Multilevel methods for mixed finite elements in three dimensions

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Dedicated to Prof. Dr. Dr.h.c. R. Bulirsch on the occasion of his 65th birthday

Summary. In this paper we consider second order scalar elliptic boundary value problems posed over three–dimensional domains and their discretization by means of mixed Raviart–Thomas finite elements [18]. This leads to saddle point problems featuring a discrete flux vector field as additional unknown.

Following Ewing and Wang [26], the proposed solution procedure is based on splitting the flux into divergence free components and a remainder. It leads to a variational problem involving solenoidal Raviart–Thomas vector fields.

A fast iterative solution method for this problem is presented. It exploits the representation of divergence free vector fields as $\mathbf{curl}$s of the $\mathbf{H}(\mathbf{curl})$–conforming finite element functions introduced by Nédélec [43]. We show that a nodal multilevel splitting of these finite element spaces gives rise to an optimal preconditioner for the solenoidal variational problem: Duality techniques in quotient spaces and modern algebraic multigrid theory [50, 10, 31] are the main tools for the proof.

1. Introduction

We are dealing with second order elliptic boundary value problems: Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ we seek u such that

$$
\begin{aligned}
-\operatorname{div}(a\,\mathbf{grad}\,u) &= f \quad \text{in } \Omega,\\
u &= g \quad \text{on } \partial\Omega\ ,
\end{aligned} \tag{1}
$$

where $\Omega \subset \mathbb{R}^3$ is a *simply connected* bounded domain with *connected* polyhedral boundary and $a \in L^\infty(\Omega)$. Further we assume $\underline{\alpha} \le a(\boldsymbol{x}) \le \overline{\alpha}$ a.e. in Ω for some $\overline{\alpha} \ge \underline{\alpha} > 0$. Without loss of generality, we may drop a altogether, i.e. $a \equiv 1$, since our central results are invariant with respect to spectrally equivalent operators.

Mixed finite element methods are attractive for discretizing (1) in cases where conservation of the flux $\boldsymbol{j} := \mathbf{grad}\,u$ is paramount. They are based on the dual variational formulation of (1) (see [18]):

Seek $(\boldsymbol{j}, u) \in \boldsymbol{H}(\operatorname{div};\Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
(\boldsymbol{j},\boldsymbol{v})_0 \quad + (\operatorname{div}\boldsymbol{v}, u)_0 &= \langle g, \langle \boldsymbol{v},\boldsymbol{n}\rangle\rangle_{H^{\frac12}\times H^{-\frac12}} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)\\
(\operatorname{div}\boldsymbol{j}, w)_0 \quad\quad\quad\quad\quad &= -\,(f, w)_0 && \forall w \in L^2(\Omega)\ .
\end{aligned} \tag{2}
$$

Here, $(\cdot,\cdot)_0$ denotes the $L^2(\Omega)$–inner product, $\langle\cdot,\cdot\rangle_{H^{\frac12}\times H^{-\frac12}}$ refers to the duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, and $\langle\cdot,\cdot\rangle$ stands for the Euclidean inner product in $\mathbb{R}^3$. Besides, $\boldsymbol{H}(B,\Omega)$, $B = \operatorname{div}, \mathbf{curl}$, designates the Hilbert spaces of vector fields in $\boldsymbol{L}^2(\Omega)$, whose images under the operator B are square integrable.

By replacing the continuous function spaces by appropriate finite element subspaces we end up with a system of linear equations in saddle point form. It will be our goal to develop a fast iterative solver for this saddle point problem. In our terminology an iterative solver qualifies as “fast” or “optimal”, if its speed of convergence does not sag, no matter how fine the mesh.

Whereas a fairly comprehensive understanding of multilevel methods for standard discretizations of (1) has been achieved, only partial breakthroughs have been scored in the field of mixed schemes. Basically, two approaches towards efficient multilevel methods for the iterative solution of the discrete mixed problem have been pursued:

The first seeks to recover a positive definite system by eliminating the flux. A sort of discrete representation of the Laplacian emerges, thus. The elimination yields a Schur–complement system, which turns out to be eligible for the treatment with special multigrid methods (see [13], Sect. 8). Standard methods fail due to the discontinuity of the finite element functions approximating u.

Another way to retreat to a positive definite setting is offered by mixed hybridization [3]. Recent research [2] revealed an equivalence between the discrete problems arising from hybridization and those obtained through particular nonconforming finite element discretizations of the primal second order elliptic problem. This paves the way for adapting multilevel techniques which have been developed for nonconforming discretizations to the mixed hybrid system [14, 38, 21, 22]. The nonconforming nature of the underlying finite element spaces entails devising special intergrid transfer operators. If this is done properly, good convergence properties have been reported (see citations above). However, if one is interested in an approximation of the flux, additional post–processing has to be applied to the mixed hybrid solution.

The second strategy retains the flux and drops the scalar unknown u. This can be achieved by penalty methods [49], augmented Lagrangian multipliers [36], a direct elimination of the divergence constraint [27, 26] or by iteration in an appropriate subspace [37]. In either case one faces variational problems set in finite element subspaces of $\boldsymbol{H}(\mathrm{div}; \Omega)$. For their solution several multilevel decompositions have been explored: In the absence of a zero order term, Ewing and Wang [26, 27] showed how a splitting of the flux in a part satisfying the divergence condition and a divergence–free remainder can pave the way to an optimal iterative solver. This has been generalized in [37] covering the case of a non–vanishing Helmholtz term. Similar in spirit is the approach of Vassilevski and Wang [49], who rely on an approximate Helmholtz–decomposition of vector fields. Then they propose separate multilevel decompositions of solenoidal and non–solenoidal parts. Later the optimality of this multilevel decomposition could be established [36].

The Helmholtz–decomposition also underlies the scheme of Arnold, Falk, and Winther [4, 5], which yields an optimal multigrid method. All these concepts are essentially confined to two dimensions, since inherently they exploit that certain variational problems for solenoidal fluxes can be recast as H^1–elliptic problems in standard finite element spaces of continuous, piecewise polynomial functions. Thus they become amenable to powerful conventional multilevel solvers. Such a simple relationship no longer holds in three dimensions.

Conversely, a valid decomposition in two and three dimensions alike is provided by the hierarchical basis method introduced by Cai, Goldstein, and Pasciak in [19]. In the 2D case, it has been shown by Wohlmuth and one of the authors [37] that this scheme leads to a slightly suboptimal growth like $O(L^2)$ of the condition number of the preconditioned system, where L is the total number of levels.

The three dimensional case is also dealt with in [20], where the authors take the cue from the work of Ewing and Wang [26] and construct an iterative scheme based on domain decomposition supplemented with a coarse grid.

Their analysis shows that an h–independent rate of convergence can be achieved.

The method presented in this paper falls into the second category, since our preconditioner acts on vector fields. It can be viewed as a consistent generalization of the ideas of Ewing and Wang: The treatment of the non-solenoidal part of the flux is rather the same. In contrast, the treatment of the solenoidal vector fields is much compounded by the fact that their representation now involves $\mathbf{curl}$s of $\boldsymbol{H}(\mathbf{curl}; \Omega)$–conforming finite element spaces. Moreover, the variational problems in these spaces are related to the degenerate bilinear form $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_0$.

"Ellipticity" of this bilinear form, which is a vital prerequisite for multilevel methods, does only hold on the orthogonal complement of the kernel $\mathrm{Ker}(\mathbf{curl})$ of the $\mathbf{curl}$–operator. More precisely, we have for $\boldsymbol{u} \in \mathrm{Ker}(\mathbf{curl})^{\perp}$ the equivalence $\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \approx |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)}$ (see [30], Theorem 3.9), provided that Ω is convex. The latter is the classical energy norm on $H_0^1(\Omega)$, for which nodal BPX–type decompositions (cf. [13]) have the desired stability properties (see [45, 11]). This hints that a similar splitting into one–dimensional subspaces spanned by localized canonical basis functions on each level might be a promising option for 3D stream functions, as well. Of course, orthogonality with respect to $\mathrm{Ker}(\mathbf{curl})$ has to be relaxed to get a computationally feasible scheme. Yet, nothing more than approximate orthogonality is sufficient for two reasons. Firstly, we are only interested in the $\mathbf{curl}$s of stream functions, so that components in $\mathrm{Ker}(\mathbf{curl})$ do not matter. Secondly, the concept of stable splittings [46] is rather flexible and permits us to absorb moderate deviations from orthogonality into constants. This paper furnishes a rigorous underpinning of these heuristic arguments for the cause of a nodal multilevel decomposition of stream function spaces.

The paper is organized as follows: In the next section we briefly describe the finite element spaces used for discretizing (2) and needed for the formulation of the multilevel algorithm. We are concerned with $\boldsymbol{H}(\mathrm{div}; \Omega)$–conforming Raviart–Thomas spaces and $\boldsymbol{H}(\mathbf{curl}; \Omega)$–conforming Nédélec spaces. A brief discussion of some of their essential properties is included.

In the third section, we specify the multilevel algorithm used to solve the discrete saddle point problem. We outline, how both the non–solenoidal parts and divergence–free parts of the flux can be computed in a multilevel framework. To this end we introduce the nodal multilevel decomposition of Nédélec spaces. A preliminary discussion of its stability is included.

The next section examines discrete and semicontinuous quotient spaces with respect to $\mathrm{Ker}(\mathbf{curl})$. They are the main tools for the proof of stability. A thorough examination of their interrelationships is given.

In the fifth section, we temporarily switch to a problem on a cube. This guarantees sufficient regularity for duality techniques which are applied in those semicontinuous spaces. Thus we arrive at a stability estimate on a cube.

Then, we employ an extension trick to return to general domains. This completes the proof of stability, which, according to the central result of algebraic multilevel theory, guarantees uniformly fast convergence of preconditioned iterative methods.

In the last but one section, the amount of computation involved in crucial steps of the algorithm is investigated. For lowest order hexahedral finite elements, we establish an operation count that is a small multiple of the number of unknowns for both the preprocessing step and the evaluation of the preconditioner.

In the final section we discuss the the algorithm and report on a few numerical experiments which illustrate the actual performance of the method in different settings.

