Geographic Wayfinders and Space-Time Algebra

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Abstract

Time Geography is a framework for describing reachable points in a (static) spatio-temporal environment. While originally devised to facilitate reasoning about an individual's or a population's living conditions, it was later adapted to many other applications. A wayfinder is an entity that moves through a space-time continuum with possible obstacles. We show how to model the pertinent notions in relational algebra (and, more abstractly, in modal semirings) with box and diamond operators. Admissible or undesired regions can be described as Boolean combinations of primitive regions such as the set of all points reachable by forward or backward movement from a given region or starting point. To derive results about the region blocked by the union of two regions we introduce an abstract algebraic view of coordinates that is largely independent of dimensional and metric aspects and thus very general. Moreover, the approach lends itself quite easily to machine-supported proofs.

Keywords: Time geography; moving objects; obstacle analysis; formal algebraic semantics; modal operators; modal semirings.

1. Introduction

Time Geography is a framework for describing reachable points in a (static) spatio-temporal environment. While originally devised to facilitate reasoning about an individual's or a population's living conditions [11], it was later adapted to many other applications (e.g., transportation planning [27], epidemiology and environmental risk assessment [15] or crime analysis [23]).

In this paper we concentrate on the description and analysis of *wayfinders*. These are entities that move through a space-time continuum with possible obstacles. Therefore it is an important task to determine regions of space-time which wayfinders may safely traverse without running the risk of getting stuck before an obstacle (obstacles are considered impenetrable).

In describing such phenomena we take up an approach by Hendricks et al. [13]. Central concepts there are modalities such as "may" and "must" when describing movement in space-time. Examples are

- "We *must* reach the plane before it leaves."
- "We *must not* pass through road X because of construction work."

While the treatment in the original paper is only semi-formal, we show how to model these notions in relational algebra with box and diamond operators and further in the general algebraic setting of modal semirings. One advantage of the more abstract treatment is that for a large part our results are independent of the number of spatial (or even temporal) dimensions and of metric considerations.

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The paper is structured as follows. In Sects. 2 and 3 we review the original approach of [13] and set up the connection with relation algebra and modal operators. In Sect. 4 we introduce the algebraic structure of modal semirings which abstracts and generalises the relational setting. Then Sect. 5 introduces algebraic formulations of transitivity, irreflexivity and reflexive closure. Sect. 6 deals with intervals and convexity. In Sect. 7 we present central notions and properties concerning regions that are blocked by obstacles and therefore have to be avoided by wayfinders. Sect. 8 defines minimal and maximal elements of a region which are used in Sect. 9 to characterise lower and upper bounds. These serve in Sect. 10 to prove a number of properties of obstacles with a bounded avoidance region. The subsequent sections deal with the problem of determining the avoidance region for a union of regions. It turns out that in case of a partial overlap this is larger than just the union of the avoidance regions. To characterise this situation, Sect. 11 introduces a notion of separatedness. Sect. 12 presents an abstract algebraic notion of coordinates, with which further boundedness assertions become possible. The above-mentioned results about the avoidance regions of unions are then presented in Sect. 13. In Sect. 14 we show how our general, dimension-independent, ideas work out in the concrete case of two spatial dimensions, with a surprising result on avoidance regions. Sect. 15 contains a brief discussion of related work. Finally, Sect. 16 lists a few further properties about transitivity and strict-orders.

2. Points and Reachability

We start with a set of very concrete definitions. First, we assume a set S of *spatial coordinates*, e.g., a subset of $\mathbb{R}^n$ for some natural number n. However, we only assume an associative and commutative addition operator $+$ on spatial coordinates, such as addition of position vectors; subtraction is not needed. Second, we assume a set T of *temporal coordinates*, linearly ordered by a partial order $\leq$, i.e., $\leq$ is reflexive, transitive and antisymmetric and satisfies $\forall\, t, t' \in T : t \leq t' \vee t' \leq t$. Moreover, we assume an element $0 \in T$ and an addition operator $+$ on T such that $(T, +, 0)$ is a cancellative commutative monoid, i.e., $+$ is associative and commutative with neutral element 0 and satisfies $t + t' = t + t'' \Rightarrow t' = t''$. Finally, we require that $t \leq t' \Leftrightarrow \exists\, t'' : t' = t + t''$. By cancellativity this t'' is unique and we denote it by $t' - t$ when $t \leq t'$. Moreover, 0 is the least element of T under $\leq$. A *point* is a pair $(s, t) \in S \times T$.

We want to describe the possible movements of a wayfinder. Without any restrictions one could not predict where the wayfinder might be at a given time. Therefore one assumes an upper bound on the velocity and analyses how far away a wayfinder can get from its starting position while staying below this speed bound all the time. The bound is supposed to be constant over time. Now, given a constant and non-negative maximum velocity vector v we define the reachability relation R_v between points by

$$(s, t)\, R_v\, (s', t') \;\Leftrightarrow_{df}\; t \leq t' \,\wedge\, s' \leq s + v \cdot (t' - t) \ ,$$

where $\leq$ on vectors is taken coordinate-wise. This means that (s', t') can be reached from starting point (s, t) by travelling for time $t' - t$ with maximal velocity v. Having v as a parameter allows modelling switches between varying velocities, e.g., shifting to a lower gear in a steeper region.

Lemma 2.1. *R_v is a partial order.*

The straightforward proof is omitted.

If besides addition there are also subtraction $-$ and a norm operator $\|\ \|$ on spatial coordinates available, a more liberal variant of the reachability relation is the following:

$$(s, t)\, S_v\, (s', t') \;\Leftrightarrow_{df}\; t \leq t' \,\wedge\, \|s' - s\| \leq v \cdot (t' - t) \ ,$$

with a scalar non-negative velocity v.

Lemma 2.2. *S_v is a partial order.*

Proof. *Reflexivity* is clear.
Transitivity: Assume $(s,t)\,S_v\,(s',t') \wedge (s',t')\,S_v\,(s'',t'')$, i.e., $t \le t' \wedge \|s'-s\| \le (t'-t)\cdot v \wedge t' \le t'' \wedge \|s''-s'\| \le v\cdot(t''-t')$. By transitivity of $\le$ on T we infer $t \le t''$. Moreover, by vector arithmetic, the triangle inequality for $\| \; \|$, distributivity and time arithmetic,

$$\|s''-s\| = \|s''-s'+s'-s\| \le \|s''-s'\| + \|s'-s\| \le v\cdot(t''-t') + v\cdot(t'-t) = v\cdot(t''-t) \ ,$$

and hence $(s,t)\,S_v\,(s'',t'')$.
Antisymmetry: Assume $(s,t)\,S_v\,(s',t') \ \wedge \ (s',t')\,S_v\,(s,t)$, i.e., $t \le t' \ \wedge \ s' \le s + v\cdot(t'-t) \ \wedge \ t' \le t \ \wedge \ s \le s' + v\cdot(t'-t)$. By antisymmetry of $\le$ on T we infer $t = t'$. Hence $t - t' = 0 = t' - t$, so that from the assumptions we obtain $\|s'-s\| \le 0\cdot v = 0$, thus $s' - s = 0$ by definiteness of the norm and hence $s = s'$. ◻

We now abbreviate points by x, x', y, y' etc.

We can compose reachability relations in the standard relational way:

$$x\,(R_v\,;\,R_w)\,y \ \Leftrightarrow_{df} \ \exists\, z : x\,R_v\,z \ \wedge \ z\,R_w\,y \ .$$

Hence $x\,(R_v\,;\,R_w)\,y$ holds iff we can move from x to y by travelling for a while with maximal velocity v to an intermediate point z and from there to y with maximal velocity w.

