A Jigsaw Puzzle of Basic Risk-adjusted Performance Measures

In 1997, Modigliani and Modigliani developed the risk-adjusted performance measure RAP (often called M-squared), which is now widely accepted in theory and practice. Their measure has further increased investor awareness of risk-adjusted performance measurement. However, this measure uses the standard deviation as the relevant measure of risk, and, therefore, is relevant only to investors who invest their entire savings into a single fund. In this article the authors present a jigsaw puzzle of basic risk-adjusted performance measures, which helps to better understand the key links between these measures. In doing so they include a hardly known measure: the market risk-adjusted performance (MRAP). While closely related to the Modigliani measure, the MRAP measures returns relative to market risk instead of total risk. Thus, the MRAP is suitable for investors who invest in many different assets.

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Modigliani and Modigliani (1997) introduced "risk-adjusted performance" (RAP) as a new measure to gauge the performance of investment funds. It uses the standard deviation as the relevant measure of risk and can be easily understood by investors. By now, the RAP (often also referred to as "M-squared" or "M^2")[1] has been widely accepted in theory and practice (see, for example, Sharpe, Alexander and Bailey (1999), Hopkins and Acton (1999)) and serves as a basis for further performance measures.[2] However, the RAP is based on total risk, taking only the fund's standard deviation of returns into account. Thus, in a strict sense, it only suits investors who exclusively invest in a portfolio consisting of a single investment fund and either investing or borrowing at the risk-free rate.[3]

We introduce a jigsaw puzzle or system of basic risk-adjusted performance measures that helps understand the key differences between these performance measures and clarifies the links between them (see Table 1). In doing so we include a still hardly known measure: the "market risk-adjusted performance" (MRAP). This performance measure is similar to RAP, except that it is based on market risk instead of total risk. Since the MRAP considers the market risk of the funds, it is also relevant for investors who invest in a variety of assets. This permits a sensible comparison of the performance of investment funds on the basis of a measure that can easily be interpreted.

In order to explain the key differences between the basic performance measures and to point out the links between them, we present an example. It is based on fictitious annualized returns of a market index and of two funds A and B (see Table 2).[4] Investors can invest or borrow at the constant risk-free rate of 3% p.a. at any time.

Definitions and Notation

Before proceeding to operational formulae, we summarize the notation to be used as:

r_f = risk-free interest rate

μ_i = average return of fund i

μ_M = average return of the market index

σ_i = standard deviation of the returns of fund i

σ_M = standard deviation of the returns of

Table 1
Jigsaw puzzle of basic risk-adjusted performance measures

interpretation \ risk	total risk	market risk
Ratio dividing the excess return of the fund by its risk	Sharpe Ratio (SR)	Treynor Ratio (TR)
Differential return between fund and risk-adjusted market index	Total Risk Alpha (TRA)	Jensen Alpha (JA)
Return of the risk-adjusted fund	Risk-adjusted Performance (RAP)	Market Risk-adjusted Performance (MRAP)
Differential return between risk-adjusted fund and market index	Differential Return based on RAP (DR^{RAP})	Differential Return based on MRAP (DR^{MRAP})

Table 2
Example (p. a.)

	Fund A	Fund B	Market Index
Average Return (μ_i)	12.0%	17.4%	11.0%
Standard Deviation (σ_i)	15%	24%	17%
Jensen Alpha (JA_i)	3.4%	4.0%	0.0%
Beta (β_i)	0.7	1.3	1

the market index

β_i = market risk of fund i (beta)