2. Finite element spaces

Let $\mathcal{T}_h := \{T_i\}_i$ denote a quasiuniform simplicial or hexahedral triangulation of Ω with meshwidth $h := \max\{\operatorname{diam} T_i\}$. Besides we write $\mathcal{F}_h$ for the set of faces of $\mathcal{T}_h$, and $\mathcal{E}_h$ for the set of edges. We demand that the tetrahedra are uniformly shape–regular. Based on this mesh we introduce several conforming finite element spaces:

$\mathcal{S}_d(\Omega;\mathcal{T}_h) \subset H^1(\Omega)$	: The space of continuous finite element functions, piecewise polynomial of degree $d \in \mathbb{N}$
$\boldsymbol{\mathcal{ND}}_d(\Omega;\mathcal{T}_h) \subset \boldsymbol{H}(\mathbf{curl};\Omega)$	: Nédélec finite element space of order $d \in \mathbb{N}$ (see [43,44])
$\boldsymbol{\mathcal{RT}}_d(\Omega;\mathcal{T}_h) \subset \boldsymbol{H}(\mathrm{div};\Omega)$	: Raviart–Thomas finite element space of order $d \in \mathbb{N}_0$ (see [43,18,47])
$\mathcal{Q}_d(\Omega;\mathcal{T}_h) \subset L^2(\Omega)$	: Space of discontinuous functions, piecewise polynomial of degree $d \in \mathbb{N}_0$

If no ambiguity can arise, the domain may be omitted. Supplemented with a subscript 0 the same notations cover the spaces equipped with homogeneous boundary conditions (in the sense of an appropriate trace operator). In addition, $\mathcal{Q}_{d,0}(\Omega;\mathcal{T}_h)$ contains only functions with zero mean value.

All finite element spaces are equipped with sets $\Xi(\mathcal{X}_k,\mathcal{T}_h)$, $\mathcal{X} = \mathcal{S}$, $\boldsymbol{\mathcal{ND}}$, $\boldsymbol{\mathcal{RT}}$, $\mathcal{Q}$, of *global degrees of freedom* which ensure conformity. They can be defined in a canonical fashion so that they remain invariant under the respective canonical transformations of finite element functions. Consequently, all finite element spaces form affine families in the sense of [23]. We refer to [43] for a comprehensive exposition. Besides, we impose a p–hierarchical structure on the sets of degrees of freedom by requiring that $\Xi(\mathcal{X}_{d-1},\mathcal{T}_h)$ is contained in $\Xi(\mathcal{X}_d,\mathcal{T}_h)$ and all functionals from $\Xi(\mathcal{X}_d,\mathcal{T}_h)/\Xi(\mathcal{X}_{d-1},\mathcal{T}_h)$ have to vanish on $\mathcal{X}_{d-1}$.

Based on the degrees of freedom, sets of *canonical nodal basis functions* can be introduced as bidual bases for $\Xi(\mathcal{X}_d,\mathcal{T}_h)$. They are locally supported and form an L^2*–frame*. For instance, in the case of Nédélec spaces we can find generic constants $\underline{C},\overline{C}>0$, independent of the meshwidth h and only depending on d, such that

(3)
$$\underline{C}\|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(\Omega)} \leq h \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_h)} \kappa(\boldsymbol{\xi})^2 \leq \overline{C}\,\|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall\boldsymbol{\xi}\in\boldsymbol{\mathcal{ND}}_d(\Omega;\mathcal{T}_h)\ .$$

Following a popular convention a capital C will be used as a token for a generic constant. Its value can vary between different occurrences, but we will always specify what it must not depend on.

Now, given the degrees of freedom, for sufficiently smooth functions the nodal projections (nodal interpolation operators) $\Pi^{\mathcal{X}_k}_{\mathcal{T}_h}$, $\mathcal{X}=\mathcal{S},\boldsymbol{\mathcal{ND}},\boldsymbol{\mathcal{RT}}$, $\mathcal{Q}$ are well defined . The nodal interpolation operators are exceptional in that they satisfy (for $d\in\mathbb{N}_0$) the following *commuting diagram property* [24,29,18]

$$\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\ \mathbf{grad}\ } & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\ \mathbf{curl}\ } & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\ \mathrm{div}\ } & C^\infty(\Omega) \\
\Big\downarrow \Pi^{\mathcal{S}_{d+1}}_{\mathcal{T}_h} & & \Big\downarrow \boldsymbol{\Pi}^{\boldsymbol{\mathcal{ND}}_{d+1}}_{\mathcal{T}_h} & & \Big\downarrow \boldsymbol{\Pi}^{\boldsymbol{\mathcal{RT}}_d}_{\mathcal{T}_h} & & \Big\downarrow \Pi^{\mathcal{Q}_d}_{\mathcal{T}_h} \\
\mathcal{S}_{d+1}(\Omega;\mathcal{T}_h) & \xrightarrow{\ \mathbf{grad}\ } & \boldsymbol{\mathcal{ND}}_{d+1}(\Omega;\mathcal{T}_h) & \xrightarrow{\ \mathbf{curl}\ } & \boldsymbol{\mathcal{RT}}_d(\Omega;\mathcal{T}_h) & \xrightarrow{\ \mathrm{div}\ } & \mathcal{Q}_d(\Omega;\mathcal{T}_h)\ ,
\end{array}$$

which links nodal projectors and differential operators. The commuting diagram property is the key to the proof of the following *representation theorem*:

Theorem 1. *The following sequences of vector spaces are exact for any* $d>0$*:*

$$\begin{aligned}
&\{\text{const.}\} \xrightarrow{Id} \mathcal{S}_d(\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{ND}}_d(\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{RT}}_{d-1}(\mathcal{T}_h) \\
&\qquad\qquad \xrightarrow{\mathrm{div}} \mathcal{Q}_{d-1}(\mathcal{T}_h) \longrightarrow \{0\} \\
&\{0\} \xrightarrow{Id} \mathcal{S}_{d,0}(\mathcal{T}_h) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{ND}}_{d,0}(\mathcal{T}_h) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{RT}}_{d-1,0}(\mathcal{T}_h) \\
&\qquad\qquad \xrightarrow{\mathrm{div}} \mathcal{Q}_{d-1,0}(\mathcal{T}_h) \longrightarrow \{0\}
\end{aligned}$$

Proof. See [30,43] and in particular [33, Theorem 20]. □

Another consequence of the commuting diagram property is that p–hierarchical surpluses are preserved when the appropriate differential operator is applied. For Nédélec spaces this reads:

$$\begin{aligned} &\mathbf{curl}\ \ \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_{d+1}} - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_d}\ \ \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_h) \\ &\quad \subset\ \ \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_d} - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{RT}}_{d-1}}\ \ \boldsymbol{\mathcal{RT}}_d(\mathcal{T}_h)\ . \end{aligned} \tag{4}$$

Remark 1. An inconvenient trait of the nodal projectors has to be stressed: Except in the case of $\mathcal{Q}_k$, they cannot be extended to the respective continuous function spaces. A slightly enhanced smoothness of the argument function is required (cf. Lemma 4.7 in [1] for the case of Nédélec's spaces), which drastically complicates the use of these projectors. Nevertheless, we cannot dispense with them; no other projectors are known, which satisfy the commuting diagram property (compare Remark 3.1 in [29]).

To cope with the projectors' need for smooth arguments, we have to resort to the following approximation property in fractional Sobolev spaces: For $d \geq 2$ from a variant of the Bramble–Hilbert lemma ([25], Theorem 6.1) we get

$$\begin{aligned} &\left\| \boldsymbol{\xi} - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_d} \boldsymbol{\xi} \right\|_{\boldsymbol{L}^2(\Omega)} \leq C\, h^s \|\boldsymbol{\xi}\|_{\boldsymbol{H}^s(\Omega)} \\ &\qquad \forall \boldsymbol{\xi} \in \boldsymbol{H}^s(\Omega), 1 < s \leq 2\ , \end{aligned} \tag{5}$$

with $C > 0$ only depending on s, d and the shape–regularity of $\mathcal{T}_h$ Another important estimate is obtained via the commuting diagram property [43,30]: For $d \geq 1$

$$\begin{aligned} &\left\| \mathbf{curl}\ \ \boldsymbol{\xi} - \boldsymbol{\Pi}_{\mathcal{T}_h}^{\boldsymbol{\mathcal{ND}}_d} \boldsymbol{\xi}\ \ \right\|_{\boldsymbol{L}^2(\Omega)} \leq C\ h\, |\mathbf{curl}\, \boldsymbol{\xi}|_{H^1(\Omega)} \\ &\qquad \forall \boldsymbol{\xi}; \mathbf{curl}\, \boldsymbol{\xi} \in H^1(\Omega)\ , \end{aligned} \tag{6}$$

with $C > 0$ independent of h.

Remark 2. Suitable finite elements are also available for prismatic elements [44,39] and the construction of isoparametric variants is straightforward. Also a variety of other $\boldsymbol{H}(\mathrm{div};\Omega)$– and $\boldsymbol{H}(\mathbf{curl};\Omega)$–conforming finite element spaces has been constructed (see e.g. [16,17,44,18]). If the commuting diagram property and the analogue of Theorem 1 hold, our method directly carries over to them. Yet, we will not elaborate on this subject any further.

3. Multilevel iterative solution procedure

To fix the multilevel setting, let $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_L, L \in \mathbb{N}$, denote a hierarchy of quasiuniform simplicial meshes, created by regular refinement of a shape–regular initial coarse mesh $\mathcal{T}_0$. See [7] for an algorithm, how this can be done in the case of a tetrahedral mesh. We write h_l for the meshwidth of $\mathcal{T}_l$, $l = 0, \dots, L$. Please note that during the refinement process each element is subdivided into eight smaller ones.

Following Ewing and Wang [26], we tackle the discrete saddle point problems arising from the $\boldsymbol{\mathcal{RT}}_d(\Omega; \mathcal{T}_L) \times \mathcal{Q}_d(\Omega; \mathcal{T}_L)$, $d \geq 0$, finite element discretization of (2) in two successive steps:

1st step. To begin with, we compute a discrete flux $\boldsymbol{j}_h^* \in \boldsymbol{\mathcal{RT}}_d(\Omega; \mathcal{T}_L)$ with divergence $\Pi_{\mathcal{T}_L}^{\mathcal{Q}_d} f$. To do so efficiently, the hierarchy of triangulations is employed. We start with an L^2–orthogonal decomposition of the source term:

$$\Pi_{\mathcal{T}_L}^{\mathcal{Q}_d} f = \Pi_{\mathcal{T}_0}^{\mathcal{Q}_d} f + \sum_{j=1}^{L} \;\; \Pi_{\mathcal{T}_j}^{\mathcal{Q}_d} - \Pi_{\mathcal{T}_{j-1}}^{\mathcal{Q}_d} \;\; f \; . \tag{7}$$

We refer to its individual terms as f_j, $j = 0, \dots, L$. Note that the canonical projections $\Pi_{\mathcal{T}_l}^{\mathcal{Q}_d}$ coincide with L^2–orthogonal projections on $\mathcal{Q}_d(\Omega; \mathcal{T}_l)$. Nevertheless, they are readily available, since functions in $\mathcal{Q}_d(\Omega; \mathcal{T}_j)$ are fully decoupled across interelement boundaries.

