(z) \psi_m(z) \, dz = B_{mn}
\]

(13)

where:

\[
B_{mn} = \frac{k_n \sin k_n H \cos \kappa_n H - \kappa_n \cos k_n H \sin \kappa_n H}{(\cos k_n H \cos \kappa_n H)(k_n^2 - \kappa_n^2)}
\]

(14)

We now solve for the function \( \rho_n(r) \). Using Laplace’s equation in polar coordinates we obtain:

\[
y^2 \frac{d^2 \rho_n}{dy^2} + y \frac{d \rho_n}{dy} - (n^2 + \mu^2) \rho_n = 0
\]

(15)

where \( \mu \) is \( k_n \) or \( \kappa_m \), depending on whether \( r \) is greater or less than \( a \). We can convert this equation to the standard form by substituting \( y = \mu r \) to obtain:

\[
y^2 \frac{d^2 \rho_n}{dy^2} + y \frac{d \rho_n}{dy} - (n^2 + y^2) \rho_n = 0
\]

(16)

The solution of this equation is a linear combination of the modified Bessel functions of order \( n \), \( I_n(y) \) and \( K_n(y) \). (See Abramowitz and Stegun, 1964.) Since the solution must be bounded, we know that under the plate the solution will be a linear combination of \( I_n(y) \), while outside the plate the solution will be a linear combination of \( K_n(y) \). Thus the potential can be expanded as:

\[
\phi(r, \theta, z) = \sum_{n=\infty}^{\infty} \sum_{m=0}^{\infty} a_{mn}K_n(k_m r)e^{i\alpha \theta} \phi_m(z), \quad r > a
\]

(17)

\[
\phi(r, \theta, z) = \sum_{n=\infty}^{\infty} \sum_{m=-2}^{\infty} b_{mn}I_n(\kappa_m r)e^{i\alpha \theta} \psi_m(z), \quad r < a
\]

(18)

where \( a_{mn} \) and \( b_{mn} \) are the coefficients of the potential in the open water and the plate-covered region, respectively.

**Incident Potential**

The incident potential is a wave of amplitude \( A \) in displacement travelling in the positive \( x \)-direction. Following Zilman and Miloh (2000), the incident potential can then be written as:

\[
\phi_i = \frac{A}{i\sqrt{\alpha}} e^{k_0 x} \phi_0(z) = \sum_{n=\infty}^{\infty} e_n I_n(k_0 r) \phi_0(z) e^{i\alpha \theta}
\]

(19)

where \( e_n = A/ (i\sqrt{\alpha}) \). (We retain the dependence on \( n \) for situations where the incident potential might take another form.)

**Boundary Conditions**

The boundary conditions for the plate also have to be considered. The vertical force and bending moment must vanish, which, following Zilman and Miloh (2000), can be written as:

\[
\left[ \hat{\lambda} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{\partial^2}{r \partial \theta^2} \right) \right] w = 0
\]

(20)
and:

\[
\left[ \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} + 1 - \nu - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] w = 0
\]  

where \( w \) is the time-independent surface displacement, \( \nu \) is Poisson’s ratio, and \( \Delta \) is the polar coordinate Laplacian:

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]  

**Displacement of Plate**

The surface displacement and the water velocity potential at the water surface are linked through the kinematic boundary condition:

\[ \phi_z = -i \sqrt{\alpha} w \quad \text{at} \quad z = 0 \]  

From Eq. 2, the potential and the surface displacement are thus related by:

\[ w = i \sqrt{\alpha} \phi, \quad r > a \]

\[ (\beta \Delta + 1 - \alpha \gamma) w = i \sqrt{\alpha} \phi, \quad r < a \]

The surface displacement can also be expanded in eigenfunctions as:

\[ w(r, \theta) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i \sqrt{\alpha} a_{nm} K_m(k_m r) e^{i n \theta}, \quad r > a \]

\[ w(r, \theta) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i \sqrt{\alpha} (\beta k_m^4 + 1 - \alpha \gamma)^{-1} b_{nm} I_m(k_m r) e^{i n \theta}, \quad r < a \]

using the fact that:

\[ \Delta (I_m(k_m r)e^{i n \theta}) = k_m^2 I_m(k_m r)e^{i n \theta} \]