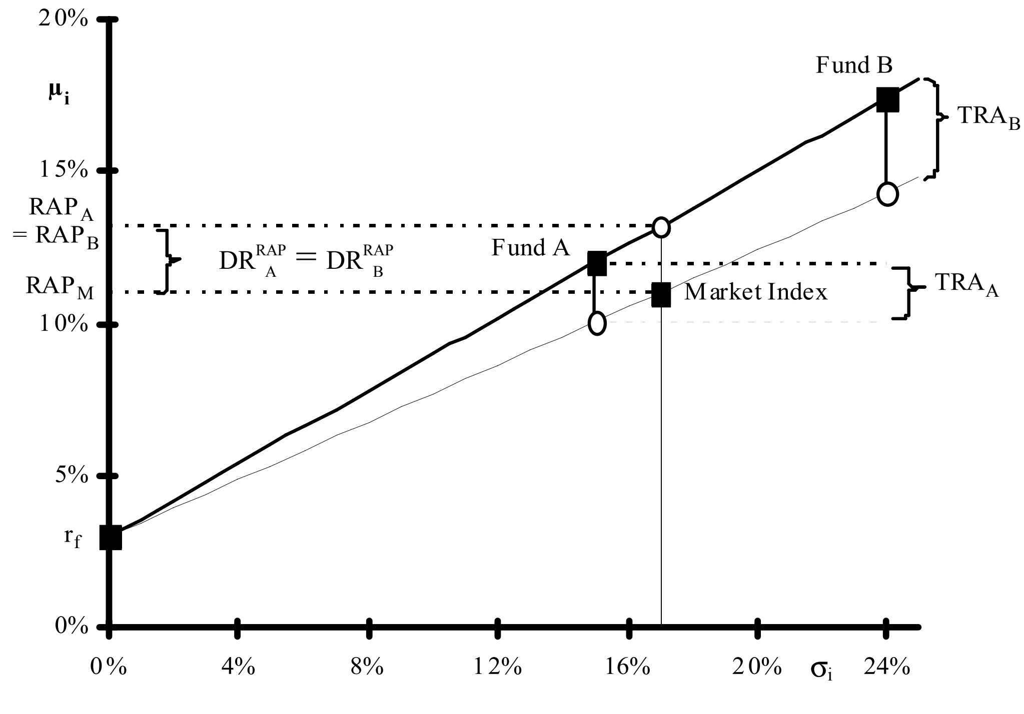
RISK-ADJUSTED PERFORMANCE MEASURES BASED ON TOTAL RISK

The Sharpe Ratio (SR), described in Sharpe (1966, 1994), is probably still the most widely employed performance measure in practice. It uses the standard deviation as the measure of total risk. Geometrically, the Sharpe Ratio of a fund i is given by the slope of the connecting line between the risk-free rate and the fund in the μ-σ -diagram. It is computed as:[5]

$$SR_i = \frac{\mu_i - r_f}{\sigma_i} \quad . \qquad (1)$$

All other things being equal, higher Sharpe Ratios translate into a higher performance. In our example,

Figure 1
Performance measures based on total risk



both funds A and B outperform the market index but have identical Sharpe Ratios.[6]

The total risk alpha (TRA), Fama (1972) calls it "net-selectivity," measures the performance of a fund by comparing its returns with those of a benchmark portfolio (BP). This benchmark portfolio represents the market index matched to the total risk of the fund.[7] Formally,

$$TRA_i = \mu_i - \mu_{BP\,i} \quad , \tag{2}$$

where

$$\mu_{BP\,i} = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_i \quad . \tag{3}$$

Since the total risk alpha measures performance in terms of basis points, it is easier to interpret than the Sharpe Ratio.

Gressis, Philippatos and Vlahos (1986) propose an alternative formula to derive the total risk alpha, which helps understand the link between the total risk alpha and the Sharpe Ratio:

$$TRA_i = \sigma_i\,(SR_i - SR_M) \ . \tag{4}$$

In our example, fund B has a higher total risk alpha than fund A. However, the TRA is not a suitable device for ranking performances. An investor in fund A could have created a portfolio which exactly matches the return and risk of fund B simply by borrowing at the risk-free rate and investing these proceeds in fund A. This effect is called the leverage bias.

The popular risk-adjusted performance (RAP) of Modigliani and Modigliani (1997) also uses the standard deviation as the relevant risk measure. Starting from the original μ-σ-position of the fund, Modigliani and Modigliani's procedure calls for creating the same degree of total risk as the market index by (de-)leverage. The RAP then simply expresses the average return of the resulting risk-adjusted fund (RAF) in basis points. Therefore, the average investor can easily understand this measure, which is calculated as

$$RAP_i = \mu_{RAF\,i} = \frac{\sigma_M}{\sigma_i}(\mu_i - r_f) + r_f$$
$$= SR_i\,\sigma_M + r_f . \tag{5}$$

Inspection of equation 5 shows that the risk-adjusted performance transforms the Sharpe Ratio into an absolute performance measure. Thus, funds can be ranked according to their RAP, and the resulting ranking does not depend on the market index chosen. The RAP identifies the fund which, in combination with r_f, achieves the highest risk-adjusted return at any given risk level. Hence, the RAP measure combines the advantage of the Sharpe Ratio (possibility of ranking) with that of the total risk alpha (measurement in basis points).