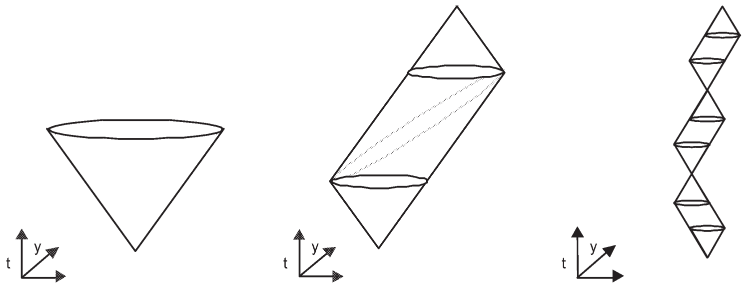
In the following sections we will abstract from this concrete relational model. However, we will frequently illustrate the definitions there using the notion of a *v-path*. This is the set of points reachable from some starting point (s,t) by moving with constant velocity v, i.e.,

$$\{(s',t') \,|\, t' \ge t \ \wedge \ s' = x + v\cdot(t'-t)\} \ .$$

Such a path describes a movement of unbounded duration along a line, disregarding possible obstacles in the way. This notion will be helpful in the definition of blocking, though. When v is irrelevant we just speak of a *path*.

3. Modalities

Now we want to characterise regions of spatio-temporal reachability. The diagrams to follow are similar to the light cones introduced by Minkowski within the Theory of Relativity [17]. In this section we restrict ourselves to the case of two spatial dimensions to allow simple depictions. The illustrations in this section are taken from [13]; time progresses upwards.



(a) (b) (c)

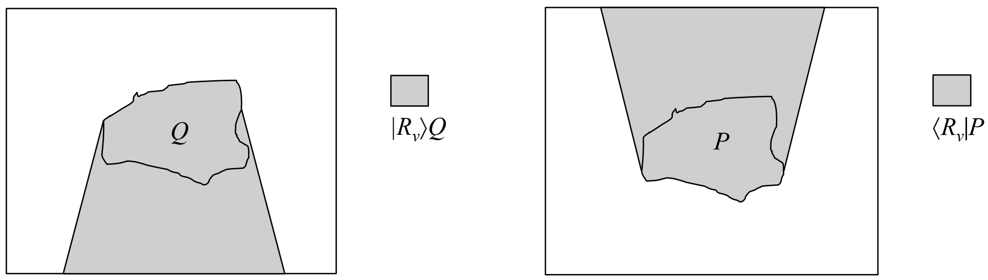
The cone in (a) models the case where a wayfinder starts at the space-time point at the tip of the cone. Since we study the points reachable under maximal velocity v, the further time progresses the farther a wayfinder can move away (in every spatial direction) from the spatial coordinate of its starting point. In (b) we find the situation where both a departure point and a destination point are given. The downward open cone at the top models the region from which the destination can be reached under maximal velocity v; it is also known as a *past cone*. Dually, a cone such as the one in (a) is called a *future cone*.

Since the speed limitation is assumed to be global, the possible region of travel for a wayfinder is the intersection of the (infinitely extended) future and past cones; it is called a *prism* [11, 13]. Finally, in (c) there is a sequence of space-time points that the considered wayfinder has to meet; they are connected by prisms. Such a structure is called a *necklace*.

We will now show how to model cones and prisms in terms of reachability relations. We use the modal forward and backward diamond operators (e.g. [21]) that compute the inverse image $|R_v\rangle Q$ and the image $\langle R_v|P$ of sets Q, P of points under some relation R_v:

$$|R_v\rangle Q \;=_{df}\; \{x \mid \exists\, y \in Q : x\, R_v\, y\}\ , \qquad\qquad \langle R_v|P \;=_{df}\; \{y \mid \exists\, x \in P : x\, R_v\, y\}\ .$$

Hence these operators are existential quantifiers about successor and predecessor points. Concerning movement, $|R_v\rangle Q$ describes the past cone spanned by Q and $\langle R_v|P$ the future cone spanned by P. Therefore, the modal notation can serve as a "calculus of diagrams".



Let us explain the notation. In many modal logics one considers only one direction of transitions and then uses a notation like $\langle R\rangle$. However, we are interested in both directions and hence use the above asymmetric notations. As a mnemonic aid, one can think of the diamonds as a simplified version of arrows: a frequent notation for transition systems R is $x \overset{R}{\mapsto} y$ when R offers a transition from x to y. Dropping the middle line of the arrow and enlarging stem and tip one obtains $x\,|R\rangle\, y$ or, the other way around, $y\langle R|\, x$. Our notations generalise that to point sets P, Q instead of points x, y: we have $x \in |R\rangle\, Q$ iff there is a point $y \in Q$ with $x\,|R\rangle\, y$. Likewise, $y \in \langle R|\, P$ iff there is a point $x \in P$ with $y\langle R|\, x$. So the set of "given" points for which we want to compute the inverse image/image is at the tip/stem of the simplified $\mapsto$ arrow.

Box operators corresponding to the diamonds are defined deploying the De Morgan duality $\forall\, z : P(z) \Leftrightarrow \neg\exists\, z : \neg P(z)$ between existential and universal quantifiers: one sets

$$|R_v]Q \;=_{df}\; \overline{|R_v\rangle\overline{Q}}\ , \qquad\qquad \langle R_v|P \;=_{df}\; \overline{\langle R_v|\,\overline{P}}\ ,$$

where $\overline{X}$ is the complement of X. Therefore

$$|R_v]Q \;=_{df}\; \{x \mid \forall\, y : x\, R_v\, y \;\Rightarrow\; y \in Q\}\ , \qquad\qquad [R_v|P \;=_{df}\; \{y \mid \forall\, x : x\, R_v\, y \;\Rightarrow\; x \in P\}\ .$$

Hence diamonds and boxes express possible and guaranteed reachability, resp.:

- $x \in |R_v\rangle Q$ iff under maximal velocity v it is possible to reach from x some point in Q;
- $y \in \langle R_v|P$ iff under maximal velocity v it is possible to reach y from some point in P;
- $x \in |R_v]Q$ iff under maximal velocity v all points reachable from x lie in Q;

– $y \in [R_v|P$ iff under maximal velocity v all points from which y is reachable lie in P.

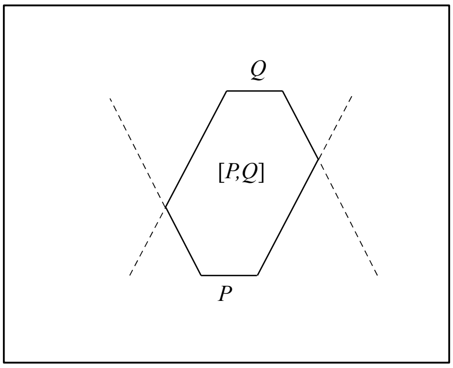
The diamond and box operators are well known from modal logic (e.g. [21]) and hence can also mirror the deontic concepts of "must" and "must not" mentioned in the introduction.

We now sketch how some of the constructions in [13] can be described in terms of the above operators. To simplify notation we identify singleton sets with their only elements. With this, the past and future cones for a starting point x, a target point y and maximal velocity v mentioned at the beginning of this section are given by $\langle R_v|x$ and $|R_v\rangle y$, resp. The space-time prism between x, y (cf. Part (b) in the diagram on p.3 is simply the *interval* $[x, y]_v =_{df} \langle R_v|x \cap |R_v\rangle y$ between x, y w.r.t. the partial order R_v, i.e.,

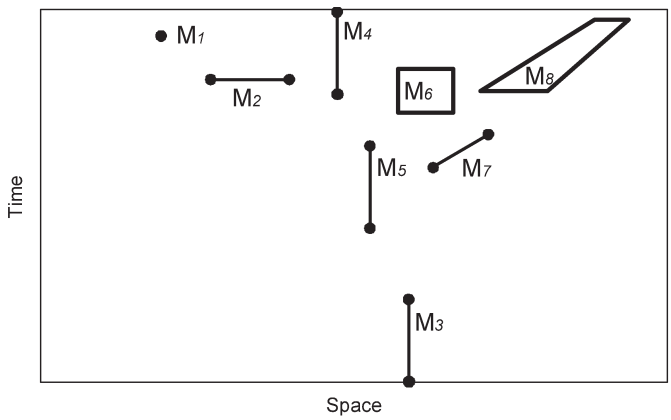
$$[x, y]_v = \{r \mid x\, R_v\, r \wedge r\, R_v\, y\} . \tag{1}$$

Note that by reflexivity of R_v we have $x, y \in [x, y]_v$ (and hence $[x, y]_v \neq \emptyset$) whenever $x\, R_v\, y$.