**Infinite Dimensional System of Equations**

The boundary conditions (Eqs. 20 and 21) can be expressed in terms of the potential using Eq. 28. Since the angular modes are uncoupled, the conditions apply to each mode, giving:

\[
\sum_{m=-\infty}^{\infty} (\beta k_m^4 + 1 - \alpha \gamma)^{-1} b_{nm} \left( k_m^2 I_m(k_m a) \right.
\]

\[ = -\frac{1 - \nu}{a} \left( \frac{\beta k_m^4}{a} - \frac{1}{a^2} I_m(k_m a) \right) = 0 \]  

\[
\sum_{n=-\infty}^{\infty} (\beta k_m^4 + 1 - \alpha \gamma)^{-1} b_{nm} \left( k_m^2 I_m(k_m a) \right.
\]

\[ + n^2 \frac{1 - \nu}{a^2} \left( \frac{\beta k_m^4}{a} - \frac{1}{a^2} I_m(k_m a) \right) \left) = 0 \]  

The potential and its derivative must be continuous across the transition from open water to the plate-covered region. Thus, the potentials and their derivatives at \( r = a \) have to be equal. Again, we know that this must be true for each angular mode, and we obtain:

\[ e_n I_n(k_n a) \phi_0(z) + \sum_{m=0}^{\infty} a_{nm} K_m(k_m a) \phi_m(z) \]

\[ = \sum_{m=-\infty}^{\infty} b_{nm} I_m(k_m a) \phi_m(z) \]  

\[ e_n k_n I_n'(k_n a) \phi_0(z) + \sum_{m=0}^{\infty} a_{nm} K_m'(k_m a) \phi_n(z) \]

\[ = \sum_{m=-\infty}^{\infty} b_{nm} K_m'(k_m a) \phi_n(z) \]

for each \( n \). We solve these equations by multiplying both equations by \( \phi_n(z) \) and integrating from \(-H\) to 0 to obtain:

\[ e_n I_n(k_n a) A_0 B_{nl} + a_{nm} K_m(k_m a) A_n = \sum_{m=-\infty}^{\infty} b_{nm} K_m'(k_m a) B_{nl} \]  

\[ e_n k_n I_n'(k_n a) A_0 B_{nl} + a_{nm} K_m'(k_m a) A_n = \sum_{m=-\infty}^{\infty} b_{nm} K_m'(k_m a) B_{nl} \]

Eq. 33 can be solved for the open-water coefficients \( a_{nm} \):

\[ a_{nm} = -e_n \frac{I_n(k_n a) A_0}{K_m(k_m a) A_n} \sum_{m=-\infty}^{\infty} b_{nm} \frac{K_m'(k_m a) B_{nl}}{K_m(k_m a) A_n} \]  

which can then be substituted into Eq. 34 to give us:

\[ \left( k_n I_n'(k_n a) - k_n I_n'(k_n a) \right) e_n A_n \delta_{nl} \]

\[ = \sum_{m=-\infty}^{\infty} \left( \frac{k_n I_n'(k_n a) - k_n I_n'(k_n a)}{K_m(k_m a) A_n} \right) B_{ml} \]

for each \( n \). Together with Eqs. 29 and 30, Eq. 36 gives the required equations to solve for the coefficients of the water velocity potential in the plate-covered region.