We call the risk-adjusted performance of fund i relative to the market index "differential return based on RAP" (DR_i^{RAP}),[8] which is calculated as

$$DR_i^{RAP} = RAP_i - RAP_M. \quad (6)$$

On the basis of the risk-adjusted performance, a fund i outperformed the market index whenever the differential return based on RAP (DR_i^{RAP}) is positive. In order to illustrate the links between the performance measures described above, we later exploit that the DR_i^{RAP} can be rewritten as:

$$DR_i^{RAP} = TRA_i \frac{\sigma_M}{\sigma_i} = \sigma_M (SR_i - SR_M). \quad (7)$$

RISK-ADJUSTED PERFORMANCE MEASURES BASED ON MARKET RISK

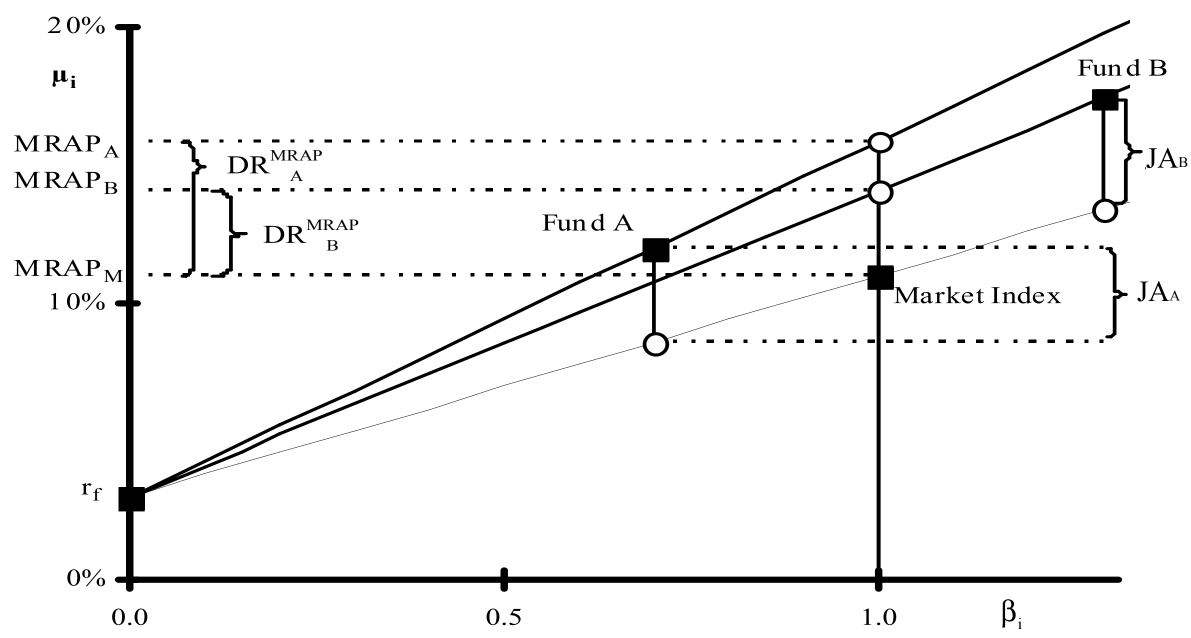
This section presents the basic risk-adjusted performance measures based on market (or systematic) risk, as measured by the beta factor. Each of these measures has a corresponding measure in the μ-σ-context (see Table 1). Market risk is relevant to investors whose portfolio contains not only the analyzed fund but also a variety of other assets. In this case, the investor may neglect the fund's unsystematic risk due to the diversification of the portfolio.[9]