The f_j, $j = 1, \dots, L$ have vanishing mean value over every element $T \in \mathcal{T}_{j-1}$. This permits us to solve Neumann problems on the elements of the next coarser grid. Thus, for each $T \in \mathcal{T}_{j-1}$ we can determine a flux $\boldsymbol{j}_{j,T}^* \in \boldsymbol{\mathcal{RT}}_d(T; \mathcal{T}_{j|T})$ such that $\operatorname{div} \boldsymbol{j}_{j,T}^* = f_j$ on T and $\langle \boldsymbol{j}_T^*, \boldsymbol{n} \rangle_{|\partial T} = 0$. On the coarsest mesh $\mathcal{T}_0$ we have to solve a saddle point problem to get $\boldsymbol{j}_0^* \in \boldsymbol{\mathcal{RT}}_d(\Omega; \mathcal{T}_0)$ with $\operatorname{div} \boldsymbol{j}_0^* = f_0$. It is easy to see that

$$\boldsymbol{j}_h^* = \boldsymbol{j}_0^* + \sum_{j=1}^{L} \sum_{T \in \mathcal{T}_{j-1}} \boldsymbol{j}_{j,T}^*$$

gives us what we have been looking for.

2nd step. At the second step we look for a divergence–free correction $\boldsymbol{j}_h^0 \in \boldsymbol{\mathcal{RT}}_d^0(\Omega; \mathcal{T}_L)$ of $\boldsymbol{j}_h^*$ as the solution of the symmetric, positive definite variational problem

Find $\boldsymbol{j}_h^0 \in \boldsymbol{\mathcal{RT}}_d^0(\Omega; \mathcal{T}_L)$ such that

$$\begin{aligned} \left(\boldsymbol{j}_h^0, \boldsymbol{v}_h^0\right)_0 = &- \left(\boldsymbol{j}_h^*, \boldsymbol{v}_h^0\right)_0 \\ &+ \left\langle g, \left\langle \boldsymbol{v}_h^0, \boldsymbol{n} \right.\right. \quad {}_{H^{\frac{1}{2}} \times H^{-\frac{1}{2}}} \quad \forall \boldsymbol{v}_h^0 \in \boldsymbol{\mathcal{RT}}_d^0(\mathcal{T}_L) \; . \end{aligned} \tag{8}$$

(The superscript 0 marks subspaces of divergence free vector fields.) Then $\boldsymbol{j}_h := \boldsymbol{j}_h^* + \boldsymbol{j}_h^0$ yields the solution for the discrete flux. The variational problem (8) would be benign, unless an explicit localized basis of $\boldsymbol{\mathcal{RT}}_d^0(\Omega; \mathcal{T}_L)$ proved elusive. Thus it is not possible to convert (8) into a sparse system of linear equations.

The representation theorem offers a remedy. It reveals that

$$\boldsymbol{\mathcal{RT}}^0_d(\Omega;\mathcal{T}_L) = \mathbf{curl}\,\boldsymbol{\mathcal{ND}}_{d+1}(\Omega;\mathcal{T}_L)\ .$$

Therefore $\boldsymbol{j}^0_h$ can be obtained as $\mathbf{curl}\,\boldsymbol{\xi}_h$ where $\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d+1}(\Omega;\mathcal{T}_L)$ satisfies

$$\begin{aligned} a(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h) = &-(\boldsymbol{j}^*_h,\mathbf{curl}\,\boldsymbol{\eta}_h)_0 \\ &+ \langle g, \langle \mathbf{curl}\,\boldsymbol{\eta}_h,\boldsymbol{n}\rangle\rangle_{H^{\frac{1}{2}}\times H^{-\frac{1}{2}}} \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_L)\ , \end{aligned} \tag{9}$$

where we have written $a(\cdot,\cdot)$ for the "energy" bilinear form $(\mathbf{curl}\,\cdot,\mathbf{curl}\,\cdot)_0$ on $\boldsymbol{H}(\mathbf{curl};\Omega)$. Occasionally, $\|\cdot\|_A$ will denote the related seminorm. On the one hand, (9) seems to be a normal linear system of equation arising from the finite element discretization of a linear variational problem. On the other hand, (9) is degenerate due to the large kernel of the $\mathbf{curl}$–operator; we confront a singular matrix and solutions of (9) are by no means unique. This does not need to worry us, since the right hand side of (9) is consistent and we are only interested in $\mathbf{curl}\,\boldsymbol{\xi}_h$, anyway.

Under these circumstances, the conjugate gradient method is well known to provide a viable iterative solver even for semidefinite systems. When applied to (9) it will return a valid approximate solution. The speed of convergence is governed by the ratio of the largest and smallest nonzero eigenvalue of the system matrix. Since this ratio grows like h^{-2} in the case of (9), the speed of the CG iteration is likely to deteriorate severely on fine meshes. Preconditioning is indispensable, hence.

Following the considerations laid out in the introduction, we opt for an additive Schwarz preconditioner based on a nodal decomposition. Writing $\boldsymbol{\psi}_{l,\kappa}$ for the canonical $\boldsymbol{\mathcal{ND}}_{d+1}$–basis function associated with the degree of freedom κ on level l, the concrete decomposition runs

$$\boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_L) = \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_0) + \sum_{l=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_{d+1},\mathcal{T}_l)} \operatorname{span}(\boldsymbol{\psi}_{l,\kappa}) \tag{10}$$

Since the subspaces involved in the decomposition (10) are small, a single evaluation of the related additive Schwarz preconditioner can be done with $O(\dim \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_L))$ operations. Obviously, we have an optimal method at our disposal, if we can show that (10) is stable uniformly in L.

It is one of the major insights of the algebraic theory of Schwarz methods (see [50,53,46]) that the proof of h–independent stability boils down to verifying the following two estimates:

The first states a geometric decrease of the constants in a strengthened Cauchy Schwarz inequality, namely the existence of $C_U > 0$ independent of L such that for all $\boldsymbol{\phi}_k \in \boldsymbol{\mathcal{V}}_{k,\kappa_1}, \boldsymbol{\eta}_l \in \boldsymbol{\mathcal{V}}_{l,\kappa_2}$ ($\boldsymbol{\mathcal{V}}_{l,\kappa} := \mathrm{span}(\boldsymbol{\psi}_{l,\kappa})$)

$$a(\boldsymbol{\phi}_k, \boldsymbol{\eta}_l) \leq \epsilon_{k,l} \|\boldsymbol{\phi}_k\|_A \|\boldsymbol{\eta}_l\|_A \ \text{ with } \ \epsilon_{k,l} \leq C_U \left(\sqrt{\frac{1}{2}}\right)^{|k-l|} . \tag{11}$$

The proof of this property for the decomposition (10) is fairly elementary. We can proceed almost exactly as in the case of piecewise linear finite elements, which is studied in [9,8,51] in great detail. We can thus skip the proof at this site.

The second estimate claims that for all $\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d+1}(\mathcal{T}_L)$

(12)
$$\inf \left\{ \sum_{l,\kappa} \|\boldsymbol{\nu}_{l,\kappa}\|_A^2 ;\ \mathbf{curl} \sum_{l,\kappa} \boldsymbol{\nu}_{l,\kappa} = \mathbf{curl}\, \boldsymbol{\xi}_h,\ \boldsymbol{\nu}_{l,\kappa} \in \boldsymbol{\mathcal{V}}_{l,\kappa} \right\} \leq C_L \|\boldsymbol{\xi}_h\|_A^2$$

with a universal constant $C_L > 0$ for all depths of refinement. To establish (12) is the hard part. The following sections are devoted to this task.

If (11) and (12) hold, we know that the generalized condition number of the multilevel preconditioned matrix related to (9) remains bounded as $L \to \infty$, $h \to 0$. So the preconditioned conjugate gradient iteration indeed meets our criterion for an optimal algorithm.

4. Discrete and semicontinuous quotient spaces

The degeneracy of the bilinear form $a(\cdot,\cdot)$ thwarts a straightforward application of the standard techniques of multilevel theory. To recover the usual setting, which requires $a(\cdot,\cdot)^{\frac{1}{2}}$ to give rise to a valid energy norm, we formally switch to quotient spaces with respect to the null spaces of $\mathbf{curl}$.

Since many basic estimates heavily rely on duality techniques, we carry out the bulk of the proof of (12) on a cube shaped domain. Further, we demand homogeneous boundary conditions for $\boldsymbol{H}(\mathbf{curl};\Omega)$–vector fields throughout. So, for the time being, assume $\Omega = \mathbf{C} :=]\mathbf{0};\mathbf{1}[^{\mathbf{3}}$. This is motivated by the ease with which results on the regularity of solutions of boundary value problems can be obtained on this particular domain. We can apply reflection principles to get:

Theorem 2. *A vector field $\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl};\mathbf{C})$ with vanishing divergence and its* $\mathbf{curl}$ *in $\boldsymbol{H}^\varepsilon(\mathbf{C})$, $0 \leq \varepsilon \leq 1$, belongs to $\boldsymbol{H}^{1+\varepsilon}(\mathbf{C})$. In addition, we have the estimate*

$$\|\boldsymbol{\xi}\|_{\boldsymbol{H}^{1+\varepsilon}(\mathbf{C})} \quad \leq \quad C\, \|\mathbf{curl}\, \boldsymbol{\xi}\|_{\boldsymbol{H}^{\varepsilon}(\mathbf{C})} , \tag{13}$$

with $C > 0$ independent of $\boldsymbol{\xi}$.

Proof. The proof uses a standard argument based on a symmetric extension and is given in [34]. By more elaborate techniques the theorem can also be proved for general convex polyhedra [48, Theorem. 2.3]. □

In a Hilbert space setting, orthogonal complements of $\mathrm{Ker}(\mathbf{curl})$ offer an isomorphic model for quotient spaces modulo $\mathrm{Ker}(\mathbf{curl})$. First we introduce the function space

$$\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C}) := \Big\{\boldsymbol{\xi} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathbf{C});\ (\boldsymbol{\xi}, \mathbf{grad}\,\varphi)_{\boldsymbol{L}^2(\mathbf{C})} = \mathbf{0}$$

$$\forall \varphi \in H_0^1(\mathbf{C})\Big\}$$

It is plain to see that $a(\cdot,\cdot)$ becomes truly $\boldsymbol{H}(\mathbf{curl};\mathbf{C})$–elliptic on $\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$.

