Multilevel iterative solution and adaptive mesh refinement for mixed finite element discretizations

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Abstract

We consider the numerical solution of elliptic boundary value problems by mixed finite element discretizations on simplicial triangulations. Emphasis is on the efficient iterative solution of the discretized problems by multilevel techniques and on adaptive grid refinement. The iterative process relies on a preconditioned conjugate gradient iteration in a suitably chosen subspace with a multilevel preconditioner of hierarchical type that can be constructed by means of appropriate multilevel decompositions of the mixed ansatz spaces. Using the Dryja–Widlund theory of additive Schwarz iterations, we show that the spectral condition number of the preconditioned stiffness matrix asymptotically exhibits a quadratic growth in the refinement level as it is the case in the standard conforming approach. The adaptive grid refinement is based on an efficient and reliable a posteriori error estimator for the total error in the flux which can be established by a defect correction in higher order mixed ansatz spaces combined with a hierarchical two-level splitting.

1. Introduction

During the past decade, the theory and application of multilevel techniques for the efficient numerical solution of elliptic boundary value problems discretized by standard conforming finite element methods has been extensively dealt with. For an overview and further references we refer to Oswald's monograph [27] and the survey articles by Xu [33], Yserentant [35] and Zhang [37]. Likewise, beginning with the pioneering work done by Babuška and Rheinboldt [3,4], adaptive grid refinement based on appropriate a posteriori error estimators has attracted considerable interest (cf., e.g., [7,8,15,26,31] and the references therein). We note that the realization of adaptivity concepts and the multilevel iterative solution go hand in hand, since in a natural way the multilevel iterative solver works on the adaptively generated hierarchy of triangulations.

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However, considerably less work has been done in the framework of adaptive multilevel techniques for nonstandard finite element approaches such as mixed finite element methods. A multigrid algorithm has been constructed and analyzed by Brenner [12] whereas multilevel iterative schemes based on multilevel decompositions of the mixed ansatz spaces have been considered in [14,18–21,23,29,32]. On the other hand, the technique of mixed hybridization involving interelement multipliers has been addressed in [2,13,22,28,32] by means of the multilevel iterative solution of equivalent primal nonconforming discretizations. As far as adaptive grid refinement is concerned, we are only aware of the recent papers by Braess et al. [10], Braess and Verfürth [11], Achchab et al. [1] and the authors [22,23,32].

In this paper, we shall be concerned with the iterative solution of the saddle point problem arising from the mixed discretization by a preconditioned cg-iteration in an appropriate subspace involving a hierarchical type multilevel preconditioner similar to that one proposed by Cai, Goldstein and Pasciak in [14]. As a main result, we shall prove that the spectral condition number of the preconditioned coefficient matrix quadratically grows with the number of refinement levels and thus shows the same asymptotic behavior as Yserentant's hierarchical preconditioner [35] in the standard conforming case. We note that this result considerably improves on the bound for the spectral condition number established in [14]. Moreover, we shall derive an a posteriori error estimator for the total error in the flux which can be used as an indicator for local refinement of the triangulations. This error estimator will be constructed by the principle of defect correction in higher order mixed ansatz spaces and an appropriate elementwise localization by a hierarchical two-level splitting of these ansatz spaces.

2. Mixed finite element discretization

We consider the following elliptic boundary value problem

$$\begin{aligned} Lu := -\mathrm{div}(a\nabla u) + bu + f &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$. We assume $f \in L^2(\Omega)$ and the coefficients to be a symmetric matrix-valued function $a = (a_{i,j})_{i,j=1}^2$, $a_{i,j} \in L^\infty(\Omega)$, $1 \leqslant i,j \leqslant 2$, and a function $b \in L^\infty(\Omega)$ satisfying for almost all $x \in \Omega$

$$\begin{aligned} &\alpha_0|\xi|^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leqslant \alpha_1|\xi|^2, \qquad \xi \in \mathbb{R}^2, \quad 0 < \alpha_0 \leqslant \alpha_1, \\ &0 \leqslant \beta_0 \leqslant b(x) \leqslant \beta_1. \end{aligned} \tag{2.2}$$

It should be noted that homogeneous Dirichlet boundary conditions have been chosen only for the ease of exposition. Other types of boundary conditions can be treated as well.

It is well known that (2.1) represents the Euler equation for the unconstrained minimization problem

$$\begin{aligned} J(u) &= \inf_{v \in H_0^1(\Omega)} J(v), \\ J(v) &:= \frac{1}{2}\left(\int_\Omega a|\nabla v|^2\,\mathrm{d}x + \int_\Omega b|v|^2\,\mathrm{d}x\right) - \int_\Omega fv\,\mathrm{d}x. \end{aligned} \tag{2.3}$$

The mixed finite element approach to (2.1) is based on the weak form of the dual Dirichlet problem. In particular, we denote by $H(\mathrm{div};\Omega)$ the flux space

$$H(\mathrm{div};\Omega) := \{\boldsymbol{q} \in (L^2(\Omega))^2 \mid \mathrm{div}\,\boldsymbol{q} \in L^2(\Omega)\}$$

which is a Hilbert space with respect to the inner product

$$(\boldsymbol{p},\boldsymbol{q})_{\mathrm{div}} := (\boldsymbol{p},\boldsymbol{q})_{0;\Omega} + (\mathrm{div}\,\boldsymbol{p},\, \mathrm{div}\,\boldsymbol{q})_{0;\Omega}$$

and the associated graph norm $\|\cdot\|_{\mathrm{div}} := (\cdot,\cdot)_{\mathrm{div}}^{1/2}$. Note that $(\cdot,\cdot)_{k;\Omega}$, $k \geqslant 0$, refers to the standard inner product on $H^k(\Omega)$ and $(H^k(\Omega))^2$ and $|\cdot|_{k;\Omega}$, $\|\cdot\|_{k;\Omega}$ denote the associated seminorms and norms, respectively.

Now, starting from (2.3) and using the duality argument

$$\frac{1}{2}\int_\Omega a\nabla v\cdot\nabla v\,\mathrm{d}x = \sup_{\boldsymbol{q}\in(L^2(\Omega))^2}\left(\int_\Omega \boldsymbol{q}\cdot\nabla v\,\mathrm{d}x - \frac{1}{2}\int_\Omega a^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,\mathrm{d}x\right), \quad v\in H_0^1(\Omega),$$

as well as Green's formula

$$\int_\Omega \boldsymbol{q}\cdot\nabla v\,\mathrm{d}x = -\int_\Omega \mathrm{div}\,\boldsymbol{q} v\,\mathrm{d}x, \qquad \boldsymbol{q}\in H(\mathrm{div};\Omega), \quad v\in H_0^1(\Omega),$$

we are led to the saddle point problem

$$L(\boldsymbol{j},u) = \inf_{\boldsymbol{q}\in H(\mathrm{div};\Omega)}\ \sup_{v\in L^2(\Omega)} L(\boldsymbol{q},v) \tag{2.4}$$

where $L: H(\mathrm{div};\Omega)\times L^2(\Omega)\to\mathbb{R}$ stands for the Lagrangian

$$L(\boldsymbol{q},v) := \frac{1}{2}\int_\Omega a^{-1}\boldsymbol{q}\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \mathrm{div}\,\boldsymbol{q}v\,\mathrm{d}x - \frac{1}{2}\int_\Omega b|v|^2\,\mathrm{d}x + \int_\Omega fv\,\mathrm{d}x.$$

The optimality conditions for (2.4) give rise to the following variational system which is referred to as the mixed formulation of (2.1):

$$\begin{aligned}&\text{Find } (\boldsymbol{j},u)\in H(\mathrm{div};\Omega)\times L^2(\Omega) \text{ such that}\\ &a(\boldsymbol{j},\boldsymbol{q}) + b(\boldsymbol{q},u) = 0, \quad \boldsymbol{q}\in H(\mathrm{div};\Omega),\\ &b(\boldsymbol{j},v) - c(u,v) = l(v), \quad v\in L^2(\Omega),\end{aligned} \tag{2.5}$$

where the bilinear forms a, b, c and the functional l are given by

$$a(\boldsymbol{p},\boldsymbol{q}) := \int_\Omega a^{-1}\boldsymbol{p}\cdot\boldsymbol{q}\,\mathrm{d}x, \quad \boldsymbol{p},\,\boldsymbol{q}\in H(\mathrm{div};\Omega),$$

$$b(\boldsymbol{q},v) := \int_\Omega \mathrm{div}\,\boldsymbol{q}v\,\mathrm{d}x, \qquad \boldsymbol{q}\in H(\mathrm{div};\Omega), \quad v\in L^2(\Omega),$$

$$c(u,v) := \int_\Omega b\,u\,v\,\mathrm{d}x, \quad u,\,v\in L^2(\Omega),$$

$$l(v) := \int_\Omega fv\,\mathrm{d}x, \quad v\in L^2(\Omega).$$

The unique solvability of (2.5) is a classical result (cf., e.g., [13, Section 2, Theorem 1.2]). We consider the standard mixed finite element discretization based on the use of the lowest order Raviart–Thomas elements with respect to a simplicial triangulation $\mathcal{T}_h$ of Ω. We denote by $\mathcal{N}_h$ and $\mathcal{E}_h$ the sets of vertices and edges of $\mathcal{T}_h$ and refer to $\boldsymbol{p}_\nu^T$ and e_ν^T, $1 \leqslant \nu \leqslant 3$, as the vertices and edges of an element $T \in \mathcal{T}_h$. Further, $P_k(D)$, $D \subseteq T$, $k \geqslant 0$, stands for the set of polynomials of degree $\leqslant k$ on D.