Treynor (1965) introduced the Treynor Ratio (TR), or "Reward-to-Volatility ratio", as the first risk-adjusted performance measure for investment funds. It is calculated as

$$TR_i = \frac{\mu_i - r_f}{\beta_i}. \quad (8)$$

In the μ-β-diagram, the Treynor Ratio gives the slope of the line connecting all possible μ-β-combinations that investors could have achieved by investing in the

Figure 2
Performance Measures Based on Market Risk



fund and investing or borrowing at the risk-free rate. The Treynor Ratio differs from the Sharpe Ratio only through the choice of the beta factor, instead of the standard deviation, as the relevant risk measure. In the form presented, and with the interpretation commonly given in the literature, both measures share the disadvantage that they do not provide any guidance for analyzing return differentials. Thus, investors who are not familiar with capital market theory and regression analysis will find the Treynor Ratio difficult to interpret.

Jensen (1968) introduced a measure whose geometric interpretation in the μ-β-diagram corresponds directly to that of the total risk alpha in the μ-σ-diagram. The Jensen Alpha (JA) computes the difference between the average return to the fund and the average return to a benchmark portfolio (BP*) whose market risk is identical to that of the fund. Formally,

$$JA_i = \mu_i - \mu_{BP^*i} , \qquad (9)$$

where

$$\mu_{BP^*i} = r_f + \beta_i (\mu_M - r_f). \qquad (10)$$

Both the Jensen Alpha and the total risk alpha share the disadvantage that results can be easily manipulated by means of leverage. Rudd/Clasing (1988, p. 429) state: "If a manager has a positive alpha, then it is easy to double it simply by doubling the active holdings. Hence, alpha itself is a meaningless parameter." A ranking of funds on the basis of their Jensen Alphas, say B > A since $JA_B > JA_A$, is misleading. For each chosen risk level $\beta > 0$ an investor could have created a combination of fund A and investing or borrowing at the risk-free rate that out-performs a combination of fund B and investing or borrowing at the risk-free rate. Because of this leverage bias, rankings should be based on the Treynor Ratio rather than on the Jensen Alpha.[10] The performance measure Jensen Alpha is related to the Treynor Ratio as follows:

$$TR_i = \frac{JA_i}{\beta_i} + \mu_M - r_f = \frac{JA_i}{\beta_i} + TR_M. \qquad (11)$$

That is, funds with identical Jensen Alphas but differing betas have different Treynor Ratios.

Summarizing, we observe that both market risk-based performance measures presented above create problems. While the Treynor Ratio is difficult to interpret for an average investor, the Jensen Alpha does not permit an appropriate ranking of funds when the funds have different levels of exposure to the market. It appears that there is a piece missing in the jigsaw puzzle of basic performance measures. This gap is filled up with the still hardly known "market risk-adjusted performance" (MRAP) as an alternative performance measure, which overcomes these problems.[11] The MRAP follows the logic underlying Modigliani and Modigliani's risk-adjusted performance measure, RAP, but it incorporates the market risk of a fund rather than its total risk.

The idea is to compare funds on the basis of a measure of market risk that is identical for all funds. A natural candidate for this benchmark is the beta factor of the market index, $\beta_M = 1$. To obtain the market risk-adjusted performance for fund i, it is (de-)levered in order to achieve a Beta equal to one. If the fund's systematic risk exceeds that of the market ($\beta_i > 1$), this procedure can be interpreted as a fictitious sale of some fraction d_i of fund holdings and then investing the proceeds at the risk-free rate ($d_i < 0$). Similarly, if the fund's systematic risk falls below that of the market index ($\beta_i < 1$), the procedure corresponds to a fictitious loan at the risk-free rate, amounting to some fraction d_i, in order to increase investments into the fund ($d_i > 0$). The fraction d_i is calculated as follows:

$$d_i = \frac{1}{\beta_i} - 1. \qquad (12)$$

In order to obtain the market risk-adjusted performance of fund i ($MRAP_i$), we average the return of the market risk-adjusted fund (MRAF):

$$\begin{aligned} MRAP_i = \mu_{MRAF\,i} &= (1 + d_i)\,\mu_i - d_i\, r_f \\ &= \mu_i + d_i (\mu_i - r_f). \qquad (13) \end{aligned}$$

On this basis, a fund, adjusted for market risk, outperformed the market index whenever its market risk-adjusted performance exceeds the return of the market index. In our example, both funds outperformed the market index. Note that fund A would have had a higher performance than fund B for any level of market risk. Ranking funds according to their MRAPs corresponds to ranking them based on their Treynor Ratios.