Thanks to the discrete representation theorem (Theorem 1), for $d \geq 1$ we have

$$\mathrm{Ker}(\mathbf{curl}) = \{\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d,0}(\mathbf{C};\mathcal{T}_l);\ \mathbf{curl}\,\boldsymbol{\xi}_h = 0\} = \mathbf{grad}\,\mathcal{S}_{d,0}(\Omega;\mathcal{T}_l)$$

so that the orthogonal complements of $\mathrm{Ker}(\mathbf{curl})$ in the finite element spaces are given by

$$\boldsymbol{\mathcal{ND}}_{d,0}^{+}(\mathbf{C};\mathcal{T}_l) := \Big\{\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d,0}(\mathbf{C};\mathcal{T}_l);\ (\boldsymbol{\xi}_h, \mathbf{grad}\,\varphi_h)_0 = 0$$

$$\forall \varphi_h \in \mathcal{S}_{d,0}(\mathbf{C};\mathcal{T}_l)\Big\}\ .$$

These spaces serve as an isometrically isomorphic model for the quotient spaces $\boldsymbol{\mathcal{ND}}_{d,0}(\mathbf{C};\mathcal{T}_l)/\mathrm{Ker}(\mathbf{curl})$. $\boldsymbol{H}(\mathbf{curl};\mathbf{C})$–ellipticity of $a(\cdot,\cdot)$ also holds on these spaces with a constant independent of h_l:

Lemma 1. *For all $l \in \mathbb{N}_0$ and $d \in \mathbb{N}$ we have the estimate*

$$\|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C\ \|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d,0}^{+}(\mathbf{C};\mathcal{T}_l)$$

with $C > 0$ independent of the level l of refinement.

Proof. This is a special case of Theorem 9 in [43] or Proposition 5.1 in [30], which cover general convex domains. □

Unfortunately, these orthogonal complements are not nested nor are they contained in the continuous space, i.e., for $0 \leq l < L$

$$\boldsymbol{\mathcal{ND}}_{d,0}^{+}(\mathbf{C};\mathcal{T}_l) \not\subset \boldsymbol{\mathcal{ND}}_{d,0}^{+}(\mathbf{C};\mathcal{T}_{l+1}) \not\subset \boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})\ .$$

This makes them an awkward environment for the intended proof, as the duality techniques we are aiming at depend on nested spaces. To steer clear of these difficulties we introduce a mapping $\theta\ :\ \boldsymbol{H}_0(\mathbf{curl};\mathbf{C}) \mapsto$

$\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$ by $\theta(\boldsymbol{\xi}) := \boldsymbol{\xi} + \mathbf{grad}\,\psi$, where $\psi \in H_0^1(\Omega)$ is uniquely characterized as the solution of

$$(\mathbf{grad}\,\psi, \mathbf{grad}\,\varphi)_0 = -\,(\boldsymbol{\xi}, \mathbf{grad}\,\varphi)_0 \quad \forall \varphi \in H_0^1(\Omega)\ .$$

Note that θ preserves the $\mathbf{curl}$ of a vector field and cannot increase its L^2–norm.

This mapping is used to define the *semicontinuous spaces*

$$\boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\mathbf{C};\mathcal{T}_l) := \theta(\boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l))\ .$$

They have a hybrid nature: Their dimensions are finite, but their elements are no longer piecewise polynomial. Their $\mathbf{curl}$s belong to finite element spaces, however. Obviously $\boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l)$ and $\boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\mathbf{C};\mathcal{T}_l)$ are isomorphic, an isomorphism given by θ. It is also immediate that the spaces $\boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\mathbf{C};\mathcal{T}_l)$ are perfectly nested.

We intend to stick to a double track strategy: we want to make use of the nestedness of the $\boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\Omega;\mathcal{T}_l)$–spaces and benefit from the properties of the finite element spaces $\boldsymbol{\mathcal{ND}}^{+}_{d,0}(\Omega;\mathcal{T}_{l+1})$ as well. So we need a link between these two families of spaces. θ is a promising candidate. Unfortunately, the inverse of the mapping $\theta : \boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l) \mapsto \boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\mathbf{C};\mathcal{T}_l)$ fails to be uniformly bounded in the L^2–norm with respect to the meshwidth h_l. A weaker estimate holds, however:

Lemma 2. *Let $d \geq 2$. With $C > 0$ independent of the depth l of refinement and the function $\boldsymbol{\xi}_h$, otherwise only depending on d and $\mathcal{T}_0$, we have*

$$\|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \quad \|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{L}^2(\Omega)} + h_l\|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\Omega)}$$

$$\forall \boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l)\ .$$

Proof. Pick an arbitrary $\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l)$. Since, evidently, the mapping $\theta : \boldsymbol{\mathcal{ND}}^{+}_{d,0}(\mathbf{C};\mathcal{T}_l) \mapsto \boldsymbol{\mathcal{ND}}^{\perp}_{d,0}(\mathbf{C};\mathcal{T}_l)$ does not affect the $\mathbf{curl}$ of its argument, $\mathbf{curl}\,\theta(\boldsymbol{\xi}_h) = \mathbf{curl}\,\boldsymbol{\xi}_h$ is piecewise polynomial.

Now, recall the important fact that any piecewise polynomial function $f \in L^2(\mathbf{C})$ belongs to $H^{\varepsilon}(\mathbf{C})$ for all $\varepsilon < 1/2$ and fulfills the inverse estimate

$$\|f\|_{H^{\varepsilon}(\mathbf{C})} \leq C(\varepsilon) h_l^{-\varepsilon}\, \|f\|_{L^2(\mathbf{C})} \tag{14}$$

with $C(\varepsilon)$ independent of f (cf. the appendix of [13]).

We conclude that $\mathbf{curl}\,\theta(\boldsymbol{\xi}_h) \in \boldsymbol{H}^{\varepsilon}(\mathbf{C})$ for $\varepsilon \in]0;1/2[$. According to Theorem 2, this means that $\theta(\boldsymbol{\xi}_h) \in \boldsymbol{H}^{1+\varepsilon}(\mathbf{C})$ and

$$\|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{H}^{\varepsilon+1}(\mathbf{C})} \leq C\|\mathbf{curl}\,\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{H}^{\varepsilon}(\mathbf{C})}\ ,$$

where we made tacit use of $\operatorname{div}\theta(\boldsymbol{\xi}_h) = 0$. This makes sure that the nodal interpolation operator $\boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}$ is well defined for $\theta(\boldsymbol{\xi}_h)$.

In addition, we can rely on the estimate (5) and the triangle inequality to see that

$$\left\|\boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\left(\theta(\boldsymbol{\xi}_h)\right)\right\|_{\boldsymbol{L}^2(\mathbf{C})} \leq \|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{L}^2(\mathbf{C})} + C\,h_l^{1+\varepsilon}\,\|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{H}^{1+\varepsilon}(\mathbf{C})}\ ,$$

where ε enters the constant C, of course. Now we fix $\varepsilon\in]0;1/2[$, so that it can be regarded as a constant from now on. Again Theorem 2 is used to proceed with the estimate:

$$\begin{aligned}\left\|\boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\left(\theta(\boldsymbol{\xi}_h)\right)\right\|_{\boldsymbol{L}^2(\mathbf{C})} &\leq\ \|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{L}^2(\mathbf{C})} + C\,h_l^{1+\varepsilon}\,\|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{H}^{\varepsilon}(\mathbf{C})} \\ &\leq\ \|\theta(\boldsymbol{\xi}_h)\|_{\boldsymbol{L}^2(\mathbf{C})} + C\,h_l\,\|\mathbf{curl}\,\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\mathbf{C})}\ . \end{aligned} \tag{15}$$

In the final step above we employed the inverse estimate (14).

From basic properties of the mapping θ we conclude

$$\mathbf{curl}\,(\boldsymbol{\xi}_h - \theta(\boldsymbol{\xi}_h)) = 0\ . \tag{16}$$

We have already found out that the nodal projection is defined for both summands in (16) so that we can infer from the commuting diagram property of the nodal projectors:

$$\mathbf{curl}\quad \boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\left(\boldsymbol{\xi}_h - \theta(\boldsymbol{\xi}_h)\right) \quad = 0\ .$$

Further, linearity and idempotence of the projector ensure

$$\mathbf{curl}\quad \boldsymbol{\xi}_h - \boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\theta(\boldsymbol{\xi}_h) \quad = 0\ .$$

Consequently, by the representation theorem we find a $\varphi_h \in \mathcal{S}_{d,0}(\mathbf{C};\mathcal{T}_l)$ such that

$$\boldsymbol{\xi}_h - \boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\left(\theta(\boldsymbol{\xi}_h)\right) = \mathbf{grad}\,\varphi_h\ .$$

Now, we are going to exploit $\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{ND}}_{d,0}^{+}(\mathbf{C};\mathcal{T}_l)$:

$$\begin{aligned}\|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\mathbf{C})}^2 &=\ \ \boldsymbol{\xi}_h, \mathbf{grad}\,\varphi_h + \boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\theta(\boldsymbol{\xi}_h)\ \ {}_{\boldsymbol{L}^2(\mathbf{C})} \\ &=\ \ \boldsymbol{\xi}_h, \boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\theta(\boldsymbol{\xi}_h)\ \ {}_{\boldsymbol{L}^2(\mathbf{C})} \\ &\leq \|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^2(\mathbf{C})}\left\|\boldsymbol{\Pi}_{\mathcal{T}_l}^{\boldsymbol{\mathcal{ND}}_d}\theta(\boldsymbol{\xi}_h)\right\|_{\boldsymbol{L}^2(\mathbf{C})}\ .\end{aligned}$$

Plugging (15) into this estimate completes the proof. $\square$

5. Stable splitting on a cube

Still, we confine ourselves to $\Omega = \mathbf{C}$. For an arbitrary $\boldsymbol{\xi}_L \in \boldsymbol{\mathcal{ND}}_{d,0}(\mathbf{C};\mathcal{T}_L)$, $d \geq 1$, $L \in \mathbb{N}$, we attempt to find a nodal decomposition according to (10) that satisfies the estimate (12). Without loss of generality, we may assume $\boldsymbol{\xi}_L \in \boldsymbol{\mathcal{ND}}^+_{k,0}(\mathbf{C};\mathcal{T}_L)$, since only $\mathbf{curl}\,\boldsymbol{\xi}_L$ is relevant,

The construction of the multilevel splitting of $\boldsymbol{\xi}_L$ and the proof of its stability loosely follows the ideas presented by Zhang in [54]. Similar techniques based on duality arguments can be found in the appendix of [52] and Sect. 3 in [53].