This definition is easily generalised to point sets P, Q by setting $[P, Q]_v =_{df} \langle R_v|P \cap |R_v\rangle Q$. This can be visualised as follows:



Next we show how *compulsions* or *barriers*, i.e., regions that must be reached or avoided, can be modelled as sets of points.



In this picture we restrict ourselves to one spatial dimension. Region M_1 consists of a single space-time point. M_2 is a spatially extended region which exists only for one single instant of time. M_3–M_5 are singleton regions in space which each exist for a certain interval of time. M_6 is a spatially extended region which exists and retains constant width for another interval of time. Finally, M_7 and M_8 are regions that, during a time interval, move from one place in space to another. While M_7 at each time during the interval occupies a singleton region in space, the spatial extent of M_8 shrinks during its time interval.

By the above explanations, the set of points from which a wayfinder can avoid a barrier B in forward direction is $[R_v|\,\overline{B}$. An analogous expression models this for the backward direction. The set of points from which a wayfinder is guaranteed to reach only compulsion C is $[R_v|C$.

Frequently, one will want a barrier or compulsion to be "connected" (i.e., not to have "holes"). For this one can use the classical mathematical notion of convexity: a set P of points is *R_v-convex* if with any two points in P also all intermediate points belong to P, i.e., $\forall\, p, q \in P : [p, q]_v \subseteq P$.

Lemma 3.1.

1. *For any transitive relation R and point set B the sets $|R\rangle B$ and $\langle R|B$ are convex.*
2. *Convex sets are closed under intersection. In particular, intervals are convex.*

A proof will be given within the algebraic treatment of the following sections.

4. Abstraction

The relational treatment already allows abstracting from the particular space in which the movements occur: all that matters is the respective reachability relation, which simply can be treated as a parameter of the whole approach. The only general requirement is that reachability be transitive.

However, we abstract even further by generalising the setting of concrete relations to the algebraic structure of modal semirings. This is a well established theory (see [4] for a survey) with many concrete instantiations, in particular relations. Using it makes our theory apply to a much larger class of models. Additionally it becomes amenable to (semi-)automatic proofs more easily (see [14] for a pioneering paper on this and [10, 22, 9] for more recent comprehensive case studies).

Definition 4.1. An *idempotent semiring* is a structure $(S, +, \cdot, 0, 1)$, where $(S, +, 0)$ forms an idempotent commutative monoid (i.e., $a+a = a$ for all $a \in S$) and $(S, \cdot, 1)$ a monoid; moreover, $\cdot$ has to distributive over $+$, and 0 has to be a multiplicative annihilator, i.e., $0 \cdot a = 0 = a \cdot 0$. The operator $+$ (not to be confused with addition of spatial coordinates) induces the *natural order* given by $a \leq b \Leftrightarrow_{df} a + b = b$, in which 0 is the least element (this should not be confused with the order $\leq$ between temporal coordinates).

In many concrete examples the elements of a semiring correspond to sets of possible transitions between states of some kind. In the case of modelling wayfinders, the states would be points and the transitions pairs of points as in the reachability relation. In informal motivations of our definitions we will often use this transition view of semiring elements; in binary relations these are one-step transitions, but in other models it is also possible to have longer transition paths. The roles of the operators can be explained as follows.

$a + b$ $\leftrightarrow$ choice between the transitions of a and b (union of a and b),
$a \cdot b$ $\leftrightarrow$ all possible sequential compositions of transitions from a followed by transitions from b,
0 $\leftrightarrow$ empty choice (empty set of transitions, modelling abortion or blocking),
1 $\leftrightarrow$ identity transition, taking each state to itself (modelling "skip" or "no-operation").

A prominent example of an idempotent semiring is provided by taking S to be the set of binary relations over some set X, with relational union as $+$, relational composition as $\cdot$, the empty relation as 0 and the identity relation $\{(x, x) : x \in X\}$ as 1. In the preceding sections X was the set of space-time points. In the relational interpretation, the natural order $\leq$ coincides with inclusion $\subseteq$.

Frequently we use the proof principle of *indirect equality* [7]:

$$a = b \Leftrightarrow (\forall\, c : c \leq a \Leftrightarrow c \leq b)\ . \tag{2}$$

The implication $\Rightarrow$ is trivial. For $\Leftarrow$ set $c = a$ to obtain $a \leq b$ and $c = b$ to obtain $b \leq a$, and use antisymmetry then. The principle may seem cumbersome at first, but the variable c offers an additional degree of freedom which often makes proofs go through more easily (or at all).

In the relational semiring, when the elements of the set X are interpreted as points, subsets of the identity relation can be used as adequate representations of subsets of X. When X is the set of space-time-points, subsets of X are also called *regions*. They can therefore be represented by subrelations of the identity relation on X. The set of all points is represented by the identity relation itself, i.e., by the element 1 in the relational semiring, while the empty set of points is represented by the empty relation, i.e., by the semiring element 0. Subrelations of the identity relation can be mimicked algebraically by so-called tests.

Definition 4.2. In a semiring, a *test* is a sub-identity element $p \leq 1$ that has a complement relative to 1, i.e., an element $\neg p$ that satisfies $p + \neg p = 1$ and $p \cdot \neg p = 0 = \neg p \cdot p$.

Actually, the condition $p \leq 1$ is redundant, since it follows from $p + \neg p = 1$. We have given it to ease the connection with the relational semiring. It is easy to see that complements are unique if they exist. Moreover, the set of tests forms a Boolean subalgebra in which $+$ coincides with the binary supremum $\sqcup$ and $\cdot$ with the binary infimum $\sqcap$. The notations $+$ and $\cdot$ for supremum and infimum are also frequently used in Boolean algebra, notably in switching theory. As a consequence of the definitions,

$$p \leq q \Leftrightarrow p = p \cdot q \ . \tag{3}$$

Every semiring contains the greatest test 1 and the least test 0. In the relational semiring the correspond to the overall set X and the empty set, resp. We also use the analogue of set difference, defined by $p - q =_{df} p \cdot \neg q$. It satisfies the *shunting rule*

$$p - q \leq r \Leftrightarrow p \leq q + r \ . \tag{4}$$

Because of their interpretation in the concrete space-time model we will write "regions" instead of "tests" in the sequel. Regions will be denoted by p, q, r etc., and the set of all regions by Reg.

Points can be modelled as atomic regions, corresponding to singleton sets.

Definition 4.3. Region p is a *point* if $p \neq 0 \wedge \forall q : q \leq p \Rightarrow q = 0 \vee q = p$. This means that there are no regions properly between 0 and p. In lattice theory the points are therefore often called *atoms*. We denote the set of all points by Pt. Moreover, we require the set Reg to be *atom-determined*; this means that every region p is the least upper bound of the set of all points $\leq p$. In the literature "atom-determined" is mostly abbreviated to just "atomic". In the sequel we will denote points by x, y, z.

Lemma 4.4. *For point x and region p we have $x \not\leq p \Leftrightarrow x \leq \neg p$.*

Proof. By (3), $p \leq 1$, hence $x \cdot p \leq x$, x a point and Boolean algebra,

$$x \not\leq p \Leftrightarrow x \cdot p \neq x \Leftrightarrow x \cdot p \leq 0 \Leftrightarrow x \leq \neg p \ . \qquad \square$$

Without the assumption that x be a point this property may fail: in set-theoretic notation, $q \not\subseteq p$ is not equivalent to $q \subseteq \overline{p}$.