An alternative representation of the MRAP and the Treynor Ratio is instructive. Replacing d_i and rearranging terms, we obtain:

$$MRAP_i = TR_i + r_f. \quad (14)$$

Hence, the Treynor Ratio also can be interpreted as the excess return of the market risk-adjusted fund expressed in basis points.

Substituting in equation 14 for the Treynor Ratio given by equation 11 yields a representation that admits several alternative interpretations of the market risk-adjusted performance:

$$MRAP_i = \frac{JA_i}{\beta_i} + \mu_M = \frac{JA_i}{\beta_i} + TR_M + r_f. \quad (15)$$

Equations 14 and 15 imply that rankings based on MRAP, alpha-beta ratio, and Treynor Ratio all lead to the same result. Figure 3 illustrates the relationships explained above.

The MRAP is particularly convenient for comparing the performance across funds or relative to the market index since the variation in performance can be measured by simply calculating the difference between the respective MRAPs. These differences are easy to interpret because they are measured in terms of basis points. We call the market risk-adjusted performance of funds i relative to the market index "differential return based on MRAP (DR_i^{MRAP})",[12] which is given by

$$DR_i^{MRAP} = MRAP_i - MRAP_M = MRAP_i - \mu_M. \quad (16)$$

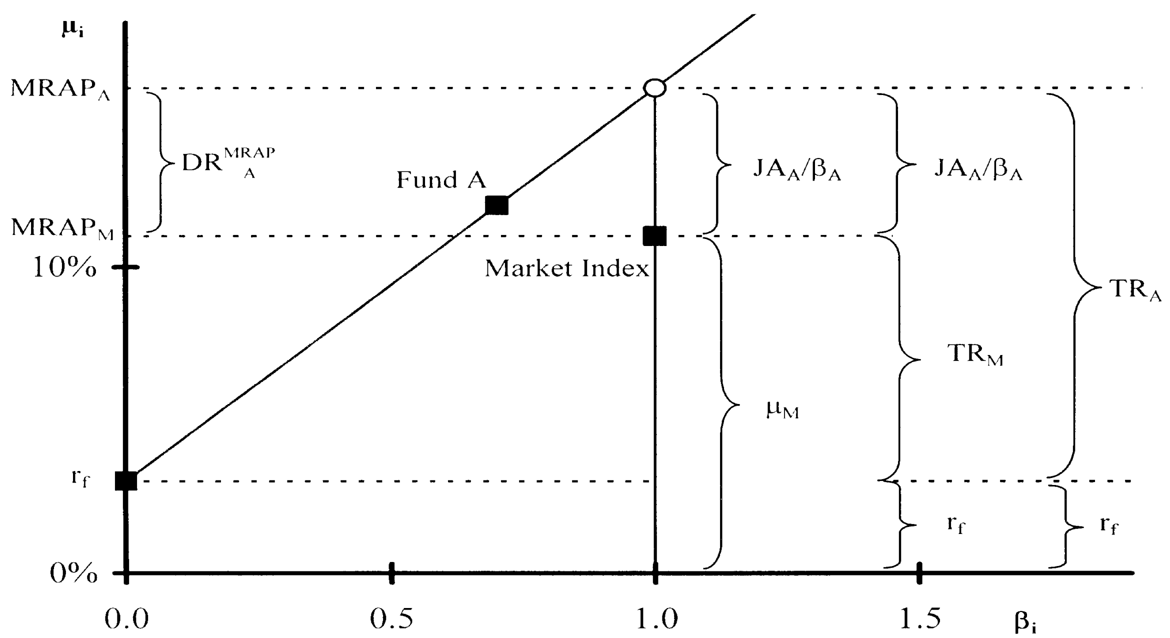
From equation 15 it follows that

$$DR_i^{MRAP} = \frac{JA_i}{\beta_i}. \quad (17)$$

That is, the differential return based on the MRAP can be calculated either by taking the difference between the appropriate MRAP measures or by calculating the Jensen Alpha-Beta Ratio. Thus, normalizing the Jensen Alpha by the market risk takes care of the ranking problem of the Jensen Alpha of funds with dissimilar market exposure.[13]

Summarizing, all measures based on market risk give the same assessment of a portfolio's performance relative to the market index. However, the Jensen Alpha can lead to different portfolio rankings than the other three measures. These results carry over to the measures based on total risk.