The primal variable u will be approximated by piecewise constants leading to the ansatz space

$$W_0(\Omega; \mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_h|_T \in P_0(T),\ T \in \mathcal{T}_h\}.$$

An appropriate approximation of the flux space $H(\mathrm{div}; \Omega)$ results from the natural requirement that the discrete fluxes $\boldsymbol{q}_h$ satisfy $\mathrm{div}\, \boldsymbol{q}_h \in W_0(\Omega; \mathcal{T}_h)$. This can be achieved by means of the Raviart–Thomas approximation

$$RT_0(\Omega; \mathcal{T}_h) := \{\boldsymbol{q}_h \in H(\mathrm{div}; \Omega) \mid \boldsymbol{q}_h|_T \in RT_0(T),\ T \in \mathcal{T}_h\}$$

where $RT_0(T)$, $T \in \mathcal{T}_h$, stands for the Raviart–Thomas element

$$RT_0(T) := P_0(T)^2 + P_0(T)\boldsymbol{x}, \quad \boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}.$$

The degrees of freedom are given by the following moments

$$\int_{\partial T} \boldsymbol{\nu} \cdot \boldsymbol{q}_h v \, \mathrm{d}\sigma, \quad v \in R_0(\partial T),$$

where $\boldsymbol{\nu}$ is the outer normal on ∂T and $R_0(\partial T) := \{p \in L^2(\partial T) \mid p|_{e_\nu^T} \in P_0(e_\nu^T),\ 1 \leqslant \nu \leqslant 3\}$.

Then, the mixed discretization of (2.1) can be stated as follows:

$$\begin{aligned}
&\text{Find } (\boldsymbol{j}_h, u_h) \in RT_0(\Omega; \mathcal{T}_h) \times W_0(\Omega; \mathcal{T}_h) \text{ such that} \\
&a(\boldsymbol{j}_h, \boldsymbol{q}_h) + b(\boldsymbol{q}_h, u_h) = 0, \quad \boldsymbol{q}_h \in RT_0(\Omega; \mathcal{T}_h), \\
&b(\boldsymbol{j}_h, v_h) - c(u_h, v_h) = l(v_h), \quad v_h \in W_0(\Omega; \mathcal{T}_h).
\end{aligned} \tag{2.6}$$

We note that the Babuška–Brezzi condition is satisfied and that (2.6) admits a unique solution (cf., e.g., [13, Section 2, Proposition 2.11]).

3. Multilevel iterative solution process

In this section, we consider the efficient iterative solution of the mixed discretization (2.6) by means of a preconditioned cg-iteration in an appropriately chosen subspace with a multilevel preconditioner of hierarchical type that can be derived from a hierarchical splitting of the mixed ansatz spaces. It should be noted that the preconditioner considered here corresponds to a fully additive multilevel Schwarz iteration in contrast to the preconditioner investigated by the authors in [23]. Hierarchical splittings of the mixed ansatz spaces have been considered before by Cai, Goldstein and Pasciak [14], Ewing and Wang [18–20] and Vassilevski and Wang [29].

We assume that $(\mathcal{T}_k)_{k=0}^l$ is a hierarchy of simplicial triangulations of Ω generated by the well known refinement process due to Bank et al. [6]. In particular, a triangle $T \in \mathcal{T}_k$, $0 \leqslant k \leqslant l-1$, either remains unrefined or is subdivided into four congruent subtriangles (regular refinement) or is bisected into two subtriangles (irregular refinement). The subtriangles are referred to as regular and irregular triangles, respectively. The following refinement rules have to be observed:

(R1) Each vertex of $\mathcal{T}_{k+1}$ that does not belong to $\mathcal{T}_k$ is a vertex of a regular triangle.

(R2) Irregular triangles must not further be refined.

(R3) Only triangles $T \in \mathcal{T}_k$ of level k, i.e., triangles that do not belong to $\mathcal{T}_{k-1}$, may be refined for the construction of $\mathcal{T}_{k+1}$.

It follows from the refinement rules that each triangle $T \in \mathcal{T}_k,\ 1 \leqslant k \leqslant l$, is geometrically similar either to an element of $\mathcal{T}_0$ or to an irregularly refined triangle of $\mathcal{T}_0$. Moreover, the sequence $(\mathcal{T}_k)_{k=0}^{l}$ is locally quasiuniform and can be uniquely reconstructed from the initial triangulation $\mathcal{T}_0$ and the final triangulation $\mathcal{T}_l$ (cf., e.g., [5,15,34]).

In the sequel, we refer to $\widehat{\mathcal{T}}$ as the set of all triangles being geometrically similar either to a triangle $T \in \mathcal{T}_0$ or to a triangle obtained from an element of $\mathcal{T}_0$ by single or double bisection. For $T \in \mathcal{T}_k,\ 0 \leqslant k \leqslant l$, we denote by h_T the diameter of T and by $h_{e_\nu^T}$ the length of the edge $e_\nu^T \in \mathcal{E}_k,\ 1 \leqslant \nu \leqslant 3$. Then, the regularity of the sequence $(\mathcal{T}_k)_{k=0}^{l}$ guarantees the existence of constants $0 < \kappa_0 \leqslant \kappa_1$ such that

$$\kappa_0 h_{e_\nu^T}^2 \leqslant |T| \leqslant \kappa_1 h_{e_\nu^T}^2, \quad 1 \leqslant \nu \leqslant 3,\ T \in \mathcal{T}_k, \tag{3.1}$$

where $|T|$ stands for the area of T.

We refer to $\alpha_\nu^T,\ \beta_\nu^T,\ 0 \leqslant \nu \leqslant 1,\ T \in \mathcal{T}_k,\ 0 \leqslant k \leqslant l$, as the local constants from (2.2), i.e., those constants which appear if (2.2) is considered on $T \subset \Omega$. We further introduce the constants

$$\gamma_R := \max_{T \in \mathcal{T}_0} \frac{h_T^2 {\beta_1}^T}{{\alpha_0}^T}, \qquad \alpha_R := \min_{T \in \mathcal{T}_0} \frac{{\alpha_0}^T}{{\alpha_1}^T} \tag{3.2}$$

reflecting the local relationship between the weighted Helmholtz term and the principal part of the operator and the local relationship between the lower and upper bounds for the eigenvalues of the principal part, respectively. Moreover, we will frequently use the following two elementary results:

Lemma 3.1. *There exist constants $0 < c_0 < C_0$ depending only on the local geometry of $\mathcal{T}_0$ such that for all $v \in P_1(T),\ T \in \widehat{\mathcal{T}}$,*

$$c_0 \sum_{\substack{\nu,\mu=1\\ \nu<\mu}}^{3} \left(v(\boldsymbol{p}_\nu^T) - v(\boldsymbol{p}_\mu^T)\right)^2 \leqslant |v|_{1;T}^2 \leqslant C_0 \sum_{\substack{\nu,\mu=1\\ \nu<\mu}}^{3} \left(v(\boldsymbol{p}_\nu^T) - v(\boldsymbol{p}_\mu^T)\right)^2. \tag{3.3}$$

Proof. We refer to [35]. □

Lemma 3.2. *There exist constants $0 < c_1 \leqslant C_1$ depending only on the local geometry of $\mathcal{T}_0$ such that for all $\boldsymbol{q} \in RT_0(T),\ T \in \widehat{\mathcal{T}}$,*

$$c_1 \sum_{\nu=1}^{3} h_{e_\nu^T}^2 (\boldsymbol{\nu} \cdot \boldsymbol{q}|_{e_\nu^T})^2 \leqslant \|\boldsymbol{q}\|_{0;T}^2 \leqslant C_1 \sum_{\nu=1}^{3} h_{e_\nu^T}^2 (\boldsymbol{\nu} \cdot \boldsymbol{q}|_{e_\nu^T})^2. \tag{3.4}$$

Proof. The proof relies on the affine equivalence of the Raviart–Thomas elements and can be easily given with respect to a reference triangle. □

We attempt to solve the mixed finite element approximation (2.6) with respect to the finest triangulation $\mathcal{T}_l$. Denoting by A_l, B_l and C_l the matrices associated with the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ and by f_l the vector representing the right-hand side of the second equation in (2.6), the problem can be stated as the following algebraic saddle point problem

$$\mathcal{A}_l z_l = \begin{pmatrix} A_l & B_l^{\mathrm{T}} \\ B_l & -C_l \end{pmatrix} \begin{pmatrix} \boldsymbol{j}_l \\ u_l \end{pmatrix} = \begin{pmatrix} 0 \\ f_l \end{pmatrix} \tag{3.5}$$

where the coefficient matrix $\mathcal{A}_l$ is symmetric, but indefinite.

Throughout the rest of this section, for ease of exposition we will assume that the coefficient function b is piecewise constant with respect to the initial triangulation with $b|_T = b_T > 0$, $T \in \mathcal{T}_0$. The simplifying assumption $b_T > 0$ allows a unified treatment, since in this case the matrix C_l is symmetric, positive definite. In the more general case of a Helmholtz term vanishing for some $T \in \mathcal{T}_0$, our approach has to be modified on such elements (cf. Remark 3.10).

The solution of (3.5) will be based on the splitting

$$z_l = z_l^p + z_l^h$$

where $z_l^p = (\boldsymbol{j}_l^p, u_l^p)$ is a particular solution of the inhomogeneous equation

$$B_l \boldsymbol{j}_l^p - C_l u_l^p = f_l \tag{3.6}$$

and $z_l^h = (\boldsymbol{j}_l^h, u_l^h)$ satisfies

$$\mathcal{A}_l z_l^h = \begin{pmatrix} A_l & B_l^{\mathrm{T}} \\ B_l & -C_l \end{pmatrix} \begin{pmatrix} \boldsymbol{j}_l^h \\ u_l^h \end{pmatrix} = \begin{pmatrix} g_l \\ 0 \end{pmatrix} \tag{3.7}$$

where $g_l := -(A_l \boldsymbol{j}_l^p + B_l^{\mathrm{T}} u_l^p)$.

