# Mathematical Argumentation and Proof – Supporting a Complex Cognitive Skill

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**Abstract:** During the last decades, research has provided evidence that handling mathematical argumentation and proof (MA&P) represents a complex cognitive skill, which requires various constituent skills. However, research on MA&P skills as well as their facilitation largely disregards this fact and effective means to foster the constituents and overall MA&P skills remain mainly unclear. Transferring research on the facilitation of complex cognitive skills from instructional design, two approaches may be effective: Fostering the constituents one by one respectively fostering them simultaneously. We therefore present an intervention study that takes a holistic approach on MA&P skills and their constituents, comparing a sequential (one-by-one) and an integrated (simultaneous) instructional approach to foster each constituent skill as well as students' overall MA&P skills. The results show that learners in the integrated condition and the sequential condition have very similar learning gains that differ only in their mathematical strategic knowledge, which is superior in the integrated condition.

**Keywords**: mathematics education, proof, argumentation, intervention, whole-task approach

## Introduction

Reasoning, argumentation and proof are of special importance within the *proving science* mathematics. Therefore, research on mathematical argumentation and proof (MA&P) skills has been a long-term focus within mathematics education (Hanna, 2000). Most MA&P research, however, disregards the fact that MA&P is a complex cognitive skill integrating a variety of domain-general and domain-specific constituents, e.g. knowledge facets, sub-skills, and beliefs. A recent review on research on MA&P within mathematics education has shown that studies taking these constituents into account and conceptualizing MA&P skills in a holistic way are rare (Sommerhoff, Ufer, & Kollar, 2015). However, research on instructional design (Anderson, Reder, & Simon, 1996; Branch & Merrill, 2011; van Merriënboer & Kester, 2007) shows that acknowledging the complex structure of MA&P skills has important implications for how MA&P skills and their constituents should be supported and which instructional designs should be used. Knowing how to best foster MA&P skills and their constituents is important since research has repeatedly revealed students' severe problems with MA&P tasks (e.g. Weber, 2001). We therefore present an intervention study focusing on several constituents of MA&P skills, comparing a sequential and an integrated instructional approach to foster these within university mathematics.

## MA&P as a complex cognitive skill

Research on complex cognitive skills, e.g. (information) problem solving, has become increasingly important in instructional design. Yet, in spite of several theoretical accounts and a multitude of studies on how complex skills can be effectively fostered in general, empirically validated approaches to foster MA&P skills are still scarce. However, most studies underline that acknowledging the constituents of the complex skill is essential for designing powerful learning environments.

Table 1: Constituents of MA&P skills investigated in the current study

| Constituent | Characterization |
|---|---|
| Mathematical knowledge base | Basic conceptual and procedural knowledge in the field of mathematics (Ufer et al., 2008) |
| Methodological knowledge | Knowledge of the nature and the functions of proof as well as the acceptance criteria for a valid proof (Healy & Hoyles, 2000) |
| Mathematical strategic knowledge | Knowledge about cues within mathematical tasks and problems that indicate which concepts and representation systems can be used productively (Weber, 2001) |
| Problem solving strategies | Domain-general and domain-specific problem solving strategies (Schoenfeld, 1985) |

MA&P skills represent a complex cognitive skill and up to now several constituents of MA&P skills have been identified and shown to be predictive for the success of MA&P processes (e.g. Chinnappan, Ekanayake, & Brown, 2011). In our study, we utilize a model that includes four main constituents of MA&P skills (Table 1). It is based on existing, models for geometry proofs (Chinnappan et al., 2011; Ufer, Heinze, & Reiss, 2008) as well as frameworks for general mathematical problem solving (Schoenfeld, 1985) and self-regulated learning (De Corte, Verschaffel, & Eynde, 2000).