To begin with, we specify the wanted splitting of $\boldsymbol{\xi}_L \in \boldsymbol{\mathcal{ND}}^+_{d,0}(\mathbf{C};\mathcal{T}_L)$ into multiples of nodal bases on all levels: Set $\boldsymbol{\xi}^\perp_L := \theta(\boldsymbol{\xi}_L) \in \boldsymbol{\mathcal{ND}}^\perp_{d,0}(\mathbf{C};\mathcal{T}_\mathbf{L})$ Then define $\boldsymbol{\zeta}^\perp_k \in \boldsymbol{\mathcal{ND}}^\perp_{d,0}(\mathbf{C};\mathcal{T}_k)$, $k \in \{0,\dots,L\}$, as solutions of the variational problems:

Seek $\boldsymbol{\eta}^\perp_k \in \boldsymbol{\mathcal{ND}}^\perp_{d,0}(\mathbf{C};\mathcal{T}_k)$ such that

$$a(\boldsymbol{\eta}^\perp_k,\boldsymbol{\phi}^\perp_k) = a(\boldsymbol{\xi}^\perp_L,\boldsymbol{\phi}^\perp_k) \quad \forall \boldsymbol{\phi}^\perp_k \in \boldsymbol{\mathcal{ND}}^\perp_{d,0}(\mathbf{C};\mathcal{T}_k)\ . \tag{17}$$

The existence of a unique solution of (17) is guaranteed, since $a(\cdot,\cdot)$ is an inner product in $\boldsymbol{H}^\perp(\mathbf{curl};\mathbf{C})$. Writing $P_k : \boldsymbol{H}^\perp_0(\mathbf{curl};\mathbf{C}) \mapsto \boldsymbol{\mathcal{ND}}^\perp_{d,0}$ $(\mathbf{C};\mathcal{T}_\mathbf{k})$, $0 \leq k \leq L$, for the $a(\cdot,\cdot)$–orthogonal projection we have $\boldsymbol{\zeta}^\perp_k = P_k\,\boldsymbol{\xi}^\perp_L$. As usual $P_{-1} = 0$, and thus we get the $a(\cdot,\cdot)$-orthogonal decomposition

$$\boldsymbol{\xi}^\perp_L \quad = \sum_{k=0}^{L} (P_k - P_{k-1})\,\boldsymbol{\xi}^\perp_L \quad = \quad \sum_{k=0}^{L} \boldsymbol{\zeta}^\perp_k - \boldsymbol{\zeta}^\perp_{k-1}\ .$$

Setting $\boldsymbol{\mu}^\perp_k := \boldsymbol{\zeta}^\perp_k - \boldsymbol{\zeta}^\perp_{k-1}$, it has the evident property

$$\sum_{k=0}^{L} \|\mathbf{curl}\,\boldsymbol{\mu}^\perp_k\|^2_{\boldsymbol{L}^2(\mathbf{C})} = \|\mathbf{curl}\,\boldsymbol{\xi}^\perp_L\|^2_{\boldsymbol{L}^2(\mathbf{C})}\ . \tag{18}$$

Thanks to the one–to–one correspondence of the spaces $\boldsymbol{\mathcal{ND}}^+_{d,0}(\mathbf{C};\mathcal{T}_k)$ and $\boldsymbol{\mathcal{ND}}^\perp_{d,0}(\mathbf{C};\mathcal{T}_k)$ it is possible to define a unique finite element function $\boldsymbol{\mu}_k$ in $\boldsymbol{\mathcal{ND}}^+_{d,0}(\mathbf{C};\mathcal{T}_k)$ formally by $\boldsymbol{\mu}_k := \theta^{-1}(\boldsymbol{\mu}^\perp_k)$, $k \in \{0,\dots,L\}$. If $\boldsymbol{\psi}_{k,\kappa}$ stands for the nodal basis function belonging to the degree of freedom $\kappa \in \Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)$, the following sum does provide a decomposition of $\boldsymbol{\mu}_k$

$$\boldsymbol{\mu}_k = \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \kappa(\boldsymbol{\mu}_k)\,\boldsymbol{\psi}_{k,\kappa} \quad 1 \leq k \leq L\ ,$$

which now consists of strictly locally supported components. In sum, the desired specific multilevel splitting reads:

$$\text{(19)}\qquad \boldsymbol{\xi}'_L := \boldsymbol{\mu}_0 + \sum_{k=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \underbrace{\kappa\ \ \theta^{-1}\left(P_k - P_{k-1}\right)\boldsymbol{\xi}_L^{\perp}\quad \boldsymbol{\psi}_{k,\kappa}}_{=:\ \boldsymbol{\mu}_{k,\kappa}} .$$

Recalling the properties of the mapping θ, we immediately confirm that $\mathbf{curl}\,\boldsymbol{\xi}'_L = \mathbf{curl}\,\boldsymbol{\xi}_L$. The proof that the stability estimate (12) holds true for (19) is divided among several lemmata. The first is the analogue of formula (8) in [54] and formula (3.7) in [53].

Lemma 3. *Using the notations introduced above for* $0 \le k \le L$ *we have*

$$\left\|\boldsymbol{\mu}_k^{\perp}\right\|_{L^2(\mathbf{C})} \le\ C\ h_k \left\|\mathbf{curl}\,\boldsymbol{\mu}_k^{\perp}\right\|_{L^2(\mathbf{C})} ,$$

with a constant $C > 0$ *not depending on* k *and* L.

Proof. The proof is carried out in two steps: *1st step:* To begin with, we probe the approximation properties of the $a(\cdot,\cdot)$–orthogonal projector P_k. To this end, we rely on classical duality techniques ("Nitsche's trick", see [15], Sect. 5.4) in the Hilbert space $\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$. For an arbitrary $\boldsymbol{v}^{\perp} \in \boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$ we set $\boldsymbol{v}_k^{\perp} := P_k \boldsymbol{v}^{\perp}$, i.e.

$$a(\boldsymbol{v}_k^{\perp},\boldsymbol{\phi}) = a(\boldsymbol{v}^{\perp},\boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{ND}}_{d,0}^{\perp}(\mathbf{C};\mathcal{T}_k) .$$

In addition, given $\boldsymbol{\omega} \in \boldsymbol{L}^2(\mathbf{C})$ with $\operatorname{div}\boldsymbol{\omega} = 0$, we write $\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp} \in \boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$ for the unique solution of:

Seek $\boldsymbol{\eta}^{\perp} \in \boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})$ such that

$$\text{(20)}\qquad a(\boldsymbol{\eta}^{\perp},\boldsymbol{\phi}) = (\boldsymbol{\omega},\boldsymbol{\phi})_{\boldsymbol{L}^2(\mathbf{C})} \quad \forall \boldsymbol{\phi} \in \boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C}) .$$

Now we apply the customary duality technique:

$$\begin{aligned}
\|\boldsymbol{v}^{\perp} - \boldsymbol{v}_k^{\perp}\|_{\boldsymbol{L}^2(\mathbf{C})} &= \sup_{\boldsymbol{\omega}\in\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})} \|\boldsymbol{\omega}\|_{\boldsymbol{L}^2(\mathbf{C})}^{-1} \cdot \left(\boldsymbol{v}^{\perp} - \boldsymbol{v}_k^{\perp},\boldsymbol{\omega}\right)_{\boldsymbol{L}^2(\mathbf{C})} \\
&= \sup_{\boldsymbol{\omega}\in\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})} \|\boldsymbol{\omega}\|_{\boldsymbol{L}^2(\mathbf{C})}^{-1} \cdot a(\boldsymbol{v}^{\perp} - \boldsymbol{v}_k^{\perp},\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}) \\
&\le \quad \left\|\mathbf{curl}\left(\boldsymbol{v}^{\perp} - \boldsymbol{v}_k^{\perp}\right)\right\|_{\boldsymbol{L}^2} \cdot \sup_{\boldsymbol{\omega}\in\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})} \\
&\quad \times \frac{\inf\limits_{\boldsymbol{\eta}_k\in\boldsymbol{\mathcal{ND}}_{d,0}^{\perp}(\mathbf{C};\mathcal{T}_k)} \left\|\mathbf{curl}\ \ \boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp} - \boldsymbol{\eta}_k\ \ \right\|_{\boldsymbol{L}^2(\mathbf{C})}}{\|\boldsymbol{\omega}\|_{\boldsymbol{L}^2(\mathbf{C})}}
\end{aligned}$$

Note that the strong form of (20) is given by

$$\begin{aligned}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp} &= \boldsymbol{\omega} && \text{in } \mathbf{C}\\ \operatorname{div}\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp} &= 0 && \text{in } \mathbf{C}\\ \boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}\times\boldsymbol{n} &= 0 && \text{on } \partial\mathbf{C}\ .\end{aligned}$$

According to formula (4.6) from [28], $\mathbf{curl}\,\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}$ belongs to $H^1(\mathbf{C})$, as both $\operatorname{div}\boldsymbol{\omega} = 0$ and $\mathbf{C}$ is convex. Furthermore,

$$|\,\mathbf{curl}\,\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}|_{H^1(\mathbf{C})} \leq C\,\|\boldsymbol{\omega}\|_{\boldsymbol{L}^2(\mathbf{C})}\ ,$$

where the constant depends only on $\mathbf{C}$. Thus estimate (6) shows

$$\inf_{\boldsymbol{\eta}_k\in\boldsymbol{\mathcal{ND}}_{d,0}^{\perp}(\mathbf{C};\mathcal{T}_k)}\left\|\mathbf{curl}\quad\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}-\boldsymbol{\eta}_k\quad\right\|_{\boldsymbol{L}^2(\mathbf{C})} \leq\ C\,h_k\quad\mathbf{curl}\,\boldsymbol{\vartheta}_{\boldsymbol{\omega}}^{\perp}\quad_{H^1(\mathbf{C})}\ .$$

This permits us to continue the estimates:

$$\begin{aligned}\|\boldsymbol{v}^{\perp}-\boldsymbol{v}_k^{\perp}\|_{\boldsymbol{L}^2(\mathbf{C})} &\leq C\,h_k\left\|\mathbf{curl}\left(\boldsymbol{v}^{\perp}-\boldsymbol{v}_k^{\perp}\right)\right\|_{\boldsymbol{L}^2(\mathbf{C})}\\ &\leq C\,h_k\left\|\mathbf{curl}\,\boldsymbol{v}^{\perp}\right\|_{\boldsymbol{L}^2(\mathbf{C})}.\end{aligned}$$

What we have shown by now is

$$\begin{aligned}&\left\|(Id-P_k)\,\boldsymbol{v}^{\perp}\right\|_{\boldsymbol{L}^2(\mathbf{C})} \leq C\,h_k\left\|\mathbf{curl}\,\boldsymbol{v}^{\perp}\right\|_{\boldsymbol{L}^2(\mathbf{C})}\\ &\qquad\forall\boldsymbol{v}^{\perp}\in\boldsymbol{H}_0^{\perp}(\mathbf{curl};\mathbf{C})\ .\end{aligned}\tag{21}$$