Since C_l is positive definite, static condensation of u_l^h in (3.7) yields the Schur complement system

$$N_l \boldsymbol{j}_l^h = g_l, \quad N_l := A_l + B_l^{\mathrm{T}} C_l^{-1} B_l. \tag{3.8}$$

The second equation in (3.7) gives rise to the definition of the following subspace

$$Z_l := \{(\boldsymbol{q}_l, v_l) \mid B_l \boldsymbol{q}_l - C_l v_l = 0\}. \tag{3.9}$$

The next result constitutes the basis for the computation of z_l^h:

Lemma 3.3. *Let $\mathcal{A}_l$ and Z_l be given by* (3.5) *and* (3.9), *respectively. Then there holds:*

(i) *$\mathcal{A}_l$ is positive definite on the subspace Z_l.*

(ii) *If $z_l^\nu = (\boldsymbol{j}_l^\nu, u_l^\nu)$, $\nu \geqslant 1$, are the iterates obtained by the cg-iteration applied to* (3.7) *with a start iterate $z_l^0 = (\boldsymbol{j}_l^0, u_l^0)$, then $z_l^0 \in Z_l$ implies $z_l^\nu \in Z_l$, $\nu \geqslant 1$.*

Proof. For $z_l = (\boldsymbol{q}_l, v_l) \in Z_l$ we have

$$z_l^{\mathrm{T}} \mathcal{A}_l z_l = \boldsymbol{q}_l^{\mathrm{T}} (A_l \boldsymbol{q}_l + B_l^{\mathrm{T}} v_l) = \boldsymbol{q}_l^{\mathrm{T}} N_l \boldsymbol{q}_l \geqslant 0 \tag{3.10}$$

with equality iff $\boldsymbol{q}_l = 0$ due to the fact that N_l is positive definite. For $\boldsymbol{q}_l = 0$ it follows that $C_l v_l = 0$ whence $v_l = 0$ thus proving (i). The second assertion (ii) can be easily shown by induction. □

As a preconditioner for the cg-iteration we choose a matrix $\widetilde{\mathcal{A}}_l$ of the same algebraic structure as $\mathcal{A}_l$, namely

$$\widetilde{\mathcal{A}}_l = \begin{pmatrix} \widetilde{A}_l & B_l^{\mathrm{T}} \\ B_l & -C_l \end{pmatrix} \tag{3.11}$$

where $\widetilde{A}_l$ is a symmetric, positive definite matrix that will be specified below. The counterpart $\widetilde{N}_l$ of the Schur complement N_l is given by

$$\widetilde{N}_l := \widetilde{A}_l + B_l^{\mathrm{T}} C_l^{-1} B_l. \tag{3.12}$$

We have the following result:

Lemma 3.4. *Let* $\mathcal{A}_l$, N_l *and* $\widetilde{\mathcal{A}}_l$, $\widetilde{N}_l$ *be given by* (3.5), (3.8) *and* (3.11), (3.12), *respectively. Then there exist positive constants* $0 < \gamma \leqslant \Gamma$ *such that for all* $z_l = (\boldsymbol{q}_l, v_l) \in Z_l$

$$\gamma z_l^{\mathrm{T}} z_l \leqslant z_l^{\mathrm{T}} \widetilde{\mathcal{A}}_l^{-1} \mathcal{A}_l z_l \leqslant \Gamma z_l^{\mathrm{T}} z_l$$

if and only if

$$\gamma \boldsymbol{q}_l^{\mathrm{T}} \boldsymbol{q}_l \leqslant \boldsymbol{q}_l^{\mathrm{T}} \widetilde{N}_l^{-1} N_l \boldsymbol{q}_l \leqslant \Gamma \boldsymbol{q}_l^{\mathrm{T}} \boldsymbol{q}_l.$$

Proof. The proof is an easy consequence of (3.10) in Lemma 3.3 which also holds true for $\mathcal{A}_l$ and N_l replaced by $\widetilde{\mathcal{A}}_l$ and $\widetilde{N}_l$. □

The construction of a suitable preconditioner $\widetilde{\mathcal{A}}$ will be done by hierarchical multilevel decompositions of the mixed ansatz spaces $RT_0(\Omega;\mathcal{T}_l)$ and $W_0(\Omega;\mathcal{T}_l)$. These decompositions also provide an appropriate tool for the computation of a particular solution z_l^p of (3.6). As in the case of the standard conforming $P1$ approximation of (2.1) (cf., e.g., [34]), the decompositions can be obtained by means of appropriate interpolation operators. In particular, we denote by

$$\rho_k : RT_0(\Omega;\mathcal{T}_l) \to RT_0(\Omega;\mathcal{T}_k), \quad 0 \leqslant k \leqslant l,$$

the interpolation operators defined locally by

$$\int_{\partial T} \boldsymbol{\nu} \cdot (\rho_k \boldsymbol{q}_l) p_0 \,\mathrm{d}\sigma = \int_{\partial T} \boldsymbol{\nu} \cdot \boldsymbol{q}_l p_0 \,\mathrm{d}\sigma, \quad p_0 \in R_0(\partial T),\ T \in \mathcal{T}_k.$$

Due to a fundamental result by Douglas and Roberts [16] we have

$$\operatorname{div}(\rho_k \boldsymbol{q}_l) = \Pi_k \operatorname{div} \boldsymbol{q}_l, \quad \boldsymbol{q}_l \in RT_0(\Omega;\mathcal{T}_l), \tag{3.13}$$

where

$$\Pi_k : W_0(\Omega;\mathcal{T}_l) \to W_0(\Omega;\mathcal{T}_k), \quad 0 \leqslant k \leqslant l,$$

denote the orthogonal L^2-projections onto $W_0(\Omega;\mathcal{T}_k)$.

Obviously, ρ_l and Π_l are the identities on $RT_0(\Omega;\mathcal{T}_l)$ and $W_0(\Omega;\mathcal{T}_l)$. Then, setting formally $\rho_{-1} := 0$ and $\Pi_{-1} := 0$, we consider the direct subspace decompositions

$$RT_0(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \widetilde{RT}_0(\Omega;\mathcal{T}_k), \quad \widetilde{RT}_0(\Omega;\mathcal{T}_k) := (\rho_k - \rho_{k-1}) RT_0(\Omega;\mathcal{T}_l), \tag{3.14}$$

$$W_0(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \widetilde{W}_0(\Omega;\mathcal{T}_k), \quad \widetilde{W}_0(\Omega;\mathcal{T}_k) := (\Pi_k - \Pi_{k-1}) W_0(\Omega;\mathcal{T}_l). \tag{3.15}$$

By construction of the level k subspaces $\widetilde{RT}_0(\Omega;\mathcal{T}_k)$ and $\widetilde{W}_0(\Omega;\mathcal{T}_k)$, for macroelements $T \in \mathcal{T}_{k-1}$, $1 \leqslant k \leqslant l$, we have

$$\int_{\partial T} \boldsymbol{\nu} \cdot \widetilde{\boldsymbol{q}}_k \, d\sigma = \int_T \operatorname{div} \widetilde{\boldsymbol{q}}_k \, dx = 0, \quad \widetilde{\boldsymbol{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k),$$

$$\int_T \widetilde{w}_k \, dx = 0, \quad \widetilde{w}_k \in \widetilde{W}_0(\Omega;\mathcal{T}_k).$$

The preceding result suggests a further splitting of $\widetilde{RT}_0(\Omega;\mathcal{T}_k)$ according to

$$\widetilde{RT}_0(\Omega;\mathcal{T}_k) = \widetilde{RT}_0^0(\Omega;\mathcal{T}_k) \oplus \widetilde{RT}_0^1(\Omega;\mathcal{T}_k), \quad 1 \leqslant k \leqslant l, \tag{3.16}$$

where $\widetilde{RT}_0^0(\Omega;\mathcal{T}_k)$ stands for the subspace of divergence-free vector fields

$$\widetilde{RT}_0^0(\Omega;\mathcal{T}_k) := \{\widetilde{\boldsymbol{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k) \mid \operatorname{div} \widetilde{\boldsymbol{q}}_k = 0\}$$

and $\widetilde{RT}_0^1(\Omega;\mathcal{T}_k)$ is given by

$$\widetilde{RT}_0^1(\Omega;\mathcal{T}_k) := \{\widetilde{\boldsymbol{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k) \mid \boldsymbol{\nu} \cdot \widetilde{\boldsymbol{q}}_k|_{\partial T} = 0, \ T \in \mathcal{T}_{k-1}^{\text{ref}}\}$$

where $\mathcal{T}_{k-1}^{\text{ref}}$, $1 \leqslant k \leqslant l$, denotes the set of refined level $k-1$ macroelements $T \in \mathcal{T}_{k-1}$.

The divergence-free subspace $\widetilde{RT}_0^0(\Omega;\mathcal{T}_k)$ admits a useful characterization with regard to Yserentant's hierarchical splitting of the level l conforming finite element space $S_1(\Omega;\mathcal{T}_l)$ of continuous, piecewise linear finite elements

$$S_1(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \widetilde{S}_1(\Omega;\mathcal{T}_k), \quad \widetilde{S}_1(\Omega;\mathcal{T}_k) := (I_k - I_{k-1}) S_1(\Omega;\mathcal{T}_l),$$

where $I_k : S_1(\Omega;\mathcal{T}_l) \to S_1(\Omega;\mathcal{T}_k)$ refers to the interpolation operator $(I_k v_l)(p) = v_l(p)$, $p \in \mathcal{N}_k$, $0 \leqslant k \leqslant l$, and $I_{-1} := 0$ (cf. [34]).

It is easy to see that

$$\widetilde{RT}_0^0(\Omega;\mathcal{T}_k) = \operatorname{curl} \widetilde{S}_1(\Omega;\mathcal{T}_k), \quad 1 \leqslant k \leqslant l. \tag{3.17}$$

Since the hierarchical surplus $\widetilde{S}_1(\Omega;\mathcal{T}_k)$ is spanned by those level k nodal basis functions $\varphi_{m_e}^k$ associated with the midpoints m_e of refined level $k-1$ edges $e \in \mathcal{E}_{k-1}$, in view of (3.17) we have the following decomposition of $\widetilde{RT}_0^0(\Omega;\mathcal{T}_k)$ into the direct sum of one-dimensional subspaces

$$\widetilde{RT}_0^0(\Omega;\mathcal{T}_k) = \bigoplus_{e \in \mathcal{E}_{k-1}^{\text{ref}}} \widetilde{RT}_0^0(e;\mathcal{T}_k) \tag{3.18}$$

where

$$\widetilde{RT}_0^0(e;\mathcal{T}_k) := \operatorname{span}\{\operatorname{curl} \varphi_{m_e}^k\}$$

and $\mathcal{E}_{k-1}^{\text{ref}}$ stands for the set of refined level $k-1$ edges $e \in \mathcal{E}_{k-1}$.