## Supporting mathematical argumentation and proof skills

Taking the perspective of MA&P skills being dependent on various constituents and thinking about an optimal way to facilitate these, at least two opposing strategies for the design of powerful instruction emerge: Supporting each constituent separately one-by-one (*sequential approach*) or handling all of these simultaneously (*integrated approach*). Although little research has been done on the promotion of constituents of MA&P skills, hints can be found in instructional design research. Within the last decades, instructional design researchers have debated about the effectiveness of part-task *vs.* whole-task approaches (Anderson et al., 1996; Branch & Merrill, 2011; Lim, Reiser, & Olina, 2009). Classical instructional design approaches assume the decomposability of complex tasks into less complex part-tasks and recommend the separate training of each of these less complex part-tasks. The decomposition theory from ACT-R (Anderson, 2002) even breaks down complex tasks to actions happening within milliseconds. The part-task approach is guided by the ideas that instruction on part-tasks is of higher instructional clarity for the students, that each part-task is easier to master and that the learning gains on the part-tasks transfer easily to learning gains on the overall task.

On the other hand, the whole-task approach (van Merriënboer & Kester, 2007) as well as the situated cognition approach (Brown, Collins, & Duguid, 1989) reject this atomization of tasks, provide evidence for the situatedness of learning, and point to difficulties that are associated with attempts to transfer from part-tasks to the overall task (Anderson et al., 1996). This implies teaching knowledge facets, sub-skills, attitudes, and beliefs constituting a complex cognitive skill in an integrated way (van Merriënboer & Kirschner, 2007).

Leveraging these two positions and transferring them back to MA&P skills and their constituents, we therefore contrast these two approaches empirically: A sequential approach*,* with students working on proof tasks with an explicit focus on only one of the required constituents at a time, compared to an integrated approach, with students working on proof tasks with a specific focus on multiple constituents at a time.

## Aim and research question

The goal of the present intervention study is to explore the effects of an integrated and a sequential instructional approach on four constituents of MA&P skills (mathematical knowledge base, methodological knowledge, mathematical strategic knowledge, and problem solving strategies). We therefore investigated whether these approaches differ in their effects on students' knowledge and skills regarding these constituents. No *a priori* hypothesis was established regarding the greater effectiveness of each approach since theoretical arguments and evidence in support of both approaches exist (Lim et al., 2009). Yet, both approaches were expected to yield positive learning gains from pretest to posttest.

## Methods

### Study design, dependent variables and procedure

We used a quasi-experimental design for our study. The intervention was offered as a voluntary course for first year mathematics university students entitled "Mathematical proof: that's how you do it". It was scheduled between first and second semester and consisted of a pretest and a posttest as well as four two-hour intervention sessions across three consecutive days. The intervention consisted of two parallel courses representing the integrated condition and the sequential condition, respectively. Two instructors with prior experience in lecturing led the courses and switched after two units to eliminate instructor effects. Both courses covered the same content, the same tasks and time on task, although tasks and content were arranged in a different order.

During the intervention, students were provided with information on all four constituents by short presentations. Additionally, they were given a short list of prompts meant to enhance the analysis of tasks according to each constituent prior to the actual solving process (e.g. "Excerpt all important objects and properties from the task, explain these in your own words and compare them to the formal definition.", "Search the task for keywords that you know from other tasks. What methods did you use there?"). The instructor afterwards demonstrated the usage of the prompts for each of the constituents. All in all, students worked on eight tasks, and each task was analyzed regarding two constituents, solved and discussed with the instructor. Both groups received guidance during their work on the tasks. For the sequential group, each session contained a presentation on one constituent as well as four tasks that the students analyzed regarding the same constituent. Each task was then picked up in another session for the analysis of a second constituent. Within the integrated condition, the presentations were divided into two larger blocks, so that most theory on each constituent was given in session 1 and only additional points were introduced in session 3. The students directly analyzed each task regarding two of the four constituents. In order to have the students of the integrated condition work two times on each task like the students in the sequential condition, tasks that had already been analyzed and solved were reconsidered briefly once more in this condition.

## Instruments

The pretest and the posttest included self-designed scales assessing the constituents of MA&P skills on limits and infinite sums, a scale of four MA&P items, as well as control variables for inferential reasoning, metacognition, and scientific reasoning and argumentation (Gormally, Brickman, & Lutz, 2012; Inglis & Simpson, 2008; Schraw & Dennison, 1994). The MA&P items covered in the course and tests were closely related to a regular proof-oriented calculus course, but novel to the students in order to avoid bias by prior experience. The pretest and posttest were created using parallel items and contained open as well as closed items. The closed items were evaluated using mark-recognition software with a subsequent manual control of the recognition results. Two raters coded the open items following a theory-based coding scheme. The interrater reliability of the coders was $\kappa > .76$ ($M = .92$; $SD = .09$). The scales used in the both tests had an overall acceptable internal consistency of $\alpha_{\text{Mean}} = .70$ ($SD = .10$) with individual values ranging from $\alpha = .58$ (mathematical strategic knowledge; 4 items) to $\alpha = .81$ (problem solving strategies; 48 items). The results for all constituents were re-scaled to values between 0 (worst) and 1(best). Additionally all documents used by the participants were gathered throughout the intervention to analyze this process data later.