**NUMERICAL SOLUTION**

To solve the system of Eq. 36 together with the boundary conditions, we set the upper limit of \( m \) as \( M \). We also set the angular expansion to be from \( n = -N \) to \( N \). This gives us:

\[ \phi(r, \theta, z) = \sum_{n=-N}^{N} \sum_{m=-M}^{M} a_{nm} K_m(k_m r) e^{i n \theta} \phi_m(z), \quad r > a \]

\[ \phi(r, \theta, z) = \sum_{n=-N}^{N} \sum_{m=-M}^{M} b_{nm} I_m(k_m r) e^{i n \theta} \psi_m(z), \quad r < a \]

Since \( l \) is an integer with \( 0 \leq l \leq M \), this leads to a system of \( M + 1 \) equations. The number of unknowns is \( M + 3 \), and the 2 extra equations are obtained from the boundary conditions for the free plate (Eqs. 29 and 30). The equations to be solved for each
For this situation it follows that we require only \( N = 0 \) for an accurate solution, which means we only need solve \( 9 \times 4 \) systems of equations. Now we fix the number of points in the angular expansion as \( N = 16 \) and vary the number of roots of the dispersion equation \( M \). Fig. 2 shows the real part of the displacement. The number of roots of the dispersion equation is \( M = 0 \) (a), 2 (b), 4 (c) and 8 (d). The depth is \( H = 25 \). It follows that we require only \( M = 2 \) for an accurate solution, which means that we only need solve a \( 5 \times 5 \) system of equations. This shows how efficient this closed form solution is.

We compare with the results presented in Meylan (2002) for an arbitrary shaped plate modified to compute the solution for finite depth. The circle is represented in this scheme by square panels which are arranged to form a circular shape, as nearly as possible. Figs. 3 and 4 show the real part (a and c) and imaginary part (b and d) of the displacement for depth \( H = 25 \) and \( H = 1 \), respectively. The number of points in the angular expansion is \( N = 16 \). The number of roots of the dispersion equation is \( M = 8 \). Plots (a) and (b) are calculated using the circular plate method described here. Plots (c) and (d) are calculated using an arbitrary shaped plate method, with the panels shown being the actual panels used in the calculation. We see the expected agreement between the 2 methods.

Finally, Tables 1 and 2 show the values of the coefficients \( b_{mn} \) for the case \( H = 25 \). The very rapid decay of the higher evanescent modes is apparent, which explains the rapid convergence in \( M \) shown in Fig. 2.

**Fig. 1** Real part of displacement for a plane wave of unit amplitude incident from negative \( x \)-direction. Parameters chosen as \( \lambda = 50, a = 100, \beta = 10^3, \gamma = 0 \) and \( H = 25 \). \( M \) fixed as 8 while \( N \) chosen as \( N = 2 \) (a), 4 (b), 8 (c) and 16 (d).

**Fig. 2** Real part of displacement for a plane wave of unit amplitude incident from negative \( x \)-direction. Parameters chosen as \( \lambda = 50, a = 100, \beta = 10^3, \gamma = 0 \) and \( H = 25 \). \( M \) fixed as 8 while \( N \) chosen as \( N = 2 \) (a), 4 (b), 8 (c) and 16 (d).
plate method.\[\text{NSLgamma}\]\[\text{NSLgamma}\]\[\text{NSLgamma}\]\[\text{NSLgamma}\]

The potential and its derivative must match under the plate edge inside and outside the plate-covered region. The number of roots matched eigenfunction expansion is then used in the 2 regions, plate of shallow draft floating on the water surface subject to

\[\text{SUMMARY}\]

We have presented a closed form solution for a circular thin plate of shallow draft floating on the water surface subject to linear wave forcing. The solution was based on decomposing the solution into angular eigenfunctions that are uncoupled. A matched eigenfunction expansion is then used in the 2 regions, inside and outside the plate-covered region. The number of roots of the dispersion equation in the regions is always chosen such that there are 2 extra roots under the plate. The conditions that the potential and its derivative must match under the plate edge are converted to a system of equations by taking an inner product with respect to the vertical eigenfunctions for the free surface. Two further equations are found by imposing the edge conditions of the plate. We have shown that the solutions agree with those found using Meylan’s method (2002) for an arbitrary shaped plate.

\[\text{REFERENCES}\]


Table 1 Coefficients \(b_{mn}\) for parameters \(\lambda = 50, a = 100, \beta = 10^5, \gamma = 0\) and \(H = 25\)