On the other hand, the level k subspace $\widetilde{RT}_0^1(\Omega;\mathcal{T}_k)$ also admits a further decomposition into low-dimensional local subspaces. For that purpose we remind that for $T \in \mathcal{T}_{k-1}^{\text{ref}}$, $1 \leqslant k \leqslant l$, we have $T = \bigcup_{\nu=1}^{\nu_T}$, $T_\nu \in \mathcal{T}_k$, with $\nu_T = 2$, if T is an irregularly refined macroelement, and $\nu_T = 4$ in case of a regularly refined triangle. Thus, denoting by $\boldsymbol{q}_e^k$ the level k basis field associated with an interior edge e of a refined macroelement $T \in \mathcal{T}_{k-1}^{\text{ref}}$, we have

$$\widetilde{RT}_0^1(\Omega;\mathcal{T}_k) = \bigoplus_{T\in\mathcal{T}_{k-1}^{\text{ref}}} \widetilde{RT}_0^1(T;\mathcal{T}_k) \tag{3.19}$$

where $\widetilde{RT}_0^1(T;\mathcal{T}_k)$, $T \in \mathcal{T}_{k-1}^{\text{ref}}$, refers to the $(\nu_T - 1)$-dimensional subspace

$$\widetilde{RT}_0^1(T;\mathcal{T}_k) := \operatorname{span}\{\boldsymbol{q}_e^k \in RT_0(\Omega;\mathcal{T}_k) \mid e \subset \partial T_\nu \cap \operatorname{int} T,\ 1 \leqslant \nu \leqslant \nu_T\}.$$

Summarizing the decompositions (3.14), (3.16), (3.18) and (3.19) we end up with the splitting

$$RT_0(\Omega;\mathcal{T}_l) = RT_0(\Omega;\mathcal{T}_0) \oplus \bigoplus_{k=1}^{l} \left(\bigoplus_{e\in\mathcal{E}_{k-1}^{\text{ref}}} \widetilde{RT}_0^0(e;\mathcal{T}_k) \oplus \bigoplus_{T\in\mathcal{T}_{k-1}^{\text{ref}}} \widetilde{RT}_0^1(T;\mathcal{T}_k)\right). \tag{3.20}$$

It is well known that multilevel subspace decompositions such as (3.20) give rise to additive and multiplicative multilevel Schwarz iterations which in turn define associated additive and multiplicative multilevel preconditioners (cf., e.g., [33,35,36]). With regard to (3.20) we consider the following additive multilevel preconditioner $\widetilde{N}_l$ for N_l

$$\begin{aligned} \widetilde{N}_l^{-1}\boldsymbol{q}_l &:= \left(P_0 + \sum_{k=1}^{l} P_k\right) N_l^{-1}\boldsymbol{q}_l, \quad \boldsymbol{q}_l \in RT_0(\Omega;\mathcal{T}_l), \\ P_k &:= \sum_{e\in\mathcal{E}_{k-1}^{ref}} P_{k;e} + \sum_{T\in\mathcal{T}_{k-1}^{\text{ref}}} P_{k;T}, \quad 1 \leqslant k \leqslant l, \end{aligned} \tag{3.21}$$

where P_0, $P_{k;e}$ and $P_{k;T}$ are the orthogonal projections onto the low-dimensional subspaces $RT_0(\Omega;\mathcal{T}_0)$, $\widetilde{RT}_0^0(e;\mathcal{T}_k)$ and $\widetilde{RT}_0^1(T;\mathcal{T}_k)$, $1 \leqslant k \leqslant l$, with respect to the weighted inner product $\langle\cdot,\cdot\rangle_{\text{div}}$ on $H(\text{div};\Omega)$ given by

$$\langle \boldsymbol{p},\boldsymbol{q}\rangle_{\text{div}} := \sum_{T\in\mathcal{T}_0} \left(\int_T a^{-1}\boldsymbol{p}\cdot\boldsymbol{q}\,\mathrm{d}x + \int_T b^{-1}\operatorname{div}\boldsymbol{p}\operatorname{div}\boldsymbol{q}\,\mathrm{d}x\right).$$

We emphasize that the preconditioner is cheaply computable. Its implementation involves the solution of a saddle point problem with the respect to the initial triangulation $\mathcal{T}_0$ whereas on levels $1 \leqslant k \leqslant l$ we only have to solve strictly local subproblems. In particular, $P_{k;T}$ amounts to the solution of $(\nu_T - 1)$-dimensional subproblems for each $T \in \mathcal{T}_{k-1}^{\text{ref}}$ and $P_{k;e}$ merely requires the solution of a scalar equation for each $e \in \mathcal{E}_{k-1}^{\text{ref}}$.

The rest of this section will be devoted to a spectral condition number estimate for $\widetilde{N}_l^{-1}N_l$ which, in view of Lemma 3.4, simultaneously provides such an estimate for $\widetilde{\mathcal{A}}_l^{-1}\mathcal{A}_l|_{Z_l}$. In particular, we will show that the spectral condition number quadratically grows with the refinement level and thus exhibits

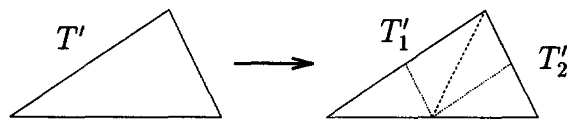

Fig. 1. Double bisection of T'.

the same asymptotic behavior as Yserentant's hierarchical preconditioner in case of the standard conforming $P1$ approximation of (2.1).

Theorem 3.5. *Let $\widetilde{N}_l$ be given by* (3.17). *Then there exist constants $0 < \gamma_{RT} \leqslant \Gamma_{RT}$ depending only on the local geometry of $\mathcal{T}_0$ and on the constants α_0^T, α_1^T and β_1^T, $T \in \mathcal{T}_0$, such that for all $\boldsymbol{q}_l \in RT_0(\Omega;\mathcal{T}_l)$*

$$\gamma_{RT}(l+1)^{-2}|||\boldsymbol{q}_l|||_{\mathrm{div}}^2 \leqslant \langle \widetilde{N}_l^{-1} N_l \boldsymbol{q}_l, \boldsymbol{q}_l \rangle_{\mathrm{div}} \leqslant \Gamma_{RT}|||\boldsymbol{q}_l|||_{\mathrm{div}}^2 \tag{3.22}$$

where $|||\cdot|||_{\mathrm{div}} := \langle\cdot,\cdot\rangle_{\mathrm{div}}^{1/2}$.

The proof of Theorem 3.5 relies on the Dryja–Widlund theory of additive Schwarz iterations (cf., e.g., [9,17]) and will be provided by a P.L. Lions type lemma and a strengthened Cauchy–Schwarz inequality (cf. Lemmas 3.7 and 3.8).

As a preliminary result we prove the following stability property of the interpolation operators ρ_k, $0 \leqslant k \leqslant l$.

Lemma 3.6. *There exists a positive constant C_s independent of $0 \leqslant k \leqslant l$ such that for all $\boldsymbol{q}_l \in RT_0(\Omega;\mathcal{T}_l)$,*

$$|||\rho_k \boldsymbol{q}_l|||_{\mathrm{div}}^2 \leqslant C_s\,(l-k+1)\,|||\boldsymbol{q}_l|||_{\mathrm{div}}^2. \tag{3.23}$$

Proof. Since (3.23) is trivially satisfied for $k = l$, we may assume $k < l$. In this case, both parts of the $|||\cdot|||_{\mathrm{div}}$-norm will be estimated separately. In particular, for $T \in \mathcal{T}_k$ we have

$$\left\|b^{-1/2}\mathrm{div}(\rho_k \boldsymbol{q}_l)\right\|_{0;T}^2 = \left\|\Pi_k\left(b^{-1/2}\mathrm{div}\,\boldsymbol{q}_l\right)\right\|_{0;T}^2 \leqslant \left\|b^{-1/2}\mathrm{div}\,\boldsymbol{q}_l\right\|_{0;T}^2. \tag{3.24}$$

On the other hand, an upper bound for $\|a^{-1/2}\rho_k \boldsymbol{q}_l\|_{0;T}^2$, $T \in \mathcal{T}_k$, will be derived based on the following construction.

For $T \in \mathcal{T}_k$ we denote by $\mathcal{T}_{l-k}^T$ the triangulation of T resulting from an $(l-k)$-fold refinement, and we refer to $\mathcal{N}_{l-k}^T$ and $\mathcal{E}_{l-k}^T$ as the set of vertices and edges of $\mathcal{T}_{l-k}^T$, respectively. The refinement process is as follows: Starting from $\mathcal{T}_0^T := \{T\}$, we obtain $\mathcal{T}_{j+1}^T$ from $\mathcal{T}_j^T$, $0 \leqslant j < l-k$, by subdividing each element $T' \in \mathcal{T}_j^T$ into four subtriangles such that the edges of T' get bisected. This will be done in two different ways: If $T' \in \mathcal{T}_j$ and $T' \notin \mathcal{T}_l$ but $T' = T_1' \cup T_2'$, $T_\nu' \in \mathcal{T}_l$, $1 \leqslant \nu \leqslant 2$, T' will be decomposed by a double bisection according to Fig. 1. In all other cases, T' will be refined regularly. The above refinement process guarantees that $\boldsymbol{q}|_T \in RT_0(T;\mathcal{T}_{l-k}^T)$ for all $\boldsymbol{q} \in RT_0(\Omega;\mathcal{T}_l)$, $T \in \mathcal{T}_k$, and that each edge e_ν^T, $1 \leqslant \nu \leqslant 3$, of T is subdivided into 2^{l-k} subedges of the same length. Moreover, each element of $\mathcal{T}_{l-k}^T$ is contained in the set $\widehat{\mathcal{T}}$. If $e_{\nu,\mu}^T \subset e_\nu^T$, $1 \leqslant \mu \leqslant 2^{l-k}$, are the subedges of e_ν^T resulting from the refinement process, we have

$$e_{\nu,\mu}^T \in \mathcal{E}_{l-k}^T, \qquad h_{e_{\nu,\mu}^T} = 2^{k-l} h_{e_\nu^T}.$$

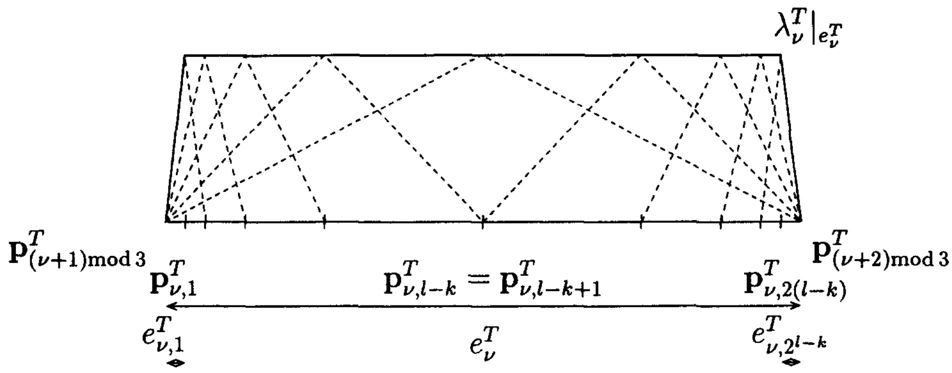

Fig. 2. Shape of $\lambda^T_{\nu,\mu}$ and λ^T_ν.