## Sample

A total of 46 students (19 male, 27 female) participated in the study. The participants were first and second year mathematics students (first year: 36, second year: 5, no indication: 5). They can be assumed to have participated in the calculus I lecture and have had prior experience with proof-based real analysis. 24 and 22 students were assigned to the integrated and sequential condition, respectively. Several participating students had failed the exam of the calculus I course, thus the sample can be assumed to be slightly lower performing than average.

# Findings

The pretest results verified that both conditions were comparable in their performance on the constituents prior to the intervention (Table 2, upper part). A Mann-Whitney U test indicated no significant differences between both conditions could be shown. Only methodological knowledge slightly approached significance ($U = 184.5$, $p = .078$). There were also no significant differences between the two conditions regarding the assessed control variables (mean final high school qualification grade, high school qualification grade in mathematics, inferential reasoning skills, metacognition, scientific reasoning and argumentation skills). Accordingly, they were not controlled for in the further analysis.

Table 2: Mean values for MA&P constituents obtained in the pre- and posttest

| | | Knowledge base | | Methodological knowledge | | Math-strategic knowledge | | Problem solving strategies | |
|---|---|---|---|---|---|---|---|---|---|
| | | *M* | *SD* | *M* | *SD* | *M* | *SD* | *M* | *SD* |
| **Pretest** | Sequential | 0.32 | 0.17 | 0.42 | 0.16 | 0.34 | 0.17 | 0.53 | 0.15 |
| | Integrated | 0.39 | 0.16 | 0.51 | 0.18 | 0.39 | 0.17 | 0.58 | 0.11 |
| **Posttest** | Sequential | 0.43 | 0.21 | 0.57 | 0.16 | 0.59 | 0.17 | 0.66 | 0.13 |
| | Integrated | 0.48 | 0.13 | 0.61 | 0.16 | 0.70 | 0.18 | 0.71 | 0.10 |

The posttest results (Table 2, lower part) showed significant ($p < .001$) learning gains for both conditions and all constituents. Nevertheless, the longitudinal effects across groups for mathematical strategic knowledge ($d_C = 1.595$) and problem solving strategies ($d_C = 1.052$) were larger than for methodological knowledge ($d_C = 0.751$) and mathematical knowledge base ($d_C = 0.582$) although the same amount of time was spent on all constituents.

Comparing the results of the integrated and sequential condition in the posttest, a significant difference can only be found for mathematical strategic knowledge ($U = 164.5$, $p = .027$), in favor of the integrated group. All other comparisons between conditions were insignificant ($U > 179$, $p > .061$).

## Conclusions and implications

The results of our study reveal that both instructional approaches differ less than implied by the theories promoting a part-task and a whole-task approach, respectively (Anderson et al., 1996; Branch & Merrill, 2011). Good arguments for both approaches exist and within this study, both approaches yielded comparable learning gains for the constituents of MA&P skills with the exception of mathematical strategic knowledge, which showed better learning outcomes for the integrated approach.

Remarkably, the results of our relatively short intervention study show large learning gains, especially for mathematical strategic knowledge and problem solving strategies. Large learning gains particularly for these two constituents are reasonable, because university instruction usually does not explicitly focus on these constituents so that little prior knowledge can be assumed. The absolute effect sizes, however, should be considered with care due to the lack of a proper control group, addressing e.g. re-testing biases or effects by the sheer engagement in proofs. Creating such a control group is challenging because approaches with students not doing proofs at all or practicing unguided both have drawbacks. Nevertheless, such a controlled study will be an important step to validate effect sizes of individual constituents. Ongoing evaluation of collected data will show the effect of the intervention on overall MA&P skills as well as overall MA&Ps relation to the constituents. The results will give further insights how to create an effective holistic approach to foster MA&P skills.

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