As mentioned, in the sequel a, b etc. can be thought of as semiring elements that represent sets of transitions between points. The natural semiring order $\leq$ corresponds to inclusion $\subseteq$ between sets of transitions. Given a region p, the product $p \cdot a$ can be used to restrict a to those transitions that start from points in region p while, symmetrically, $a \cdot p$ restricts a to those transitions that end in points of p.

Definition 4.5. With these concepts we axiomatise the *backward and forward modal diamond operators* $\langle\ |$ and $|\ \rangle$ introduced in Sect. 3:

$$\begin{array}{ll} \langle a|p \leq q \Leftrightarrow p \cdot a \cdot \neg q \leq 0 \ , & |a\rangle p \leq q \Leftrightarrow \neg q \cdot a \cdot p \leq 0 \ , \\ \langle a \cdot b|p = \langle b|\langle a|p \ , & |a \cdot b\rangle p = |a\rangle|b\rangle p \ . \end{array}$$

The first axiom says that the image of p under a lies fully in q iff the restriction $p \cdot a$ does not contain pairs that end in $\neg q$. The second axiom is the analogous characterisation of the inverse image of p under a. The axioms in the second line stipulate that the diamonds are well behaved w.r.t. sequential composition: The image of p under $a \cdot b$ is the image under b of the image of p under a, and analogously for the inverse image.

The diamonds of regions work out as follows: for regions p, q we have

$$|p\rangle q = p \cdot q = \langle p|q \tag{5}$$

which entails the import/export laws

$$\langle a \cdot p|q = p \cdot \langle a|q \ , \qquad |p \cdot b\rangle q = p \cdot |b\rangle q \ . \tag{6}$$

Also, the diamonds are *fully strict*:

$$\langle a|q = 0 \iff q \cdot a = 0 \ , \qquad |a\rangle q = 0 \iff a \cdot q = 0 \ . \tag{7}$$

Points and regions interact as follows.

Lemma 4.6. *Let x be a point, p a region and a an arbitrary element. Then*

$$x \leq \langle a|p \Leftrightarrow p \cdot a \cdot x \neq 0 \ , \qquad x \leq |a\rangle p \Leftrightarrow x \cdot a \cdot p \neq 0 \ .$$

Since, by the above remarks on restriction, $p \cdot a \cdot x \neq 0$ is the subset of a-transitions that lead from some p-element to x, the first equivalence expressess that x is in the image of p under a iff there is at least one transition from p to x. An analogous explanation applies to the second equivalence.

Proof. We show the first claim, the second being symmetric. Here and in the derivations to come, we use the symbols $\{\![$ and $]\!\}$ to bracket a comment that justifies the step from the line above it to the line below it. Some authors use { and } for that purpose, but this might lead to confusion when set comprehensions occur.

$$\begin{array}{ll}
 & x \leq \langle a|p \\
\Leftrightarrow & \quad \{\![\ (3)\ ]\!\} \\
 & x = x \cdot \langle a|p \\
\Leftrightarrow & \quad \{\![\ x \cdot \langle a|p \leq x \text{ and } x \text{ a point }]\!\} \\
 & x \cdot \langle a|p \neq 0 \\
\Leftrightarrow & \quad \{\![\ \text{by import/export (6) }]\!\} \\
 & \langle a \cdot x|p \neq 0 \\
\Leftrightarrow & \quad \{\![\ \text{definition and full strictness of diamond (7) }]\!\} \\
 & p \cdot a \cdot x \neq 0 \ .
\end{array}$$

□

Corollary 4.7. *For points x, y and arbitrary a we have $y \leq \langle a|x \Leftrightarrow y \leq |a\rangle x$.*

This expresses that y is in the image of x under a iff x is in the inverse image of y under a.

Definition 4.8. As in the relational case, the box operators are defined as De Morgan duals of the diamonds:

$$[a|q =_{df} \neg\langle a|\neg q \ , \qquad |a]q =_{df} \neg|a\rangle\neg q \ .$$

By the first diamond axioms and Boolean algebra the modal operators satisfy the *swapping rules*

$$\langle a|p \le q \;\Leftrightarrow\; p \le |a]q \quad\text{and}\quad |a\rangle p \le q \;\Leftrightarrow\; p \le [a|q \;. \tag{8}$$

This means that diamonds and boxes form Galois connections [8].

Diamonds are *disjunctive* in both arguments, while boxes are *conjunctive* in their test arguments and *antidisjunctive* in their transition arguments:

$$\begin{array}{ll} \langle a|(p+q) = \langle a|p + \langle a|q \;, & |a\rangle(p+q) = |a\rangle p + |a\rangle q \;, \\ \langle a+b|p = \langle a|p + \langle b|p \;, & |a+b\rangle p = |a\rangle p + |b\rangle p \;, \\ [a|(p\cdot q) = [a|p \cdot [a|q \;, & |a](p\cdot q) = |a]p \cdot |a]q \;, \\ [a+b|p = [a|p \cdot [b|p \;, & |a+b]p = |a]p \cdot |b]p \;. \end{array} \tag{9}$$

This entails that diamonds and boxes are *isotone*, i.e., monotonically increasing, in their test arguments. In the transition argument, diamond is isotone, while box is *antitone*, i.e., monotonically decreasing:

$$\begin{array}{lcl} p \le q & \Rightarrow & \langle a|p \le \langle a|q \;\wedge\; |a\rangle p \le |a\rangle q \;\wedge\; [a|p \le [a|q \;\wedge\; |a]p \le |a]q \;, \\ a \le b & \Rightarrow & \langle a|p \le \langle b|p \;\wedge\; |a\rangle p \le |b\rangle p \;\wedge\; [b|p \le [a|p \;\wedge\; |b]p \le |a]p \;. \end{array} \tag{10}$$

Another important property is the following.

Lemma 4.9. *For arbitrary b and regions r, s we have $|b\rangle r - |b\rangle s \le |b\rangle(r-s)$ and $\langle b|r - \langle b|s \le \langle b|(r-s)$.*

Proof.

$$\begin{array}{ll} & |b\rangle r - |b\rangle s \le |b\rangle(r-s) \\ \Leftrightarrow & \quad \{\!\![\text{ shunting (4) }]\!\!\} \\ & |b\rangle r \le |b\rangle s + |b\rangle(r-s) \\ \Leftrightarrow & \quad \{\!\![\text{ disjunctivity of diamond (9) }]\!\!\} \\ & |b\rangle r \le |b\rangle(s+(r-s)) \\ \Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\ & |b\rangle r \le |b\rangle(r+s) \\ \Leftrightarrow & \quad \{\!\![\; r \le r+s \text{ and isotony of diamond (10) }]\!\!\} \\ & \mathsf{TRUE}\;. \end{array}$$

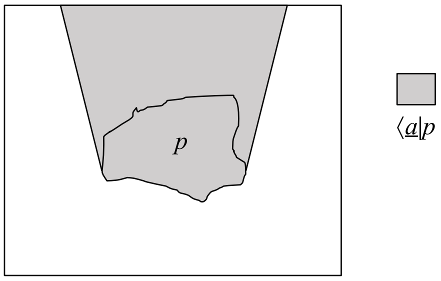
□

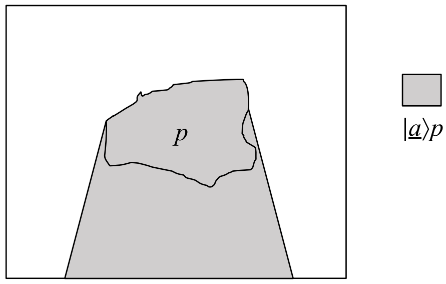
Many further properties can be found in [4, 5].