Furthermore, denoting by $\boldsymbol{p}_\nu^T$ the vertex opposite to e_ν^T so that e_ν^T has the vertices $\boldsymbol{p}^T_{(\nu+1)\bmod 3}$ and $\boldsymbol{p}^T_{(\nu+2)\bmod 3}$, the subedges are ordered lexicographically according to

$$e^T_{\nu,\mu} \cap e^T_{\nu,\mu+1} \in \mathcal{N}^T_{l-k}, \quad 1 \leqslant \mu \leqslant 2^{l-k} - 1,$$

$$\boldsymbol{p}^T_{(\nu+1)\bmod 3} \in e^T_{\nu,1}, \qquad \boldsymbol{p}^T_{(\nu+2)\bmod 3} \in e^T_{\nu,2^{l-k}}.$$

We refer to $\boldsymbol{p}^T_{\nu,l-k+\mu}$ and $\boldsymbol{p}^T_{\nu,l-k-\mu+1}$, $1 \leqslant \mu \leqslant l-k$, as those vertices generated during the μth refinement step which have shortest distance to $\boldsymbol{p}^T_{(\nu+2)\bmod 3}$ and $\boldsymbol{p}^T_{(\nu+1)\bmod 3}$, respectively. Further, we denote by $\lambda^T_{\nu,\mu}$, $1 \leqslant \mu \leqslant 2(l-k)$, the local hierarchical basis functions in $S_1(T; \mathcal{T}^T_{l-k})$ associated with the vertices $\boldsymbol{p}^T_{\nu,\mu}$ and set

$$\lambda_\nu^T := \frac{1}{2} \sum_{\mu=1}^{2(l-k)} \lambda^T_{\nu,\mu}.$$

Observe that the first refinement step only generates a single vertex so that $\boldsymbol{p}^T_{\nu,l-k} = \boldsymbol{p}^T_{\nu,l-k+1}$ and $\lambda^T_{\nu,l-k} = \lambda^T_{\nu,l-k+1}$. Moreover, $\lambda_\nu^T|_{e_\mu^T} = 0$ for $\nu \neq \mu$. For an illustration of $e^T_{\nu,\mu}$, $\boldsymbol{p}^T_{\nu,\mu}$, $\lambda^T_{\nu,\mu}$ and λ^T_ν we refer to Fig. 2.

Using the preceding construction of λ_ν^T, we get

$$\begin{aligned}
h_{e_\nu^T} \boldsymbol{\nu} \cdot \rho_k \boldsymbol{q}_l|_{e_\nu^T} &= 2^{k-l} h_{e_\nu^T} \sum_{\mu=1}^{2^{l-k}} \boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,\mu}} \\
&= \int_{\partial T} \lambda_\nu^T \boldsymbol{\nu} \cdot \boldsymbol{q}_l \,\mathrm{d}\sigma + 2^{k-l-1} h_{e_\nu^T} \big(\boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,1}} + \boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,2^{l-k}}} \big) \\
&= \int_T \lambda_\nu^T \operatorname{div} \boldsymbol{q}_l \,\mathrm{d}x + \int_T \boldsymbol{q}_l \cdot \operatorname{grad} \lambda_\nu^T \,\mathrm{d}x + 2^{k-l-1} h_{e_\nu^T} \big(\boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,1}} + \boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,2^{l-k}}} \big) \\
&\leqslant |T|^{1/2} \|\operatorname{div} \boldsymbol{q}_l\|_{0;T} + |\lambda_\nu^T|_{1;T} \|\boldsymbol{q}_l\|_{0;T} + \tfrac{1}{2} h_{e^T_{\nu,1}} \big(\boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,1}} + \boldsymbol{\nu} \cdot \boldsymbol{q}_l|_{e^T_{\nu,2^{l-k}}} \big). \qquad (3.25)
\end{aligned}$$

An upper bound for $|\lambda_\nu^T|_{1;T}$ can be obtained by means of the hierarchical splitting

$$\lambda_\nu^T = \sum_{\mu=1}^{l-k} (I_{k+\mu} - I_{k+\mu-1})\lambda_\nu^T$$

and the estimate

$$|\lambda_\nu^T|_{1;T}^2 \leqslant C_H \sum_{\mu=1}^{l-k} \left|(I_{k+\mu} - I_{k+\mu-1})\lambda_\nu^T\right|_{1;T}^2$$

known from [35] where C_H is a positive constant depending only on the local geometry of $\mathcal{T}_0$. If we further use the obvious relationship

$$\tfrac{1}{2}\left(\lambda_{\nu,l-k+\mu}^T + \lambda_{\nu,l-k-\mu+1}^T\right) = (I_{k+\mu} - I_{k+\mu-1})\lambda_\nu^T,$$

observe that the functions $\lambda_{\nu,l-k+\mu}^T$ and $\lambda_{\nu,l-k-\mu+1}^T$, $2 \leqslant \mu \leqslant l-k$, have no overlapping supports and take into account the basic norm equivalence (3.3), we obtain

$$|\lambda_\nu^T|_{1;T}^2 \leqslant \frac{C_H}{4} \sum_{\mu=1}^{l-k} \left|\lambda_{\nu,\mu}^T + \lambda_{\nu,2(l-k)-\mu+1}^T\right|_{1;T}^2 \leqslant C_0 C_H (l-k+1). \tag{3.26}$$

Now, from (3.4), (3.25) and (3.26) it follows that

$$\int_T a^{-1}\rho_k \boldsymbol{q}_l \cdot \rho_k \boldsymbol{q}_l \,\mathrm{d}x \leqslant 9C_1 \frac{\alpha_1^T}{\alpha_0^T}\left(C_0 C_H (l-k+1) + \frac{1}{6c_1}\right) \int_T a^{-1}\boldsymbol{q}_l \cdot \boldsymbol{q}_l \,\mathrm{d}x$$
$$+ 9\frac{C_1 |T| \beta_1^T}{\alpha_0^T} \int_T b^{-1} \operatorname{div} \boldsymbol{q}_l \operatorname{div} \boldsymbol{q}_l \,\mathrm{d}x.$$

Finally, summarizing over all $T \in \mathcal{T}_k$ yields

$$|||\rho_k \boldsymbol{q}_l|||_{\mathrm{div}}^2 \leqslant C_2(l-k+1) \int_\Omega a^{-1}\boldsymbol{q}_l \cdot \boldsymbol{q}_l \,\mathrm{d}x + C_3 \int_\Omega b^{-1} \operatorname{div} \boldsymbol{q}_l \operatorname{div} \boldsymbol{q}_l \,\mathrm{d}x$$

where $C_2 := 9C_1\alpha_R^{-1}(C_0 C_H + (6c_1)^{-1})$ and $C_3 := 1 + (9/2)C_1\gamma_R$. The asserted upper bound (3.23) follows with $C_s := \max(C_2, C_3)$. □

The preceding stability result enables us to establish the following P.L. Lions type lemma:

Lemma 3.7. *Let $\boldsymbol{q}_l \in RT_0(\Omega; \mathcal{T}_l)$ and consider the following decomposition*

$$\boldsymbol{q}_l = \sum_{k=0}^{l} \widetilde{\boldsymbol{q}}_k, \qquad \widetilde{\boldsymbol{q}}_k := \widetilde{\boldsymbol{q}}_k^1 + \sum_{e \in \mathcal{E}_{k-1}^{\mathrm{ref}}} \widetilde{\boldsymbol{q}}_k^e, \quad 1 \leqslant k \leqslant l,$$

where $\widetilde{\boldsymbol{q}}_0 \in RT_0(\Omega; \mathcal{T}_0)$, $\widetilde{\boldsymbol{q}}_k^1 \in \widetilde{RT}_0^1(\Omega; \mathcal{T}_k)$ and $\widetilde{\boldsymbol{q}}_k^e \in \widetilde{RT}_0^0(e; \mathcal{T}_k)$, $1 \leqslant k \leqslant l$. Then there exists a positive constant γ_{RT} depending only on $\mathcal{T}_0$ and the constants α_0^T, α_1^T and β_1^T, $T \in \mathcal{T}_0$, such that

$$|||\widetilde{\boldsymbol{q}}_0|||_{\mathrm{div}}^2 + \sum_{k=1}^{l}\left(|||\widetilde{\boldsymbol{q}}_k^1|||_{\mathrm{div}}^2 + \sum_{e \in \mathcal{E}_{k-1}^{\mathrm{ref}}} |||\widetilde{\boldsymbol{q}}_k^e|||_{\mathrm{div}}^2\right) \leqslant \gamma_{RT}^{-1}(l+1)^2 |||\boldsymbol{q}_l|||_{\mathrm{div}}^2. \tag{3.27}$$

Proof. In view of Lemma 3.6

$$\sum_{k=0}^{l} |||\widetilde{\boldsymbol{q}}_k|||_{\mathrm{div}}^2 = \sum_{k=0}^{l} \left|\left|\left|(\rho_k - \rho_{k-1})\boldsymbol{q}_l\right|\right|\right|_{\mathrm{div}}^2 \leqslant \tfrac{5}{2} C_s (l+1)^2\, |||\boldsymbol{q}_l|||_{\mathrm{div}}^2 .$$

On the other hand, using Young's inequality and (3.4)

$$|||\widetilde{\boldsymbol{q}}_k^1|||_{\mathrm{div}}^2 + \sum_{e\in\mathcal{E}_{k-1}^{\mathrm{ref}}} |||\widetilde{\boldsymbol{q}}_k^e|||_{\mathrm{div}}^2 \leqslant 7\frac{C_1}{\alpha_{RC1}} |||\widetilde{\boldsymbol{q}}_k|||_{\mathrm{div}}^2$$

which gives the assertion with $\gamma_{RT}^{-1} := 35C_1C_s/(2\alpha_{RC1})$. □

The upper bound in Theorem 3.5 will be proved by a strengthened Cauchy–Schwarz inequality which gives evidence of the coupling of the subspaces involved in the multilevel splitting of $RT_0(\Omega;\mathcal{T}_l)$.