2nd step: The previous result is applied to $\boldsymbol{\mu}_k^{\perp}$. Owing to the basic properties of projections, we conclude from (21):

$$\begin{aligned}\|\boldsymbol{\mu}_k^{\perp}\|_{\boldsymbol{L}^2(\mathbf{C})} &= \|(P_k-P_{k-1})\ \boldsymbol{\xi}_L^{\perp}\|_{\boldsymbol{L}^2(\mathbf{C})}\\ &= \|(Id-P_{k-1})\,(P_k-P_{k-1})\ \boldsymbol{\xi}_L^{\perp}\|_{\boldsymbol{L}^2(\mathbf{C})}\\ &\leq C\,h_{k-1}\left\|\mathbf{curl}\quad(P_k-P_{k-1})\,\boldsymbol{\xi}_L^{\perp}\quad\right\|_{\boldsymbol{L}^2(\mathbf{C})}\\ &\leq C\,h_k\left\|\mathbf{curl}\,\boldsymbol{\mu}_k^{\perp}\right\|_{\boldsymbol{L}^2(\mathbf{C})}.\qquad\square\end{aligned}$$

Lemma 4. *For any $d\geq 2$, arbitrary $\boldsymbol{\mu}_k\in\boldsymbol{\mathcal{ND}}_d(\mathbf{C},\mathcal{T}_k)$ and $\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)$ the following estimate holds with constants independent of κ, $\boldsymbol{\mu}_k$, and $k\in\{1,\dots,L\}$:*

$$\left\|\mathbf{curl}\left(\kappa(\boldsymbol{\mu}_k)\,\boldsymbol{\psi}_{k,\kappa}\right)\right\|_{\boldsymbol{L}^2(\Omega_\kappa)}^2 \leq C\left(\|\mathbf{curl}\,\boldsymbol{\mu}_k\|_{\boldsymbol{L}^2(\Omega_\kappa)}^2 + h_k^{-2}\|\boldsymbol{\mu}_k\|_{\boldsymbol{L}^2(\Omega_\kappa)}^2\right),$$

where Ω_κ stands for the support of the canonical basis function associated with the degree of freedom κ.

Proof. The proof relies on standard scaling arguments for affine families of finite elements. □

Theorem 3. *There exists a constant $C > 0$, independent of L and $\boldsymbol{\xi}_L$, otherwise only depending on $d \geq 2$, such that*

$$\|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2(\mathbf{C})} + \sum_{k=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \left\|\mathbf{curl}\,\boldsymbol{\mu}_{k,\kappa}\right\|^2_{\boldsymbol{L}^2(\mathbf{C})} \leq C \left\|\mathbf{curl}\,\boldsymbol{\xi}'_L\right\|^2_{\boldsymbol{L}^2(\mathbf{C})} .$$

Proof. To begin with we apply Lemma 4. Afterwards, we take advantage of uniform shape regularity, because it ensures that any element belongs to only a small number of at most N of supports Ω_κ of canonical basis functions. Thus, initially we obtain

$$\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2(\mathbf{C})} &+ \sum_{k=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \left\|\mathbf{curl}\,\boldsymbol{\mu}_{k,\kappa}\right\|^2_{\boldsymbol{L}^2(\mathbf{C})} \\
&\leq \|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2(\mathbf{C})} + C \sum_{k=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \\
&\times(\|\mathbf{curl}\,\boldsymbol{\mu}_k\|^2_{\boldsymbol{L}^2(\Omega_\kappa)} + h_k^{-2}\|\boldsymbol{\mu}_k\|^2_{\boldsymbol{L}^2(\Omega_\kappa)}) \\
&\leq \|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2(\mathbf{C})} + NC \sum_{k=0}^{L} \|\mathbf{curl}\,\boldsymbol{\mu}_k\|^2_{\boldsymbol{L}^2(\mathbf{C})} \\
&+ h_k^{-2}\|\boldsymbol{\mu}_k\|^2_{\boldsymbol{L}^2(\mathbf{C})} .
\end{aligned}$$

Now Lemma 2 bridges the gap to the semicontinuous spaces:

$$\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2(\mathbf{C})} &+ \sum_{k=1}^{L} \sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_d,\mathcal{T}_k)} \left\|\mathbf{curl}\,\boldsymbol{\mu}_{k,\kappa}\right\|^2_{\boldsymbol{L}^2(\mathbf{C})} \\
&\leq \|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2} + NC \sum_{k=1}^{L} \left\|\mathbf{curl}\,\boldsymbol{\mu}_k^\perp\right\|^2_{\boldsymbol{L}^2} \\
&+ C\, h_k^{-2} \left\|\boldsymbol{\mu}_k^\perp\right\|^2_{\boldsymbol{L}^2} + h_k^2 \left\|\mathbf{curl}\,\boldsymbol{\mu}_k^\perp\right\|^2_{\boldsymbol{L}^2} \Big) \\
&\leq \|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2} + N \sum_{k=1}^{L} C, h_k^{-2} \left\|\boldsymbol{\mu}_k^\perp\right\|^2_{\boldsymbol{L}^2} \\
&+ (1+C) \left\|\mathbf{curl}\,\boldsymbol{\mu}_k^\perp\right\|^2_{\boldsymbol{L}^2} \Big)
\end{aligned}$$

$$\leq C \quad \|\mathbf{curl}\,\boldsymbol{\mu}_0\|^2_{\boldsymbol{L}^2} + \sum_{k=1}^{L} \left\|\mathbf{curl}\,\boldsymbol{\mu}_k^{\perp}\right\|^2_{\boldsymbol{L}^2} \Bigg)$$
$$= C\left\|\mathbf{curl}\,\boldsymbol{\xi}_L^{\perp}\right\|^2_{\boldsymbol{L}^2} .$$

The final steps relied on Lemma 3 and (18). □

So far, stability of the decomposition has been shown for higher order Nédélec spaces, since Lemma 3 played a crucial role in the proof. By a simple trick the result can be extended to lowest order finite element spaces:

Corollary 1. *The decomposition (10) of* $\boldsymbol{\mathcal{ND}}_{1,0}(\mathbf{C};\mathcal{T}_L)$ *is stable in the sense of estimate (12).*

Proof. Pick an arbitrary $\boldsymbol{\xi}_L \in \boldsymbol{\mathcal{ND}}_{1,0}(\mathbf{C};\mathcal{T}_L)$. Since $\boldsymbol{\mathcal{ND}}_{1,0}(\mathbf{C};\mathcal{T}_L) \subset \boldsymbol{\mathcal{ND}}_{2,0}(\mathbf{C};\mathcal{T}_L)$, it can be regarded as a vector field from the second order Nédélec space. Then, the previous theorem confirms the stability of the splitting (19), which, of course, contains multiples of $\boldsymbol{\mathcal{ND}}_2$ basis functions. But $\boldsymbol{\xi}'_L$ from (19) is not a generic $\boldsymbol{\mathcal{ND}}_{2,0}(\mathbf{C};\mathcal{T}_L)$ vector field, because its $\mathbf{curl}$ coincides with that of the $\boldsymbol{\mathcal{ND}}_1$ finite element function $\boldsymbol{\xi}_L$. We now contend that switching back to the lowest order Nédélec spaces can be done by simply dumping the higher order components in the nodal splitting (19). Remember that the degrees of freedom in $\Xi(\boldsymbol{\mathcal{ND}}_2,\mathcal{T}_k)$ are by default arranged in a hierarchical fashion. Writing Ξ^{HB} for the set $\Xi(\boldsymbol{\mathcal{ND}}_2,\mathcal{T}_k)/\Xi(\boldsymbol{\mathcal{ND}}_1,\mathcal{T}_k)$, we can recast (19) as

$$\underbrace{\boldsymbol{\mu}_0^{(1)}}_{\in\boldsymbol{\mathcal{ND}}_{1,0}(\mathbf{C};\mathcal{T}_L)} + \underbrace{\boldsymbol{\mu}_0^{(2)}}_{\in\boldsymbol{\mathcal{ND}}^{\mathrm{HB}}_{2,0}(\mathbf{C};\mathcal{T}_L)} + \sum_{k=1}^{L} \underbrace{\sum_{\kappa\in\Xi(\boldsymbol{\mathcal{ND}}_1,\mathcal{T}_k)} \boldsymbol{\mu}_{k,\kappa}}_{\in\boldsymbol{\mathcal{ND}}_{1,0}(\mathbf{C};\mathcal{T}_L)}$$
$$+ \underbrace{\sum_{\kappa\in\Xi^{HB}} \boldsymbol{\mu}_{k,\kappa}}_{\in\boldsymbol{\mathcal{ND}}^{\mathrm{HB}}_{2,0}(\mathbf{C};\mathcal{T}_L)} = \boldsymbol{\xi}_L + \mathbf{grad}\; \underbrace{\phi_1}_{\in\mathcal{S}_1(\mathbf{C};\mathcal{T}_L)} + \underbrace{\phi_2}_{\in\mathcal{S}_2^{\mathrm{HB}}(\mathbf{C};\mathcal{T}_L)} ,$$

where the superscript HB tags p–hierarchical components. Then, we can apply (4), which shows that $\mathbf{grad}\,\phi_2 \in \boldsymbol{\mathcal{ND}}^{\mathrm{HB}}_{2,0}(\mathbf{C};\mathcal{T}_L)$. Thanks to the uniqueness of the p–hierarchical decomposition we can relate

$$\boldsymbol{\mu}_0^{(2)} + \sum_{k=1}^{L} \sum_{\kappa\in\Xi^{HB}} \boldsymbol{\mu}_{k,\kappa} = \mathbf{grad}\,\phi_2 .$$

This teaches us that the overall sum of higher order components is $\mathbf{curl}$–free. Dropping them maintains the crucial equality of $\mathbf{curl}$s, hence. In addition, the left hand side of the stability estimate can only decrease in the process. In sum, (12) remains true for the resulting genuine $\boldsymbol{\mathcal{ND}}_1$ splitting. □

6. Stability on general domains

The generalization of the stability result of the previous section to arbitrary bounded domains Ω can be done by extending solenoidal vector fields to a large cube engulfing Ω. For the H^1–elliptic case this trick is used in [54,9]. It hinges on the following *extension theorem*:

Theorem 4. *Let Ω and $\tilde{\Omega}$ be two bounded polyhedral domains in $\mathbb{R}^3$ such that $\bar{\Omega} \subset \tilde{\Omega}$. Given a quasiuniform shape–regular simplicial triangulation $\mathcal{T}_h$ of Ω, we assume that it can be extended to a triangulation $\widetilde{\mathcal{T}}_h$ of $\tilde{\Omega}$ with the same meshwidth and without loss of shape–regularity. Then, there exists a linear extension operator*

$$E_{\mathcal{RT}} : \mathcal{RT}_d^0(\Omega; \mathcal{T}_h) \mapsto \mathcal{RT}_{d,0}^0(\widetilde{\Omega}; \widetilde{\mathcal{T}}_h)$$

such that

$$\|E_{\mathcal{RT}} \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\tilde{\Omega})} \leq C \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \mathcal{RT}_d^0(\Omega; \mathcal{T}_h) \ ,$$

with $C > 0$ independent of the meshwidth h, otherwise only depending on the shape–regularity of the mesh $\widetilde{\mathcal{T}}_h$ and the polynomial order $d \geq 0$ of the finite element spaces.