5. Abstract Reachability

For the rest of the paper we restrict ourselves to the case of a single reachability relation represented by a transition element a of a semiring. Although, as mentioned in the beginning of Sect. 4, our treatment is completely independent of the concrete space or time domain used, we visualise our definitions, as in Sect. 3, for the case of one-dimensional space and time to keep the pictures simple. As in that section we assume that time flows from bottom to top.

Then the values of the diamond operators for transition element a and a region p can be visualised as follows.





If p is actually a point then exactly the future and past cones of Sect. 3 result.

Differing from Sect. 2 we only consider *proper* reachability, i.e., the starting point is not properly reachable from itself. This will facilitate the definition of extremal points later. Therefore the transition element a is required to be irreflexive and transitive.

Definition 5.1.

1. a is *irreflexive* if for all points x we have $x \cdot |a\rangle x = 0$, equivalently, if $x \cdot \langle a|x = 0$ for all points x, equivalently, if $x \cdot a \cdot x = 0$ for all points x. This means that a does not contain a self loop for any point.
2. a is *transitive* if $|a\rangle|a\rangle p \leq |a\rangle p$ for all regions p. If, concretely, a is a relation, this means that all points x, y, z with $x\,a\,y$ and $y\,a\,z$ also satisfy $x\,a\,z$, the standard notion of transitivity of a relation. The algebraic formulation is equivalent to $\langle a|\langle a|p \leq \langle a|p$ as well as to $|a]p \leq |a]|a]p$ and $[a|p \leq [a|[a|p$ (see Sect. 16.1 in the Appendix).
3. An irreflexive and transitive element a is called a *strict-order*.

By $\underline{a} =_{df} a + 1$ we denote the *reflexive closure* of a, the least reflexive element $\geq a$. The notation follows the mathematical convention of denoting the reflexive closure of a strict-order $<$ by $\leq$.

This entails, by the definition of $\underline{a}$, disjunctivity of diamond (9) and diamond of 1,

$$|\underline{a}\rangle p = |a + 1\rangle p = |a\rangle p + |1\rangle p = |a\rangle p + p \ . \tag{11}$$

Using the theory of partial orders and (11) we obtain

$$|a\rangle p \leq |a\rangle p + p = |\underline{a}\rangle p \ . \tag{12}$$

Symmetric laws hold for the backward diamond.

Lemma 5.2. *Assume a to be transitive.*

1. $|\underline{a}\rangle|a\rangle p = |a\rangle p = |a\rangle|\underline{a}\rangle p$.
2. *$\underline{a}$ is transitive as well; we have even* $|\underline{a}\rangle|\underline{a}\rangle p = |\underline{a}\rangle p$.

Analogous properties hold for the backward diamond and the box operators.

Proof.

1. By definition of $\underline{a}$, disjunctivity of diamond (9), diamond of 1 and transitivity,

$$|\underline{a}\rangle|a\rangle p = |a + 1\rangle|a\rangle p = |a\rangle|a\rangle p + |1\rangle|a\rangle p = |a\rangle|a\rangle p + |a\rangle p = |a\rangle p \ .$$

The second equation is proved symmetrically.

2. By the definition of $\underline{a}$, disjunctivity of diamond (9) with (5), Part 1 and (12),

$$|\underline{a}\rangle|\underline{a}\rangle p = |a+1\rangle|\underline{a}\rangle p = |a\rangle|\underline{a}\rangle p + |\underline{a}\rangle p = |a\rangle + |\underline{a}\rangle p = |\underline{a}\rangle p \ .$$

$\square$

6. Intervals and Convexity

From now on we assume the transition element a to be irreflexive and transitive, i.e., a strict-order, unless explicitly stated otherwise.

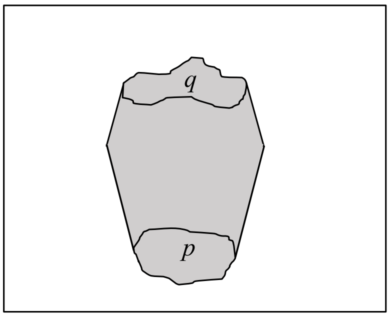
Next we generalise the definition of intervals in (1) to the algebraic setting with the help of the diamond operators. By the assumption that our transition elements a are irreflexive we can define (half) open and closed intervals using the reflexive closure of $\underline{a}$ of a when appropriate.

Definition 6.1. The *intervals* spanned by regions p, q are

$$\begin{array}{llllll}]p,q[& =_{df} & \langle a|p \cdot |a\rangle q \ , & \qquad [p,q[& =_{df} & \langle \underline{a}|p \cdot |a\rangle q \ , \\ ]p,q] & =_{df} & \langle a|p \cdot |\underline{a}\rangle q \ , & \qquad [p,q] & =_{df} & \langle \underline{a}|p \cdot |\underline{a}\rangle q \ . \end{array}$$

Strictly speaking, these intervals should be indexed by the transition element a; we omit this to reduce notational "noise".

For instance, a closed interval can be visualised like this:



interval $[p,q]$

With the help of intervals we can give an algebraic definition of convexity, as already mentioned just before Lm. 3.1.

Definition 6.2. A region p is *convex* if for all regions $q, r \leq p$ the closed interval spanned by q and r is contained in p, i.e., $[q,r] \leq p$.

Lemma 6.3.

1. *The intersection of convex regions is convex.*
2. *For transitive a and arbitrary region p the regions $\langle a|p$ and $|a\rangle p$ are convex.*
3. *Every interval is convex.*

Proof.

1. Immediate from the definitions and isotony of $\cdot$.
2. Assume $r, s \leq |a\rangle p$. By $\langle \underline{a}|r \leq 1$, isotony (10) with $s \leq |a\rangle p$ and Lm. 5.2.1,

$$\langle \underline{a}|r \cdot |\underline{a}\rangle s \leq |\underline{a}\rangle s \leq |\underline{a}\rangle|a\rangle p = |a\rangle p \ .$$

The proof for $\langle a|p$ is symmetric.

3. Immediate from the definition of intervals and Parts 1 and 2. $\square$

7. Barriers and Blocking

Definition 7.1. A *barrier* is a region p that a wayfinder must avoid. It is considered to be impenetrable, i.e., the wayfinder has no possibility of "permeating" through it and hence has to go around it.

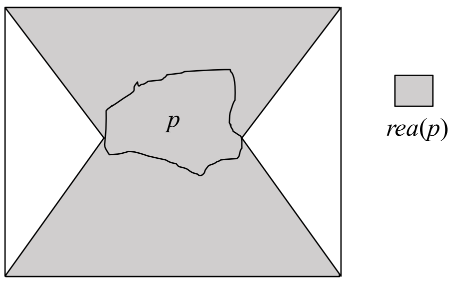
A barrier blocks movement in two respects:

– a wayfinder must not hit the barrier when moving
– and it cannot come from certain points behind the barrier, because barriers are considered to be impenetrable.

As an auxiliary notion we define the region reachable from a region p as

$$rea(p) =_{df} \langle \underline{a}|p + |\underline{a}\rangle p \ .$$

In the transition model this is the union of the past and future cones of p and consists of all points on paths to or from p.



Now we want to define the region $fbl(p)$ ("forward blocked by p") before a barrier p. If a wayfinder would start in that region it would necessarily hit p after a while. To characterise it, we observe that any subregion q of $fbl(p)$ must satisfy the following conditions:

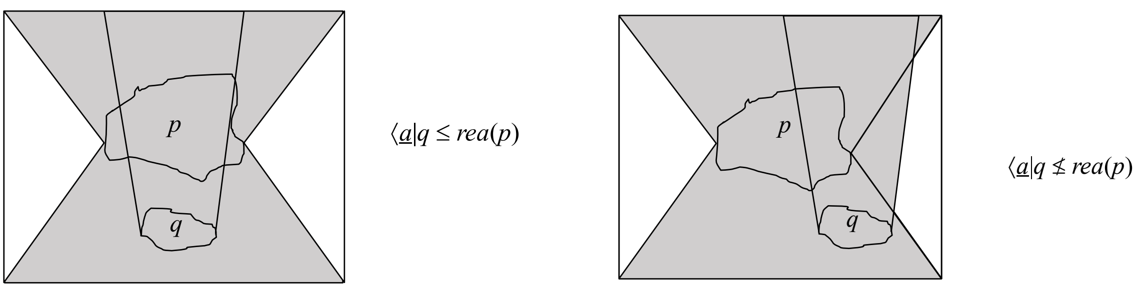
1. q must lie before p.
2. All paths starting from q must lead through p.