Lemma 3.8. *There exists a positive constant C_I depending only on the local geometry of $\mathcal{T}_0$ and on the constants α_0^T, α_1^T and β_1^T, $T\in\mathcal{T}_0$, such that for all $\widetilde{\boldsymbol{q}}_i \in \widetilde{RT}_0(\Omega;\mathcal{T}_i)$, $\widetilde{\boldsymbol{q}}_j \in \widetilde{RT}_0(\Omega;\mathcal{T}_j)$, $0\leqslant i\leqslant j\leqslant l$,*

$$\langle\widetilde{\boldsymbol{q}}_i,\widetilde{\boldsymbol{q}}_j\rangle_{\mathrm{div}} \leqslant C_I 2^{-(j-i)/2} |||\widetilde{\boldsymbol{q}}_i|||_{\mathrm{div}} |||\widetilde{\boldsymbol{q}}_j|||_{\mathrm{div}}. \tag{3.28}$$

Proof. Since (3.28) is trivially satisfied for $i=j$ with $C_I=1$, we may assume $j>i$. Decomposing $\widetilde{\boldsymbol{q}}_j$ according to $\widetilde{\boldsymbol{q}}_j = \widetilde{\boldsymbol{q}}_j^1 + \operatorname{curl}\widetilde{\varphi}_j$ where $\widetilde{\boldsymbol{q}}_j^1 \in \widetilde{RT}_0^1(\Omega;\mathcal{T}_k)$ and $\widetilde{\varphi}_j\in S_1(\Omega;\mathcal{T}_j)$, the assertion follows from

$$\int_T \widetilde{\boldsymbol{q}}_i\cdot\operatorname{curl}\widetilde{\varphi}_j\,\mathrm{d}x \leqslant \frac{2^{-(j-i)/2}}{2\kappa_0}\sqrt{\frac{3}{c_0c_1}}\|\widetilde{\boldsymbol{q}}_i\|_{0;T}\|\operatorname{curl}\widetilde{\varphi}_j\|_{0;T},\quad T\in\mathcal{T}_i, \tag{3.29}$$

$$\int_T \widetilde{\boldsymbol{q}}_i\cdot\widetilde{\boldsymbol{q}}_j^1\,\mathrm{d}x \leqslant 2^{-(j-i)}\sqrt{2C_1|T|}\|\widetilde{\boldsymbol{q}}_i\|_{0;T}\|\operatorname{div}\widetilde{\boldsymbol{q}}_j\|_{0;T},\quad T\in\mathcal{T}_i, \tag{3.30}$$

$$\|\operatorname{curl}\widetilde{\varphi}_j\|_{0;T} \leqslant \sqrt{5\frac{C_1}{c_1}}\|\widetilde{\boldsymbol{q}}_j\|_{0;T},\quad T\in\mathcal{T}_i. \tag{3.31}$$

For the proof of (3.29) we denote by $\boldsymbol{t} := (-\nu_2,\nu_1)$ the tangential vector along the edges of $T\in\mathcal{T}_i$. Then, it follows that

$$\int_T \widetilde{\boldsymbol{q}}_i\cdot\operatorname{curl}\widetilde{\varphi}_j\,\mathrm{d}x = -\int_{\partial T}\boldsymbol{t}\cdot\widetilde{\boldsymbol{q}}_i\widetilde{\varphi}_j\,\mathrm{d}\sigma \leqslant \left(\int_{\partial T}|\boldsymbol{t}\cdot\widetilde{\boldsymbol{q}}_i|^2\,\mathrm{d}\sigma\right)^{1/2}\left(\int_{\partial T}|\widetilde{\varphi}_j|^2\,\mathrm{d}\sigma\right)^{1/2}. \tag{3.32}$$

Observing $\widetilde{\varphi}_j(\boldsymbol{p})=0$ for $\boldsymbol{p}\in\mathcal{N}_{j-1}\cap\partial T$ and taking into account the basic norm equivalence (3.3), we get

$$\int_{\partial T}|\widetilde{\varphi}_j|^2\,\mathrm{d}\sigma = \frac{2}{3}\sum_{\nu=1}^{3} h_{e_\nu^T}2^{-(j-i)}\sum_{\boldsymbol{p}\in\mathcal{N}_j\cap e_\nu^T}|\widetilde{\varphi}_j(\boldsymbol{p})|^2 \leqslant 2^{-(j-i)}\left(3c_0\sqrt{\kappa_0}\right)^{-1}\sqrt{|T|}\,|\widetilde{\varphi}_j|_{1;T}^2. \tag{3.33}$$

On the other hand, in view of the explicit representation

$$\widetilde{\boldsymbol{q}}_i = \frac{1}{2|T|} \sum_{\nu=1}^{3} h_{e_\nu^T} \boldsymbol{\nu} \cdot \widetilde{\boldsymbol{q}}_i|_{e_\nu^T} \left(\boldsymbol{x} - \boldsymbol{p}_\nu^T\right),$$

we get by means of (3.1) and (3.4)

$$\int_{\partial T} |\boldsymbol{t} \cdot \widetilde{\boldsymbol{q}}_i|^2 \, \mathrm{d}\sigma \leqslant \frac{9}{4c_1 \sqrt{\kappa_0^3 |T|}} \|\widetilde{\boldsymbol{q}}_i\|_{0;T}^2. \tag{3.34}$$

Using (3.33), (3.34) in (3.32) yields (3.29).

The proof of (3.30) follows from

$$\begin{aligned} \|\widetilde{\boldsymbol{q}}_j^1\|_{0;T}^2 &= \sum_{\substack{T' \in \mathcal{T}_{j-1}^{\mathrm{ref}} \\ T' \subset T}} \|\widetilde{\boldsymbol{q}}_j^1\|_{0;T'}^2 \leqslant 2C_1 \sum_{\substack{T' \in \mathcal{T}_{j-1}^{\mathrm{ref}} \\ T' \subset T}} \sum_{\substack{e \in \mathcal{E}_j \\ e \subset T' \setminus \partial T'}} h_e^2 |\boldsymbol{\nu}_e \cdot \widetilde{\boldsymbol{q}}_j^1|^2 \\ &\leqslant \frac{1}{2} C_1 \sum_{\substack{T' \in \mathcal{T}_{j-1}^{\mathrm{ref}} \\ T' \subset T}} |T'| \|\operatorname{div} \widetilde{\boldsymbol{q}}_j^1\|_{0;T'}^2 = 2C_1 4^{-(j-i)} |T| \|\operatorname{div} \widetilde{\boldsymbol{q}}_j^1\|_{0;T}^2. \end{aligned}$$

Finally, observing $\widetilde{\boldsymbol{q}}_j^1 \cdot \boldsymbol{\nu}|_e = 0$, $e \in \mathcal{E}_{j-1}^{\mathrm{ref}}$, (3.31) is an easy consequence of (3.4).

If we use (3.29), (3.30) and (3.31), take into account that $\int_T \operatorname{div} \widetilde{\boldsymbol{q}}_i \cdot \operatorname{div} \widetilde{\boldsymbol{q}}_j \, \mathrm{d}x = 0$ and summarize over all $T \in \mathcal{T}_i$, we may conclude with

$$C_I := \sqrt{2C_1} \max\left((2\alpha_R \kappa_0 c_1)^{-1} \sqrt{\frac{15}{c_0}}, \gamma_R \sqrt{2\alpha_R} \right). \qquad \square$$

The preceding Lemmas 3.7 and 3.8 are the basic ingredients for the proof of Theorem 3.5.

Proof of Theorem 3.5. The lower bound in (3.22) is a direct consequence of Lemma 3.7. On the other hand, Lemma 3.8 implies

$$\sum_{i,j=0}^{l} \langle \widetilde{\boldsymbol{q}}_i, \widetilde{\boldsymbol{q}}_j \rangle_{\mathrm{div}} \leqslant C_I \left(3 + 2\sqrt{2}\right) \sum_{i=0}^{l} |||\widetilde{\boldsymbol{q}}_i|||_{\mathrm{div}}^2. \tag{3.35}$$

Moreover, in view of the splitting

$$\widetilde{\boldsymbol{q}}_i = \widetilde{\boldsymbol{q}}_i^1 + \sum_{e \in \mathcal{E}_{i-1}^{\mathrm{ref}}} \widetilde{\boldsymbol{q}}_i^e, \quad 1 \leqslant i \leqslant l,$$

we get

$$|||\widetilde{\boldsymbol{q}}_i|||_{\mathrm{div}}^2 \leqslant \frac{7C_1}{3\alpha_R c_1} \left(|||\widetilde{\boldsymbol{q}}_i^1|||_{\mathrm{div}}^2 + \sum_{e \in \mathcal{E}_{i-1}^{\mathrm{ref}}} |||\widetilde{\boldsymbol{q}}_i^e|||_{\mathrm{div}}^2 \right). \tag{3.36}$$

Then, the upper bound in (3.22) follows from (3.35) and (3.36) with

$$\Gamma_{RT} := \frac{7C_1}{3\alpha_R c_1} C_I \left(3 + 2\sqrt{2}\right). \qquad \square$$

The result of Theorem 3.5 will be discussed by the subsequent remarks.

Remark 3.9. The lower and upper bounds in (3.22) of order $O((l+1)^{-2})$ and $O(1)$ significantly improve on the corresponding bounds of order $O(2^{-l})$ and $O(l+1)$ established in [14].

Remark 3.10. In case of a vanishing Helmholtz term $b \equiv 0$ in (2.1), we have $Z_l = RT_0^0(\Omega;\mathcal{T}_l) \times W_0(\Omega;\mathcal{T}_l)$ where the divergence-free subspace $RT_0^0(\Omega;\mathcal{T}_l)$ can be identified with $\operatorname{curl} S_1(\Omega;\mathcal{T}_l)$. Then $\boldsymbol{j}_l^h$ turns out to be the orthogonal projection of $-\boldsymbol{j}_l^p$ onto $\operatorname{curl} S_1(\Omega;\mathcal{T}_l)$ with respect to the weighted L^2-inner product $(\cdot,\cdot)_{0;a^{-1}} := (a^{-1}\cdot,\cdot)_0$. This special case has been treated in detail in [18].