Proof. The proof is inspired by ideas from Lemma 3.2 in [12]. We start with an arbitrary $\boldsymbol{v}_h \in \mathcal{RT}_d^0(\Omega; \mathcal{T}_h)$ and denote by $\mu \in H^{-1/2}(\Gamma)$ its normal trace $\langle \boldsymbol{v}_h, \boldsymbol{n} \rangle_{|\Gamma}$. Since μ is piecewise polynomial, as in the proof of Lemma 2 we conclude that $\mu \in H^\sigma(\Gamma)$ for a $0 < \sigma < 1/2$. Consider the boundary value problem

$$\begin{aligned} -\Delta \Phi &= 0 && \text{in } \widetilde{\Omega}/\overline{\Omega} \\ \frac{\partial \Phi}{\partial \boldsymbol{n}} &= -\mu && \text{on } \partial\Omega \\ \frac{\partial \Phi}{\partial \boldsymbol{n}} &= 0 && \text{on } \partial\widetilde{\Omega} \ . \end{aligned}$$

Now, keep in mind that functions in $H^\sigma(\Gamma)$ are fully decoupled ([40], Theorem 11.4), i.e., no compatibility conditions over edges of $\mathcal{T}_{h|\Gamma}$ need to be enforced. The trace mapping

$$H^{3/2+\sigma}(\widetilde{\Omega}/\overline{\Omega}) \mapsto H^\sigma(\Gamma) \quad ; \quad u \mapsto \frac{\partial u}{\partial \boldsymbol{n}}$$

is surjective, hence. Then, a regularity result (Corollary 2.6.7 in [32]) states that $\Phi \in H^{3/2+\varepsilon}(\widetilde{\Omega}/\overline{\Omega})$ with $0 < \varepsilon \leq \sigma$ and $\|\Phi\|_{H^{3/2+\varepsilon}(\widetilde{\Omega}/\overline{\Omega})} \leq C\|\mu\|_{H^\varepsilon(\Gamma)}$. We fix a suitable ε.

Now, we set $\widetilde{\boldsymbol{v}} := \mathbf{grad}\, \Phi$ and observe $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^0(\mathrm{div}; \widetilde{\Omega}/\overline{\Omega})$. This is also a vector field in $\boldsymbol{H}^{1/2+\varepsilon}(\widetilde{\Omega}/\overline{\Omega})$ and, therefore, the interpolating finite element function $\widetilde{\boldsymbol{v}}_h := \boldsymbol{\Pi}_{\widetilde{\mathcal{T}}_l}^{\boldsymbol{\mathcal{RT}}_d}\widetilde{\boldsymbol{v}}$ restricted to $\widetilde{\Omega}/\overline{\Omega}$ is well defined. Thanks to the commuting diagram property it belongs to $\boldsymbol{\mathcal{RT}}_d^0(\widetilde{\Omega}/\overline{\Omega};\widetilde{\mathcal{T}}_l)$. Moreover, it satisfies homogeneous boundary conditions on $\partial\widetilde{\Omega}$.

An approximation estimate in fractional Sobolev spaces (see formula (1.5) in [41]) gives

$$\begin{aligned}\left\|\widetilde{\boldsymbol{v}} - \boldsymbol{\Pi}_{\widetilde{\mathcal{T}}_l}^{\boldsymbol{\mathcal{RT}}_d}\widetilde{\boldsymbol{v}}\right\|_{L^2(\widetilde{\Omega}/\overline{\Omega})} &\leq Ch_l^{1/2+\varepsilon}\|\widetilde{\boldsymbol{v}}\|_{H^{1/2+\varepsilon}(\widetilde{\Omega}/\overline{\Omega})}\\ &\leq Ch_l^{1/2+\varepsilon}\|\Phi\|_{H^{3/2+\varepsilon}(\widetilde{\Omega}/\overline{\Omega})}\\ &\leq Ch_l^{1/2+\varepsilon}\|\mu\|_{H^{\varepsilon}(\Gamma)} \leq C\,\|\mu\|_{H^{-1/2}(\Gamma)}\\ &\leq C\,\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)}\ .\end{aligned}$$

Again, the inverse estimate (14) has been applied to the piecewise polynomial function μ. Moreover, we used a trace theorem for $\boldsymbol{H}(\mathrm{div};\Omega)$.

Finally, we set

$$(E_{\boldsymbol{\mathcal{RT}}}\boldsymbol{v}_h)(\boldsymbol{x}) := \begin{array}{ll} \boldsymbol{v}_h(\boldsymbol{x}) & \text{, for } \boldsymbol{x}\in\Omega \\ \widetilde{\boldsymbol{v}}_h(\boldsymbol{x}) & \text{, for } \boldsymbol{x}\in\widetilde{\Omega}/\overline{\Omega}\ . \end{array}$$

By definition of $\widetilde{\boldsymbol{v}}$ and $\widetilde{\boldsymbol{v}}_h$, the normal components of this vector field are continuous across the boundary of Ω. Hence, the patched–together function actually belongs to $\boldsymbol{H}(\mathrm{div};\widetilde{\Omega})$. The uniform continuity of the extension operators with respect to h is then an easy consequence of the above estimates.

The following core result is now an almost trivial consequence of earlier theorems:

Theorem 5. *Under the assumptions on the domain Ω and the hierarchy of meshes $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_l$ stated before, for any polynomial oder $d\in\mathbb{N}$ the nodal multilevel decomposition of the Nédélec finite element space $\boldsymbol{\mathcal{ND}}_d(\Omega;\mathcal{T}_L)$,$d\geq 1$, is stable in the sense of (12) with a constant $C_L>0$ independent of the depth L of the refinement.*

Proof. First, Ω is embedded into a cube $\mathbf{C}$. Then, the hierarchy of meshes is extended to a hierarchy of triangulations $\widetilde{\mathcal{T}}_0, \widetilde{\mathcal{T}}_1, \dots, \widetilde{\mathcal{T}}_L$ on this cube. This can be done in many ways, preserving shape regularity in the process.

Given $\boldsymbol{\xi}_L \in \boldsymbol{\mathcal{ND}}_d^+(\Omega;\mathcal{T}_L)$, its $\mathbf{curl}$ is a vector field $\boldsymbol{v}_L \in \boldsymbol{\mathcal{RT}}_{d-1}^0(\Omega;\mathcal{T}_L)$. Employing the extension operator of Theorem 4, we arrive at $\widetilde{\boldsymbol{v}}_L \in \boldsymbol{\mathcal{RT}}_{d-1,0}^0(\mathbf{C};\widetilde{\mathcal{T}})_L$ such that $\widetilde{\boldsymbol{v}}_{L|\Omega} = \boldsymbol{v}_L$ and $\|\widetilde{\boldsymbol{v}}_L\|_{\boldsymbol{L}^2(\mathbf{C})} \leq C\|\boldsymbol{v}_L\|_{\boldsymbol{L}^2(\Omega)}$. The representation theorem (Theorem 1) gives us $\widetilde{\boldsymbol{\xi}}_L \in \boldsymbol{\mathcal{ND}}_{d,0}^+(\mathbf{C};\widetilde{\mathcal{T}}_L)$ with $\mathbf{curl}\,\widetilde{\boldsymbol{\xi}}_L = \widetilde{\boldsymbol{v}_L}$.

According to Theorem 3 and Corollary 1 we can find a nodal multilevel decomposition of $\widetilde{\boldsymbol{\xi}}_L$ such that the estimate (12) is satisfied. Restricting all components of the splitting to Ω provides the desired stable nodal multilevel decomposition of $\boldsymbol{\xi}_L$. $\square$

7. Analysis of the algorithm

Fast convergence of an iterative scheme is only necessary, but not sufficient for the overall efficiency of the method. It also hinges on the computational costs of preprocessing steps and the application of the preconditioner. These must not exceed a fixed small number of operations per unknown on the finest grid.

For the sake of simplicity, we perform a detailed investigation of the computational effort involved in the algorithm for the case of lowest order elements ($d = 0$) on a hexahedral grid only. The arguments remain essentially valid for higher order elements and on simplicial meshes.

Recall that finite element functions in $\boldsymbol{\mathcal{RT}}_0(\Omega; \mathcal{T}_h)$ are characterized by degrees of freedom located on faces, whereas the nodal values for $\boldsymbol{\mathcal{ND}}_1(\Omega; \mathcal{T})_h$ are associated with edges. Further, we assume the canonical definition of degrees of freedom (see [33]).