Condition 1 is easily expressed as $q \leq |\underline{a}\rangle p$.

For Condition 2 things are not as simple. Note that we cannot recast it in the form "all paths starting in q must lead only up to p", since in the transition model every path can in principle be extended indefinitely (unless the domain of time is finite). Therefore, to capture Condition 2. we define the *forward cone $fcone(p)$ spanned by p* using the formula

$$q \leq fcone(p) \Leftrightarrow_{df} \langle \underline{a}|q \leq rea(p) \ . \tag{13}$$

The idea is that p acts as a kind of gate through which all paths starting in q must pass. We visualise the intended behaviour.

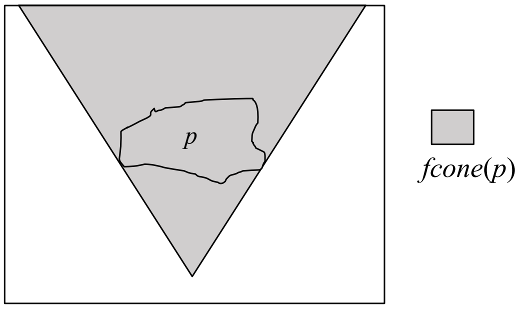


By the swapping rule (8) the right hand side is equivalent to $q \leq |\underline{a}](rea(p))$. Therefore, the principle of indirect equality (2) leads to the following definition.

Definition 7.2. The *forward cone* of p is

$$fcone(p) \ =_{df} \ |\underline{a}]rea(p) \ . \tag{14}$$

This can be visualised as follows.



In general a forward cone could also be truncated by the bottom boundary of the overall space-time region.

Lemma 7.3.

1. *The function fcone is isotone.*
2. $p \leq fcone(p)$.
3. $fcone(p) = |\underline{a}]fcone(p)$.
4. $q \leq fcone(p) \Leftrightarrow \langle\underline{a}|q \leq fcone(p)$. *In particular,* $\langle\underline{a}|fcone(p) \leq fcone(p)$ *and hence* $\langle\underline{a}|fcone(p) = fcone(p)$.
5. $fcone(p) \leq rea(p)$.
6. $fcone(p)$ *is convex.*

Part 1 speaks for itself. Part 2 says that the above picture is correct, i.e., p is contained in its own *fcone*. This also meets the original condition that every path starting in $fcone(p)$ must lead through p. Part 3 means that paths starting in $fcone(p)$ remain there throughout. Part 4 provides another closure property, namely that $fcone(p)$ contains all future cones of its points. Part 5 says that all points in $fcone(p)$ are indeed reachable from p. Part 6 again speaks for itself.

Proof.

1. The function is a composition of the isotone functions $\langle a|, |\underline{a}\rangle, \langle\underline{a}|$ and $+$.
2. By (13) we have $p \leq fcone(p) \Leftrightarrow \langle a|p \leq rea(p) = \langle\underline{a}|p + |\underline{a}\rangle p$, which holds trivially.
3. By the definition, Lm. 5.2 and the definition again,

$$fcone(p) = |\underline{a}]rea(p) = |\underline{a}]|\underline{a}]rea(p) = |\underline{a}]fcone(p) \ .$$

4. By Part 3, the swapping rules (8) and (13),

$$q \leq fcone(p) \Leftrightarrow q \leq |\underline{a}]fcone(p) \Leftrightarrow \langle\underline{a}|q \leq fcone(p) \ .$$

5. By the definitions of *fcone* and $\underline{a}$, antidisjunctivity of box (9), box of 1 and $|a]rea(p) \leq 1$,

$$\begin{array}{rcl} fcone(p) & = & |\underline{a}]rea(p) = |a+1]rea(p) = |a]rea(p) \cdot |1]rea(p) \\ & = & |a]rea(p) \cdot rea(p) \leq rea(p) \ . \end{array}$$

6. Assume $q, r \leq fcone(p)$. By (13), $|\underline{a}\rangle s \leq 1$ with isotony of diamond (10) and Lm. 5.2.2,

$$\begin{array}{rcl} \langle\underline{a}|r \cdot |\underline{a}\rangle s \leq fcone(p) & \Leftrightarrow & \langle\underline{a}|(\langle\underline{a}|r \cdot |\underline{a}\rangle s) \leq rea(p) \Leftarrow \langle\underline{a}|\langle\underline{a}|r \leq rea(p) \\ & \Leftrightarrow & \langle\underline{a}|r \leq rea(p)\ , \end{array}$$

which holds trivially. □

Definition 7.4. The region *forward blocked by* p is the part of $fcone(p)$ including p and before p, i.e.,

$$fbl(p) =_{df} |\underline{a}\rangle p \cdot fcone(p)\ . \tag{15}$$

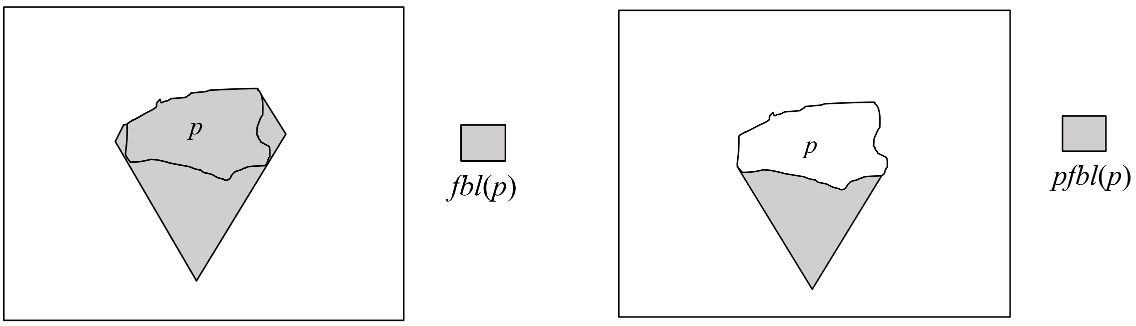
Corollary 7.5. *$p \leq fbl(p)$ and $fbl(p)$ is convex. Moreover, for all q we have $q \leq fbl(p) \Leftrightarrow q \leq |\underline{a}\rangle p \wedge \langle\underline{a}|q \leq rea(p)$.*

The second property offers a simpler way to check whether a region lies inside $fbl(p)$ than using the original definition (15), since it avoids an explicit calculation of $fcone(p)$.

Proof. The first claim is immediate from $p \leq |\underline{a}\rangle p$ and Lm. 7.3.3, the second one from Lm. 6.3.2 and Lm. 7.3.6 and Lm. 6.3.1. The final claim follows from the fact that $\cdot$ coincides with the intersection of regions and from (13). □

If one is interested in the part of $fbl(p)$ properly before p one can use

$$pfbl(p) =_{df} fbl(p) - \langle a|p\ .$$



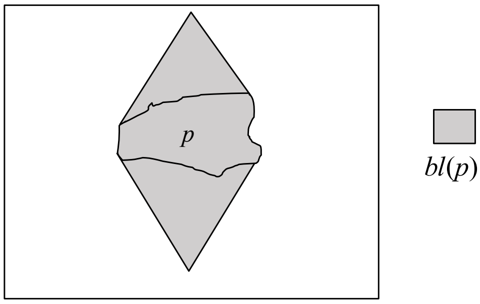
Definition 7.6.