Moreover, the case $b_T = 0$ for some $T \in \mathcal{T}_0$ can be treated as follows: Since the constants γ_{RT} and Γ_{RT} in (3.22) do not depend on $\min_{T\in\mathcal{T}_0} b_T$ and $\operatorname{div} \boldsymbol{q}_l|_T = 0$, if $b_T = 0$ and $(\boldsymbol{q}_l, v_l) \in Z_l$, the multilevel preconditioner can be applied formally by setting $b^{-1}|_T := 1$ in case $b_T = 0$.

Remark 3.11. Lemma 3.8 also provides an improved lower bound for the method presented in [20]. Instead of the lower bound of order $O((l+1)^{-1})$ we get an estimate of order $O(1)$.

We conclude this section with a procedure for the computation of a particular solution z_l^p of the inhomogeneous equation (3.6) based on the multilevel splitting of $RT_0(\Omega;\mathcal{T}_l)$. First, we compute $(\boldsymbol{j}_0, u_0) \in RT_0(\Omega;\mathcal{T}_0) \times W_0(\Omega;\mathcal{T}_0)$ as the solution of the saddle point problem with respect to the initial triangulation $\mathcal{T}_0$. Then, for each $T \in \mathcal{T}_{k-1}^{\text{ref}}$, $1 \leqslant k \leqslant l$, we determine $\tilde{\boldsymbol{j}}_{k;T} \in \widetilde{RT}_0^1(T;\mathcal{T}_k)$ as the solution of the $(\nu_T - 1)$-dimensional subproblem

$$\operatorname{div} \tilde{\boldsymbol{j}}_{k;T} = \big(\Pi_k - \Pi_{k-1}\big) f_l|_T$$

and set $\tilde{\boldsymbol{j}}_k := \sum_{T\in\mathcal{T}_{k-1}^{\text{ref}}} \tilde{\boldsymbol{j}}_{k;T}$. Obviously, $z_l^p := (\boldsymbol{j}_l^p, u_{l-1})$ where $\boldsymbol{j}_l^p := \boldsymbol{j}_{l-1} + \tilde{\boldsymbol{j}}_l$ satisfies (3.6).

4. Adaptive mesh refinement and numerical results

In this section, we briefly introduce an adaptive refinement process controlled by an easily computable a posteriori error estimator and present some numerical results obtained by the adaptive multilevel algorithm.

We will construct an efficient and reliable a posteriori error estimator for the total error in the flux measured in some weighted $H(\text{div};\Omega)$-norm. This error estimator can be derived by the principle of defect correction in higher order mixed ansatz spaces combined with an appropriate localization based on a hierarchical two-level splitting of the higher order ansatz spaces. We note that in contrast to the vast literature on error estimators for standard conforming finite element methods (cf., e.g., [3,4,7,8,15,30,31,38]), in case of mixed finite element techniques there is only some work by Braess and Verfürth [11], Braess et al. [10] and Achchab et al. [1]. We refer to [24] for a detailed presentation of various concepts and their comparison concerning estimators for mixed methods.

According to the notation introduced in the preceding section we denote by $(\boldsymbol{j}, u) \in H(\text{div};\Omega) \times L^2(\Omega)$ the unique solution of (2.5) and by $(\tilde{\boldsymbol{j}}_l, \tilde{u}_l) \in RT_0(\Omega;\mathcal{T}_l) \times W_0(\Omega;\mathcal{T}_l)$ an iterative solution of the lowest order mixed approximation $(\boldsymbol{j}_l, u_l) \in RT_0(\Omega;\mathcal{T}_l) \times W_0(\Omega;\mathcal{T}_l)$. Then, it can be easily seen that the total error $(\boldsymbol{j} - \tilde{\boldsymbol{j}}_l, u - \tilde{u}_l)$ satisfies the saddle point problem

$$
\begin{aligned}
&a(\boldsymbol{j}-\widetilde{\boldsymbol{j}}_l,\boldsymbol{q})+b(\boldsymbol{q},u-\widetilde{u}_l)=r(\boldsymbol{q}), \quad \boldsymbol{q}\in H(\mathrm{div};\Omega),\\
&b(\boldsymbol{j}-\widetilde{\boldsymbol{j}}_l,v)-c(u-\widetilde{u}_l,v)=(f-\Pi_0 f,v)_{0;\Omega}, \quad v\in L^2(\Omega),
\end{aligned}
\tag{4.1}
$$

where r stands for the residual given by $r(\boldsymbol{q}) := -a(\widetilde{\boldsymbol{j}}_l,\boldsymbol{q})-b(\boldsymbol{q},\widetilde{u}_l)$, $\boldsymbol{q}\in H(\mathrm{div};\Omega)$, and $\Pi_0 f$ is the L^2-projection of f onto $W_0(\Omega;\mathcal{T}_l)$.

The defect problem (4.1) will be approximated by means of the higher order Raviart–Thomas ansatz space

$$
\begin{aligned}
&RT_1(\Omega;\mathcal{T}_l) := \{\boldsymbol{q}\in H(\mathrm{div};\Omega) \mid \boldsymbol{q}|_T\in RT_1(T),\ T\in\mathcal{T}_l\},\\
&RT_1(T) := \big(P_1(T)\big)^2+\boldsymbol{x}P_1(T), \quad T\in\mathcal{T}_l,
\end{aligned}
$$

for the flux and the associated higher order ansatz space

$$
W_1(\Omega;\mathcal{T}_l) := \{v\in L^2(\Omega) \mid v|_T\in P_1(T),\ T\in\mathcal{T}_l\}
$$

for the primal variable. In particular, we refer to $(\boldsymbol{e_j},e_u)\in RT_1(\Omega;\mathcal{T}_l)\times W_1(\Omega;\mathcal{T}_l)$ as the unique solution of (4.1) when restricted to these higher order mixed ansatz spaces. Since the computation of $(\boldsymbol{e_j},e_u)$ is too expensive, we look for an appropriate simplification which can be realized using the hierarchical two-level splittings

$$
\begin{aligned}
&W_1(\Omega;\mathcal{T}_l)=W_0(\Omega;\mathcal{T}_l)\oplus\widetilde{W}_1(\Omega;\mathcal{T}_l),\\
&RT_1(\Omega;\mathcal{T}_l)=RT_0(\Omega;\mathcal{T}_l)\oplus\widetilde{RT}_1(\Omega;\mathcal{T}_l)
\end{aligned}
\tag{4.2}
$$

where

$$
\widetilde{W}_1(\Omega;\mathcal{T}_l) := (\mathrm{Id}-\Pi_0)W_1(\Omega;\mathcal{T}_l), \qquad \widetilde{RT}_1(\Omega;\mathcal{T}_l) := \big(\mathrm{Id}-\rho^0\big)RT_1(\Omega;\mathcal{T}_l).
$$

We note that

$$
\widetilde{W}_1(\Omega;\mathcal{T}_l)=\bigoplus_{T\in\mathcal{T}_l}\widetilde{W}_1(T), \quad \widetilde{W}_1(T) := \left\{v\in P_1(T) \mid \int_T v\,\mathrm{d}x=0\right\}.
$$

Moreover, we refer to $\widetilde{S}_2(\Omega;\mathcal{T}_l)$ as the hierarchical surplus in the hierarchical two-level splitting of the standard conforming P_2 ansatz space $S_2(\Omega;\mathcal{T}_l)$ (cf., e.g., [15]). Then, the hierarchical surplus $\widetilde{RT}_1(\Omega;\mathcal{T}_l)$ in (4.2) can be further decomposed by means of

$$
\widetilde{RT}_1(\Omega;\mathcal{T}_l)=\widetilde{RT}_1^0(\Omega;\mathcal{T}_l)\oplus\widetilde{RT}_1^1(\Omega;\mathcal{T}_l),
\tag{4.3}
$$

where

$$
\widetilde{RT}_1^0(\Omega;\mathcal{T}_l)=\mathrm{curl}\,\widetilde{S}_2(\Omega;\mathcal{T}_l)
$$

refers to the divergence-free part and

$$
\widetilde{RT}_1^1(\Omega;\mathcal{T}_l)=\bigoplus_{T\in\mathcal{T}_l}\widetilde{RT}_1^1(T), \quad \widetilde{RT}_1^1(T) := \{\boldsymbol{q}\in RT_1(\Omega;\mathcal{T}_l) \mid \boldsymbol{q}|_{\Omega\setminus T}=0\}
$$

is spanned by those basis fields associated with interior degrees of freedom.

If we consider the defect correction problem restricted to $\widetilde{RT}_1(\Omega;\mathcal{T}_l)$ and $\widetilde{W}_1(\Omega;\mathcal{T}_l)$ and take into account the splitting (4.3), we are led to an approximation of the total error $\boldsymbol{e_j}$ in the flux which can be obtained by the solution of the local low dimensional saddle point problems

$$a|_T(\widetilde{\boldsymbol{e}}^1_{\boldsymbol{j}_1}, \boldsymbol{q}) + b|_T(\boldsymbol{q}, \widetilde{e}_{u_1}) = r|_T(\boldsymbol{q}), \quad \boldsymbol{q} \in \widetilde{RT}^1_1(T),$$
$$b|_T(\widetilde{\boldsymbol{e}}^1_{\boldsymbol{j}_1}, v) - c|_T(\widetilde{e}_{u_1}, v) = (f, v)_{0;T}, \quad v \in \widetilde{W}_1(T),$$

and the solution of the scalar equations

$$\alpha_e \int_\Omega a^{-1} \mathrm{curl}\, \Phi_e \cdot \mathrm{curl}\, \Phi_e \, \mathrm{d}x = r(\mathrm{curl}\, \Phi_e), \quad e \in \mathcal{E}_l,$$

where Φ_e stands for the quadratic nodal basis function associated with the midpoint of e.