We first take a look at the preprocessing step, needed to determine $\boldsymbol{j}_h^*$ (cf. Sect. 3). We single out $T \in \mathcal{T}_j$, $j = 0, \dots, L-1$, and temporarily assume that f_{j+1} from (7) is already available. Remember that f_{j+1} is piecewise constant on $\mathcal{T}_{j+1}$; we can write $f_{j+1}(T_{k,T})$, $k = 1, \dots, 8$, for the value of f_{j+1} on $T_{k,T}$. Let $T_{1,T}, \dots, T_{8,T}$ denote the elements on level $j+1$ created by regular refinement of T. There are 12 faces $F_{1,T}, \dots, F_{12,T}$ of $\mathcal{F}_{j+1}$ contained in T. Their associated $\boldsymbol{\mathcal{RT}}_0$–basis functions can all contribute to $\boldsymbol{j}_{j,T}^*$ and the corresponding nodal values $\kappa_1, \dots, \kappa_{12}$ can to be computed as follows:

$$\begin{aligned}
f_j(T) &\leftarrow \textstyle\sum_{k=1}^{8} f_{j+1}(T_{k,T}) \\
\kappa_1, \kappa_2, \kappa_3, \kappa_4 &\leftarrow \tfrac{1}{4} \textstyle\sum_{k=1}^{4} f_{j+1}(T_{k,T}) - \tfrac{1}{8} f_j(T) \\
\kappa_5, \kappa_6 &\leftarrow \tfrac{1}{2} \left(f_{j+1}(T_{1,T}) + f_{j+1}(T_{2,T}) - \kappa_1 - \kappa_2 \right) - \tfrac{1}{8} f_j(T) \\
\kappa_2, \kappa_8 &\leftarrow \tfrac{1}{2} \left(f_{j+1}(T_{5,T}) + f_{j+1}(T_{6,T}) + \kappa_1 + \kappa_2 \right) - \tfrac{1}{8} f_j(T) \\
\kappa_9 &\leftarrow f_{j+1}(T_{1,T}) - \kappa_1 - \kappa_5 - \tfrac{1}{8} f_j(T) \\
\kappa_{10} &\leftarrow f_{j+1}(T_{3,T}) - \kappa_3 + \kappa_5 - \tfrac{1}{8} f_j(T) \\
\kappa_{11} &\leftarrow f_{j+1}(T_{5,T}) + \kappa_1 - \kappa_7 - \tfrac{1}{8} f_j(T) \\
\kappa_{12} &\leftarrow f_{j+1}(T_{7,T}) + \kappa_1 + \kappa_7 - \tfrac{1}{8} f_j(T)
\end{aligned}$$

Any other numbering of degrees of freedom can be taken into account by just changing indices. Consequently we have to reckon with about 40 elementary operations per element. Exploiting the geometric decrease of the number of elements on the coarser grids, we arrive at a total operation count of about $\frac{1}{7} \cdot 40 \cdot \sharp\mathcal{T}_L$ for the computation of the $\boldsymbol{j}_{j,T}^*$, $j = 0, \dots, L-1$ ($\sharp$ designates the numbers of elements in a set). The costs for $\boldsymbol{j}_0^*$ are negligible, if $\mathcal{T}_0$ is a reasonable coarse grid.

To obtain $\boldsymbol{j}_h^*$ we rely on $\boldsymbol{\mathcal{RT}}_0$–prolongation based upon the embedding $\boldsymbol{\mathcal{RT}}_0(\Omega;\mathcal{T}_j) \subset \boldsymbol{\mathcal{RT}}_0(\Omega;\mathcal{T}_{j+1})$. It means distributing the weighted nodal values from coarse grid faces to neighboring fine grid faces of the same orientation, which takes at most 24 operations per coarse grid face. In sum, we get $\boldsymbol{j}_h^*$ with $\frac{1}{7} \cdot 24 \cdot \sharp\mathcal{F}_L$ operations. Note that $\sharp\mathcal{F}_L \approx 3\sharp\mathcal{T}_L$ to see that, neglecting the corse grid solve, we have to pay about $10 \cdot \sharp\mathcal{F}_L$ operations for $\boldsymbol{j}_h^*$.

In the second step of the algorithm we first have to set up the algebraic system arising from (9). In the case of locally constant coefficient function a the assembly of the stiffness matrix takes about $144 \cdot \sharp\mathcal{T}_L$ operations. The vector on the right hand side of (9) has an entry for each edge. This is available through gathering the nodal values of $\boldsymbol{j}_h^*$ (and of the boundary data) from all faces that share an element with the edge. This can be done with as little as 16 operations per edge.

The preconditioner is used in the framework of a PCG method. Since the stiffness matrix from (9) is sparse — thanks to the existence of a neatly localized basis in $\boldsymbol{\mathcal{ND}}_{d+1}(\Omega;\mathcal{T}_L)$ it has no more than 40 entries per row — we face about $50 \cdot \sharp\mathcal{E}_L$ operations plus the costs for the preconditioner in each iteration (see [6]).

The implementation of the multilevel preconditioner is hardly different from that of the classical BPX–preconditioner [13] for standard finite elements; only local intergrid transfer operators, prolongation and restriction, and scalings of nodal values have to be carried out. The intergrid transfer operator arise from the embedding $\boldsymbol{\mathcal{ND}}_1(\Omega;\mathcal{T}_j) \subset \boldsymbol{\mathcal{ND}}_1(\Omega;\mathcal{T}_{j+1})$. Restriction means that every edge on level j collects weighted nodal values from all those edges on level $j+1$ that lie inside adjacent elements of $\mathcal{T}_j$. This amounts to 36 elementary operations per edge in $\mathcal{E}_j$. The same applies to prolongation which boils down to distributing values to edges on the finer level. Due to the geometric decline of the number of edges of coarser grids, all transfers within the preconditioner require about $2 \cdot \frac{1}{7} \cdot 36 \cdot \sharp\mathcal{E}_L$ operations.

Another $\frac{8}{7} \cdot \sharp\mathcal{E}_L$ operations have to spent on scaling. Please note that the scaling factors can be computed once and for all beforehand, which costs about $3 \cdot \sharp\mathcal{E}_L$ operations in the case of piecewise constant coefficient function a. Neglecting the coarse grid solve, we end up with about $3 \cdot \sharp\mathcal{E}_L$ essential operations for the evaluation of the preconditioner, which is hardly significant in the whole iteration.

Afer the iterations have terminated, the correction $\boldsymbol{j}_h^*$, which is represented in $\boldsymbol{\mathcal{ND}}_1(\Omega;\mathcal{T}_L)$ has to be expressed in terms of $\boldsymbol{\mathcal{RT}}_0$ nodal values. Relying on the embedding $\mathbf{curl}\boldsymbol{\mathcal{ND}}_1(\Omega;\mathcal{T}_L) \subset \boldsymbol{\mathcal{RT}}_0(\Omega;\mathcal{T}_L)$ this can be achieved by distributing the nodal values from the edges to all adjacent faces. This requires $4 \cdot \sharp\mathcal{E}_L$ operations.

Compared to the multilevel method for the mixed hybrid system presented in [22] the current preconditioner is slightly cheaper, chiefly due to the simpler transfer operators. Conversely, the stiffness matrix contains more nonzero entries in our case, which about offsets the gain.

Remark 3. We point out to the local nature of all computations; they do not rely on "geometrically distant" information when updating a nodal value. This permits us to stick to local data access patterns, which is highly desirable in a finite element code.

8. Numerical experiments

Even if the algorithm is economical, asymptotic optimality of the preconditioner falls short of ensuring the practical efficiency of the iterative scheme. In addition, the ominous constants occurring in the estimates have to be reasonably small. Unfortunately the theory totally fails to provide information about the size of those constants. By and large, even for simple model problems they remain elusive; only numerical experiments can provide clues: To this end we conducted a few numerical experiments in which we examined the convergence of the multilevel preconditioned CG–method when applied to the variational problem

$$(\alpha\,\mathbf{curl}\,\boldsymbol{\xi}_h, \mathbf{curl}\,\boldsymbol{\eta}_h)_0 = f(\mathbf{curl}\,\boldsymbol{\eta}_h) \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{N}\mathcal{D}}_1(\Omega; \mathcal{T}_L)\ ,$$

where $f \in \boldsymbol{H}(\mathrm{div}; \Omega)'$ and $\alpha = \alpha(\boldsymbol{x})$. All the computations relied on hexaedral meshes and regular refinement was employed to created the nested meshes $\mathcal{T}_1, \ldots, \mathcal{T}_L$. The right hand side f and the initial guess were chosen at random and we monitored the number of iterations required to achieve a reduction of the Euclidean norm of the residual by a factor of 10^{-6}. Three runs and subsequent averaging were done in each case to offset the impact of randomness.

The *first experiment* was carried out on the unit cube $\Omega :=]0; 1[^3$ and for constant coefficient function $\alpha \equiv 1$. The coarsest grid $\mathcal{T}_0$ comprised eight equal cubes. In Table 1 the average number of PCG iterations for different depths L of refinement are recorded. To highlight the superiority of the multilevel preconditioned (PCG–BPX) iteration, the performance of a plain symmetric Gauß–Seidel preconditioner (PCG–GS) was also examined. The figures clearly confirm the asymptotic optimality of the multilevel preconditioner.

Compared to the iteration counts reported in [22] (Table 3 in Sect. 6) for a very similar numerical experiment *in 2D*, the current method turns out to be about 30% faster.

Table 1. Average number of CG steps for Exp. 1

L	2	3	4	5	6
PCG–BPX	15	17	18	18	18
PCG–GS	46	86	145	208.3	316.6

Table 2. Average number of CG steps for Exp. 2

L	2	3	4	5	6
PCG–BPX	15	17	18	18	18.3
PCG–GS	46	86.3	142.3	210.3	325.3

Table 3. Average number of CG steps for Exp. 3

α_0	10^5	10^4	10^3	10^2	10^1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$L = 2$	14	14	15	15	16	17	17	17	17	17
$L = 3$	16	16	17	17	17	18	19	19	19	19
$L = 4$	17	17	17	18	18	18	19	20	19.6	19.6
$L = 5$	18	18	18	18	18	18.3	19.3	20	20	19.6

The *second experiment* involved the same investigations as before on a 3D "L–shaped" domain $\Omega :=]0;1[^3/[0;\frac{1}{2}]^3$. The results are given in Table 2. We see that the conclusions from Exp. 1 carry over even to the case of a more complex domain.

The *third experiment* was intended to probe the impact of strongly varying coefficient functions. It relied on the setting of the first numerical experiment and used

$$\alpha(\boldsymbol{x}) := \begin{array}{rl} \alpha_0 & ;\ \text{for } \boldsymbol{x} \in]1/3, 2/3[^3 \\ 0 & ;\ \text{elsewhere in }]0,1[^3 \end{array} \tag{22}$$

with values $\alpha_0 \in \{10^5, 10^4, 10^3, 10^2, 10^1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$. The average number of CG iterations resulting for different numbers L of refinement levels are reported in Table 3. The figures give evidence that the multilevel preconditioned CG method can well cope with strongly varying coefficients.

9. Conclusion

A novel multilevel decomposition of $\boldsymbol{H}(\mathbf{curl}; \Omega)$–conforming Nédélec spaces has been presented. Its stability with respect to the $\|\,\mathbf{curl}\cdot\|_0$–seminorm could be established with constants independent on the depth of uniform refinement. We immediately obtained an L^2–stable multilevel decomposition of solenoidal Raviart–Thomas vector fields giving rise to an optimal additive Schwarz preconditioner. No schemes with such properties had been known previously. The preconditioner can be used for the fast iterative solution of saddle point problems arising from the mixed discretization of second order elliptic boundary value problems in three dimensions. Besides, the results obtained in this paper can serve as the starting point for the design of multilevel methods for mixed problems in $\boldsymbol{H}(\mathbf{curl};\Omega)$. Examples are problems from electromagnetism (see [43,42,48]) and nonstandard schemes for Stokes' equations (see [30]). This has been addressed in more detail in [35].

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