1. Symmetrically to above, we define for a region p the *backward cone* $bcone(p)$, *backward blocked* region $bbl(p)$ and the *proper backward blocked* region $pbbl(p)$ as

$$bcone(p) =_{df} [\underline{a}|rea(p)\ , \qquad bbl(p) =_{df} \langle\underline{a}|p \cdot bcone(p)\ , \qquad pbbl(p) =_{df} bbl(p) - |a\rangle p\ .$$

2. Finally, we define the region blocked forward and backward by p as

$$bl(p) =_{df} pfbl(p) + p + pbbl(p)\ . \tag{16}$$



Corollary 7.7. *For $f \in \{fcone, fbl, bcone, bbl, \}$ we have $f(p) + f(q) \leq f(p + q)$.*

Proof. Immediate from isotony of these functions. □

In [20] the difference $f(p + q) - (f(p) + f(q))$ is called the *emerging* blocked region induced by the union $p + q$ of regions. We will return to this topic in Sect. 13.

8. Extremal Elements

Next we define sets of extremal points, see also [25, 6, 19].

Definition 8.1. The *maximal* and *minimal* points of region p are given by

$$\mathsf{max}_a\, p =_{df} p - |a\rangle p \ , \qquad \mathsf{min}_a\, p =_{df} p - \langle a|p \ .$$

I.e., the maximal points are those points in p that are not strictly before any p-point; likewise for the minimal points.

Here we see why a better be irreflexive: otherwise points with a self-loop would be removed by the max/min operators, even if they are related to no other points.

Lemma 8.2. [19] *Consider elements a, b, a test p and a point x. We have the following properties for* max *and symmetric ones for* min.

1. $\mathsf{max}_a\, p \leq p$.
2. $\mathsf{max}_a\, (\mathsf{max}_a\, p) = \mathsf{max}_a\, p$.
3. $p = \mathsf{max}_a\, p + p \cdot |a\rangle p$.
4. *If x is a point then* $\mathsf{max}_a\, x = x = \mathsf{max}_{\underline{a}}\, |\underline{a}\rangle x$.

For the reader's benefit we include the proof.

Proof.

1. By the definitions and $\neg|a\rangle p \leq 1$ we have $\mathsf{max}_a\, p = p - |a\rangle p = p \cdot \neg|a\rangle p \leq p$.
2. By Part 1 and isotony of diamond (10) we obtain $|a\rangle(\mathsf{max}_a\, p) \leq |a\rangle p$. Now antitony of $\neg$ shows $\neg|a\rangle p \leq \neg|a\rangle(\mathsf{max}_a\, p)$ and thus

$$\mathsf{max}_a\, (\mathsf{max}_a\, p) = \mathsf{max}_a\, p \cdot \neg|a\rangle(\mathsf{max}_a\, p) = p \cdot \neg|a\rangle p \cdot \neg|a\rangle(\mathsf{max}_a\, p) = p \cdot \neg|a\rangle p = \mathsf{max}_a\, p \ .$$

3. By neutrality of 1, the definition of regions and distributivity,

$$p = p \cdot 1 = p \cdot (\neg|a\rangle p + |a\rangle p) = p \cdot \neg|a\rangle p + p \cdot |a\rangle p = \mathsf{max}_a\, p + p \cdot |a\rangle p\ .$$

4. For the first equation we obtain by Part 3, irreflexivity of a and neutrality of 0,

$$x = \mathsf{max}_a\, x + x \cdot |a\rangle x = \mathsf{max}_a\, x + 0 = \mathsf{max}_a\, x\ .$$

   For the second equation we get by definition of max, disjunctivity of diamond (9) with Lm. 5.2.2, Boolean algebra, definition of max and by the first equation,

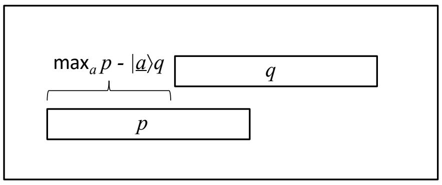
$$\mathsf{max}_a\, |\underline{a}\rangle x = |\underline{a}\rangle x - |a\rangle|\underline{a}\rangle x = (|a\rangle x + x) - |a\rangle x = x - |a\rangle x = \mathsf{max}_a\, x = x\ .$$

□

Next we determine the extremal points of a sum.

Lemma 8.3. $\mathsf{max}_a\,(p+q) = (\mathsf{max}_a\, p - |a\rangle q) + (\mathsf{max}_a\, q - |a\rangle p)$ *and*
$\mathsf{min}_a\,(p+q) = (\mathsf{min}_a\, p - \langle a|q) + (\mathsf{min}_a\, q - \langle a|p)$.

In our depictions of regions, dominated points are drawn below the dominating ones. Hence the maximal elements of a rectangular region constitute the upper border line of the rectangle. To illustrate the terms in the above lemma we choose a sample case where all of region q is "behind" all of region p, with part of the points in p dominated by elements of q. These form the inverse image $|\underline{a}\rangle q$ of q under $\underline{a}$. Hence the first summand $\mathsf{max}_a\, p - |a\rangle q$ can be visualised as follows.



Proof.

$$\begin{array}{cl}
 & \mathsf{max}_a\,(p+q) \\
= & \quad \{\!\![\ \text{definition}\ ]\!\!\} \\
 & (p+q) - |a\rangle(p+q) \\
= & \quad \{\!\![\ \text{distributivity}\ ]\!\!\} \\
 & (p - |a\rangle(p+q)) + (q - |a\rangle(p+q)) \\
= & \quad \{\!\![\ \text{disjunctivity of diamond (9)}\ ]\!\!\} \\
 & (p - (|a\rangle p + |a\rangle q)) + (q - (|a\rangle p + |a\rangle q)) \\
= & \quad \{\!\![\ \text{Boolean algebra}\ ]\!\!\} \\
 & ((p - |a\rangle p) - |a\rangle q) + ((q - |a\rangle q) - |a\rangle p) \\
= & \quad \{\!\![\ \text{definition}\ ]\!\!\} \\
 & (\mathsf{max}_a\, p - |a\rangle q) + (\mathsf{max}_a\, q - |a\rangle p)\ .
\end{array}$$

□

9. Boundedness

Using the extremal points we can define boundedness.

Definition 9.1. Region p is *up-bounded* if $p \leq |\underline{a}\rangle \mathsf{max}_a\, p$. This means that every point in p is dominated by a maximal point of p. Therefore, if $\mathsf{max}_a\, p = 0$ then p is up-bounded iff $p = 0$. *Down-boundedness* of p is defined symmetrically as $p \leq \langle \underline{a}| \mathsf{min}_a\, p$. Region p is *bounded* if it is up-bounded and down-bounded.

Corollary 9.2.

1. *p is bounded iff $p \leq [\mathsf{min}_a\, p, \mathsf{max}_a\, p]$.*
2. *p is bounded and convex iff $p = [\mathsf{min}_a\, p, \mathsf{max}_a\, p]$.*

Proof.

1.
$$\begin{array}{ll} & p \text{ bounded} \\ \Leftrightarrow & \{\!\![\ \text{definition of boundedness}\ ]\!\!\} \\ & p \leq \langle \underline{a}|\mathsf{min}_a\, p \wedge p \leq |\underline{a}\rangle \mathsf{max}_a\, p \\ \Leftrightarrow & \{\!\![\ \text{lattice algebra}\ ]\!\!\} \\ & p \leq \langle \underline{a}|\mathsf{min}_a\, p \cdot |\underline{a}\rangle \mathsf{max}_a\, p \\ \Leftrightarrow & \{\!\![\ \text{definition of intervals}\ ]\!\!\} \\ & p \leq [\mathsf{min}_a\, p, \mathsf{max}_a\, p]\ . \end{array}$$

2. ($\Rightarrow$) By Part 1 we have $p \leq [\mathsf{min}_a\, p, \mathsf{max}_a\, p]$. Morever, by definition, $\mathsf{min}_a\, p \leq p$ and $\mathsf{max}_a\, p \leq p$, and convexity of p implies $[\mathsf{min}_a\, p, \mathsf{max}_a\, p] \leq p$.
($\Leftarrow$) By Lm. 6.3.3 p as an interval is convex, while boundedness follows from Part 1.