<table>
<thead>
<tr>
<th>(m)</th>
<th>(n = 0)</th>
<th>(n = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(1.32 \cdot 10^{-1} - 9.71 \cdot 10^{-2}i)</td>
<td>(6.85 \cdot 10^{-1} - 6.37 \cdot 10^{-1}i)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-6.38 \cdot 10^{-5} + 1.47 \cdot 10^{-5}i)</td>
<td>(-3.92 \cdot 10^{-3} + 3.99 \cdot 10^{-3}i)</td>
</tr>
<tr>
<td>(0)</td>
<td>(-3.29 \cdot 10^{-4} + 1.43 \cdot 10^{-3}i)</td>
<td>(4.26 \cdot 10^{-3} - 3.62 \cdot 10^{-3}i)</td>
</tr>
<tr>
<td>(1)</td>
<td>(4.31 \cdot 10^{-7} - 3.16 \cdot 10^{-6}i)</td>
<td>(-6.64 \cdot 10^{-5} - 7.14 \cdot 10^{-5}i)</td>
</tr>
<tr>
<td>(2)</td>
<td>(6.79 \cdot 10^{-11} - 5.01 \cdot 10^{-11}i)</td>
<td>(-5.78 \cdot 10^{-11} - 6.21 \cdot 10^{-11}i)</td>
</tr>
<tr>
<td>(3)</td>
<td>(1.35 \cdot 10^{-18} - 9.95 \cdot 10^{-18}i)</td>
<td>(-9.69 \cdot 10^{-18} - 1.04 \cdot 10^{-17}i)</td>
</tr>
</tbody>
</table>

Table 2 Coefficients \(b_{mn}\) for parameters \(\lambda = 50, a = 100, \beta = 10^5, \gamma = 0\) and \(H = 25\)

<table>
<thead>
<tr>
<th>(m)</th>
<th>(n = 2)</th>
<th>(n = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(2.95 \cdot 10^{-1} - 1.12 \cdot 10^{-2}i)</td>
<td>(6.09 \cdot 10^{-1} - 4.95 \cdot 10^{-1}i)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(1.41 \cdot 10^{-3} + 2.82 \cdot 10^{-4}i)</td>
<td>(-4.28 \cdot 10^{-3} + 3.89 \cdot 10^{-3}i)</td>
</tr>
<tr>
<td>(0)</td>
<td>(-2.62 \cdot 10^{-3} + 1.76 \cdot 10^{-3}i)</td>
<td>(4.68 \cdot 10^{-3} - 3.39 \cdot 10^{-3}i)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2.07 \cdot 10^{-7} - 7.89 \cdot 10^{-7}i)</td>
<td>(-6.30 \cdot 10^{-6} - 7.74 \cdot 10^{-6}i)</td>
</tr>
<tr>
<td>(2)</td>
<td>(8.87 \cdot 10^{-11} - 3.38 \cdot 10^{-11}i)</td>
<td>(-5.54 \cdot 10^{-12} - 6.81 \cdot 10^{-12}i)</td>
</tr>
<tr>
<td>(3)</td>
<td>(1.94 \cdot 10^{-18} - 7.39 \cdot 10^{-18}i)</td>
<td>(-9.37 \cdot 10^{-19} - 1.15 \cdot 10^{-18}i)</td>
</tr>
</tbody>
</table>

Fig. 3 Real part (a and c) and imaginary part (b and d) of displacement for a plane wave of unit amplitude incident from negative x-direction. Parameters chosen as \(\lambda = 50, a = 100, \beta = 10^5, \gamma = 0\) and \(H = 25\). Number of points in angular expansion: \(N = 16\) and \(M = 8\). Plots (c) and (d) calculated using arbitrary shaped plate method.

Fig. 4 Real part (a and c) and imaginary part (b and d) of displacement for a plane wave of unit amplitude incident from negative x-direction. Parameters chosen as \(\lambda = 50, a = 100, \beta = 10^5, \gamma = 0\) and \(H = 1\). Number of points in angular expansion: \(N = 16\) and \(M = 8\). Plots (c) and (d) calculated using arbitrary shaped plate method.