We refer to $\|\cdot\|_{a^{-1};\mathrm{div}}$ as the weighted $H(\mathrm{div};\Omega)$-norm

$$\|\boldsymbol{q}\|^2_{a^{-1};\mathrm{div}} := \sum_{T\in\mathcal{T}_l} \left(\int_T a^{-1} \boldsymbol{q} \cdot \boldsymbol{q} \, \mathrm{d}x + \|\mathrm{div}\, \boldsymbol{q}\|^2_{0;T} \right), \quad \boldsymbol{q} \in H(\mathrm{div};\Omega),$$

and we define

$$(e^H_{RT})^2 := \sum_{T\in\mathcal{T}_l} (e^H_T)^2,$$
$$(e^H_T)^2 := \|\widetilde{\boldsymbol{e}}^1_{\boldsymbol{j}_1}|_T\|^2_{a^{-1};\mathrm{div}} + \sum_{\nu=1}^{3} w_\nu \|\alpha_{e_\nu} \mathrm{curl}\, \Phi_{e_\nu}\|^2_{a^{-1};\mathrm{div}}, \quad T \in \mathcal{T}_l,$$

where e_ν, $1 \leqslant \nu \leqslant 3$, denote the edges of T, and w_ν is a weighting factor given by $w_\nu := 1/2$, if $e_\nu \in \mathcal{E}^0_l$, and $w_\nu := 1$, if $e_\nu \in \mathcal{E}^\Gamma_l$.

The next result states that e^H_{RT} gives rise to an efficient and reliable a posteriori error estimator for the total error in the flux provided the iteration error is sufficiently small.

Theorem 4.1. *Under the saturation assumption*

$$\|\boldsymbol{j} - \boldsymbol{j}^1_l\|_{a^{-1};\mathrm{div}} \leqslant \beta_l \|\boldsymbol{j} - \boldsymbol{j}_l\|_{a^{-1};\mathrm{div}}, \quad 0 < \beta_l \leqslant \beta_\infty < 1,$$

where $(\boldsymbol{j}^1_l, u^1_l)$ *is the unique solution of* (2.5) *on* $RT_1(\Omega;\mathcal{T}_l) \times W_1(\Omega;\mathcal{T}_l)$, *there exist constants* c_{hier}, $C_{\mathrm{hier}} > 0$ *and* γ_{hier}, $\Gamma_{\mathrm{hier}} > 0$ *independent of the refinement level such that*

$$(1+\beta_\infty)^{-1}\left(c_{\mathrm{hier}} e^H_{RT} - \gamma_{\mathrm{hier}} |||\boldsymbol{j}_l - \widetilde{\boldsymbol{j}}_l|||_{\mathrm{div}}\right) \leqslant \|\boldsymbol{j} - \widetilde{\boldsymbol{j}}_l\|_{a^{-1};\mathrm{div}},$$
$$(1+\beta_\infty)^{-1}\left(C_{\mathrm{hier}} e^H_{RT} + \Gamma_{\mathrm{hier}} |||\boldsymbol{j} - \widetilde{\boldsymbol{j}}_l|||_{\mathrm{div}}\right) \geqslant \|\boldsymbol{j}_l - \widetilde{\boldsymbol{j}}_l\|_{a^{-1};\mathrm{div}}.$$

For the proof of Theorem 4.1 and further details we refer to [24,32].

Numerical experiments have been carried out for the following two test examples:

Example 1. Eq. (2.1) with $a = 1$, $b = 0$ on $\Omega = (0,1)^2$, homogeneous Dirichlet boundary data and f according to the solution $u(x,y) = x(x-1)y(y-1)\exp(-100((x-0.5)^2 + (y-0.117)^2))$. The solution has a sharp peak in the point (0.5, 0.117).

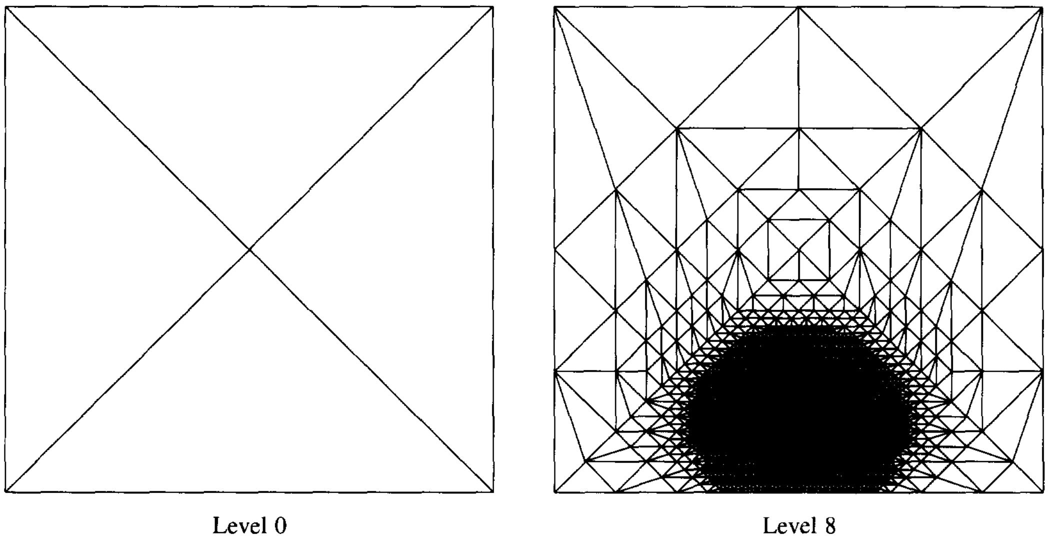

Fig. 3. Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_8$ (Example 1).

Example 2. We consider the Poisson equation on a hexagon with corners $(1, 1/2)$, $(7/8, 1)$, $(1/8, 1)$, $(0, 1/2)$, $(1/8, 0)$ and $(7/8, 0)$ and the solution given by $u(x, y) = \Phi(x)\Phi(y)$ where

$$\Phi(t) = \begin{cases} 1, & \text{if } t \leqslant 0.4, \\ 0, & \text{if } t \geqslant 0.6, \\ -(5t-3)^3\big(150t^2 - 105t + 19\big), & \text{elsewhere.} \end{cases}$$

The characteristic feature of the solution is a wavefront along the lines $x = 0.5,\ 0 < y < 0.5$ and $y = 0.5,\ 0 < x < 0.5$.

Starting from a coarse triangulation, on each refinement level the discretized problems are solved by the multilevel preconditioned cg-iterations as outlined in the previous section. The iteration process on level $l + 1$ is stopped when the estimated iteration error ε_{l+1} is less than $\varepsilon_{l+1}^2 \leqslant 0.01\varepsilon_l^2 N_l/N_{l+1}$ where ε_l denotes the estimated error on level l and N_l, N_{l+1} stand for the number of nodes on level l and $l + 1$, respectively.

Figs. 3 and 4 show the initial and the final triangulations for Examples 1 and 2. The refinement process is stopped when the estimated error is less than a safety factor times the prescribed tolerance. In both cases we observe a significant adaptive refinement in the neighborhood of the peak of the solution (Example 1) and the wavefront (Example 2).

The performance of the flux-based a posteriori hierarchical basis error estimator is shown in Figs. 5 and 6 representing the efficiency index $E = \varepsilon_{\text{est}}/\varepsilon_{\text{true}} - 1$ as a function of the total number of nodes where ε_{est} and $\varepsilon_{\text{true}}$ stand for the estimated and true error, respectively.

For both examples, at the beginning of the refinement process we observe a slight underestimation of the error, but in each case the estimated error rapidly approaches the true error with increasing refinement.

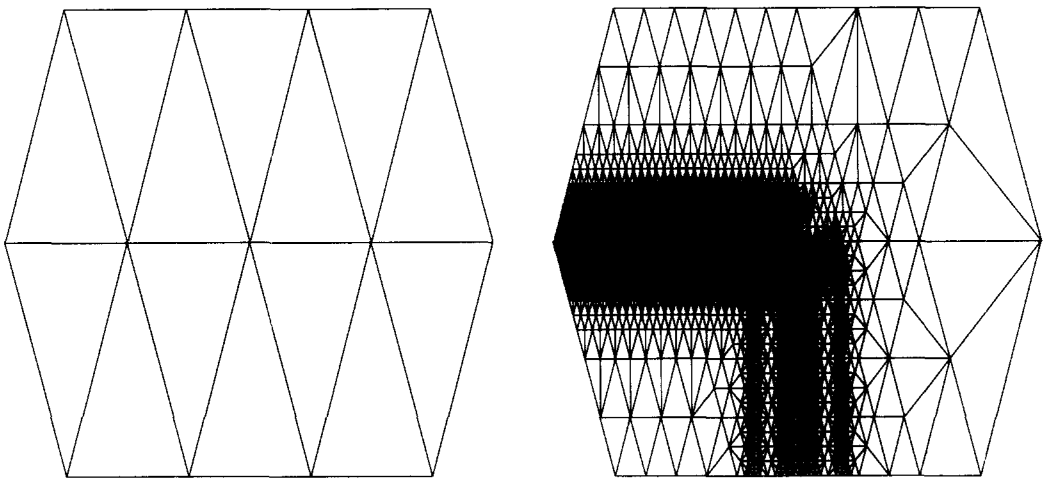

Fig. 4. Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_6$ (Example 2).

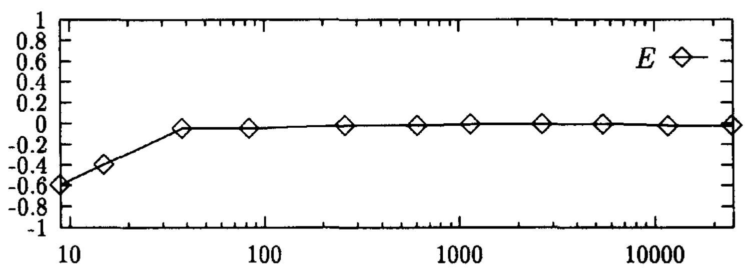

Fig. 5. Performance of the error estimator (Example 1).

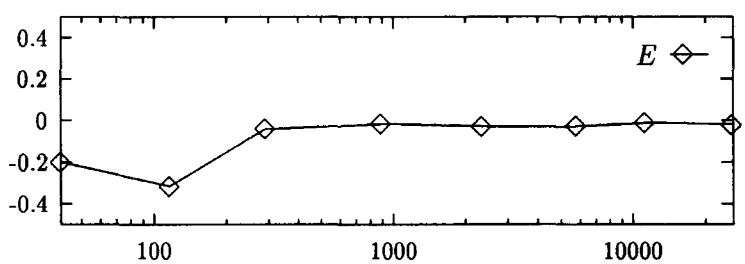

Fig. 6. Performance of the error estimator (Example 2).

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