Figure 3
Alternative Interpretations of the MRAP and the Treynor Ratio



CONCLUSION

This paper presents a system of basic performance measures which helps understand the links between a number of basic performance measures and illustrates the key differences between them. To complete this system we present the hardly known performance measure "market risk-adjusted performance" (MRAP) that measures the market risk-adjusted performance of funds in terms of basis points. Geometrically, the MRAP in the μ-β-diagram is the analogue of Modigliani and Modigliani's RAP in the μ-σ-diagram. A convenient feature of the MRAP is that it combines the advantage of the Treynor Ratio (appropriate ranking) with that of the Jensen Alpha (interpretability). In practice, the appropriate risk measure on which to base performance measurement depends on the portfolio of an investor and can either be total risk, market risk, or both.[14] Therefore, the MRAP, a measure based on market risk, is the logical extension of the RAP, which is based on total risk.

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ENDNOTES

[1] We avoid calling the RAP measure "M^2" in order not to use the same term as in Fong and Vasicek (1984) for a risk measure of an immunized fixed-income portfolio, see also Liano (2000).

[2] Lobosco (1999) modified the RAP, creating the "style/risk-adjusted performance" as a measure that accounts for investment style. Muralidhar (2000) proposes a new measure, calling it the M-3 measure, capturing not only the fund's standard deviation of returns but also the correlation of the fund's returns with the benchmark returns.

[3] See *e.g.,* Bodie, Kane, and Marcus (2005), pp. 870-872.

[4] As usual, we assume time-invariant, independent and identical distributions of the funds' returns. Combined with the assumption that portfolio weights are rebalanced at the beginning of each period, this requires the returns of the individual securities and of the market index to be time-invariant and independently and identically distributed.

[5] Sharpe Ratio determined use of the ex-post estimated distributions of the funds' returns can be distorted by the "market climate bias." To overcome this bias, so-called "normalized" Sharpe Ratios can be calculated; see Scholz and Wilkens (2004a).

[6] Table 3 summarizes the numerical results for all performance measures of funds A and B as well as the market index, respectively.

Table 3
Risk-adjusted Performance Measures in the Example

	Fund A	Fund B	Market Index
SR_i	0.6000	0.6000	0.4706
TRA_i	1.94%	3.11%	0.00%
RAP_i	13.20%	13.20%	11.00%
DR_i^{RAP}	2.20%	2.20%	0.00%
TR_i	0.1286	0.1108	0.0800
JA_i	3.40%	4.00%	0.00%
$MRAP_i$	15.86%	14.08%	11.00%
$DR_i^{MRAP} = JA_i/\beta_i$	4.86%	3.08%	0.00%

[7] In Figures 1 through 3, the symbol ■ denotes the locations of the two funds, the market index, and the risk-free rate (r_f), respectively. Moreover, by the symbol ○ we denote the benchmark portfolios or the (market) risk-adjusted funds, which are needed to compute the performance measures.

[8] Sometimes the differential return based on the RAP is called the "M^2 measure;" see Bodie, Kane, and Marcus (2005), pp. 869-870.

[9] See, *e.g.,* Bodie, Kane, and Marcus (2005), pp. 872-874.

[10] See Haugen (2001), pp. 276-279.

[11] Analogous to RAP we might dub this measure MRAP. In Modigliani (1997) it is called "M^2-for-beta".

[12] Bodie, Kane, and Marcus (2005), p. 874, call this differential return the "T^2 measure" (analogous to their definition of the "M^2 measure," see endnote 8).

[13] Originally, Smith and Tito (1969) introduced the alpha-beta ratio as a performance measure, calling it the "modified Jensen."

[14] See Scholz. Hendrick, and Marco Wilkens (2004b).