□

Lemma 9.3. *If p is up-bounded then for every transitive element $b \geq a$ one has $|b\rangle p = |b\rangle \mathsf{max}_a\, p$. In particular, $|a\rangle p = |a\rangle \mathsf{max}_a\, p$ and $|\underline{a}\rangle p = |\underline{a}\rangle \mathsf{max}_a\, p$. A symmetric property holds for down-boundedness.*

Proof. ($\geq$) follows by $p \geq \mathsf{max}_a\, p$ and isotony of diamond (10). For ($\leq$) we calculate

$$\begin{array}{ll} & |b\rangle p \\ \leq & \{\!\![\ p \text{ up-bounded and isotony of diamond (10)}\ ]\!\!\} \\ & |b\rangle|\underline{a}\rangle(\mathsf{max}_a\, p) \\ = & \{\!\![\ \text{definition of } \underline{a} \text{ and disjunctivity of diamond (9) twice}\ ]\!\!\} \\ & |b\rangle|a\rangle(\mathsf{max}_a\, p) + |b\rangle|1\rangle(\mathsf{max}_a\, p) \\ \leq & \{\!\![\ |1\rangle q = q \text{ and assumption } a \leq b \text{ with isotony of diamond (10)}\ ]\!\!\} \\ & |b\rangle|b\rangle(\mathsf{max}_a\, p) + |b\rangle(\mathsf{max}_a\, p) \\ \leq & \{\!\![\ \text{transitivity of } b \text{ and isotony of diamond (10)}\ ]\!\!\} \\ & |b\rangle(\mathsf{max}_a\, p) + |b\rangle(\mathsf{max}_a\, p) \\ = & \{\!\![\ \text{idempotence of } +\ ]\!\!\} \\ & |b\rangle(\mathsf{max}_a\, p)\ . \end{array}$$

Now, a is transitive by assumption, from which transitivity of $\underline{a}$ follows by Lm. 5.2, and a and $\underline{a}$ are $\geq a$. □

Theorem 9.4. *If p, q are up-bounded then $\mathsf{max}_a\,(p+q) = \mathsf{max}_a\,(\mathsf{max}_a\, p + \mathsf{max}_a\, q)$.*

Proof. We can re-use the proof from [19], where this property was shown under the assumption that a is *normal*, i.e., that *all* regions are up-bounded.

For the right hand side of the claim we first obtain

$$
\begin{array}{ll}
 & \mathsf{max}_a\,(\mathsf{max}_a\,p + \mathsf{max}_a\,q) \\
= & \quad \{\!\![\ \text{Lm. 8.3}\ ]\!\!\} \\
 & (\mathsf{max}_a\,(\mathsf{max}_a\,p) - |a\rangle(\mathsf{max}_a\,q)) + (\mathsf{max}_a\,(\mathsf{max}_a\,q) - |a\rangle(\mathsf{max}_a\,p)) \\
= & \quad \{\!\![\ \text{idempotence of } \mathsf{max}\ ]\!\!\} \\
 & (\mathsf{max}_a\,p - |a\rangle(\mathsf{max}_a\,q)) + (\mathsf{max}_a\,q - |a\rangle(\mathsf{max}_a\,p))\ .
\end{array}
$$

Since by Lm. 9.3 up-boundedness entails $|a\rangle q = |a\rangle(\mathsf{max}_a\,q)$ and $|a\rangle p = |a\rangle(\mathsf{max}_a\,p)$, we are done. □

Next we show the important property that the sum of bounded regions is bounded again.

Theorem 9.5. *If p, q are up-bounded then so is $p + q$.*

Proof. We show $p \leq |\underline{a}\rangle\mathsf{max}_a\,(p+q)$. By commutativity of $+$ and swapping the roles of p and q also $q \leq |\underline{a}\rangle\mathsf{max}_a\,(p+q)$, which shows the claim. First,

$$
\begin{array}{ll}
 & p \\
\leq & \quad \{\!\![\ \text{up-boundedness}\ ]\!\!\} \\
 & |\underline{a}\rangle(\mathsf{max}_a\,p) \\
= & \quad \{\!\![\ \text{Boolean splitting}\ ]\!\!\} \\
 & |\underline{a}\rangle(((\mathsf{max}_a\,p)\cdot|a\rangle q) + ((\mathsf{max}_a\,p) - |a\rangle q))) \\
= & \quad \{\!\![\ \text{disjunctivity of diamond (9)}\ ]\!\!\} \\
 & |\underline{a}\rangle((\mathsf{max}_a\,p)\cdot|a\rangle q) + |\underline{a}\rangle((\mathsf{max}_a\,p) - |a\rangle q)\ .
\end{array}
$$

The second summand is $\leq |\underline{a}\rangle\mathsf{max}_a\,(p+q)$ by Lm. 8.3 and isotony of diamond (10). For the first one we continue as follows.

$$
\begin{array}{ll}
 & |\underline{a}\rangle((\mathsf{max}_a\,p)\cdot|a\rangle q) \\
= & \quad \{\!\![\ \text{definition of } \mathsf{max}\ ]\!\!\} \\
 & |\underline{a}\rangle(p\cdot\neg(|a\rangle p)\cdot|a\rangle q) \\
\leq & \quad \{\!\![\ p \leq 1 \text{ and isotony of diamond (10)}\ ]\!\!\} \\
 & |\underline{a}\rangle(\neg(|a\rangle p)\cdot|a\rangle q) \\
= & \quad \{\!\![\ \text{commutativity of } \cdot \text{ on tests and definition of } -\ ]\!\!\} \\
 & |\underline{a}\rangle(|a\rangle q - |a\rangle p) \\
= & \quad \{\!\![\ \text{By Lm.9.3}\ ]\!\!\} \\
 & |\underline{a}\rangle(|a\rangle(\mathsf{max}_a\,q) - |a\rangle p) \\
\leq & \quad \{\!\![\ \text{transitivity of } a \text{ and shunting (4)}\ ]\!\!\} \\
 & |\underline{a}\rangle(|a\rangle(\mathsf{max}_a\,q) - |a\rangle|a\rangle p) \\
\leq & \quad \{\!\![\ \text{Lm. 4.9}\ ]\!\!\} \\
 & |\underline{a}\rangle|a\rangle((\mathsf{max}_a\,q) - |a\rangle p) \\
= & \quad \{\!\![\ \text{transitivity of } a \text{ and Lm. 5.2}\ ]\!\!\} \\
 & |a\rangle((\mathsf{max}_a\,q) - |a\rangle p)) \\
\leq & \quad \{\!\![\ \text{Lm. 8.3 and isotony of diamond (10)}\ ]\!\!\} \\
 & |\underline{a}\rangle\mathsf{max}_a\,(p+q)\ .
\end{array}
$$

□

10. Bounds and Cones

We call p *strongly up-bounded* if it is up-bounded and it is not possible to bypass its maximal points from non-maximal ones, i.e., if $p \cdot |a\rangle p \leq \mathit{fcone}(\mathsf{max}_a\, p)$. Symmetrically, p is *strongly down-bounded* if it is down-bounded and it is not possible to bypass its minimal points starting from non-minimal ones, i.e., if $p \cdot \langle a|p \leq \mathit{bcone}(\mathsf{min}_a\, p)$.

Lemma 10.1. *Assume p to be strongly up-bounded.*

1. $\mathit{fcone}(p) = \mathit{fcone}(\mathsf{max}_a\, p)$.
2. $\mathit{fbl}(p) = \mathit{fbl}(\mathsf{max}_a\, p)$.

A symmetrical property holds for strongly down-bounded regions.

Proof.

1. We show $\mathit{rea}(p) = \mathit{rea}(\mathsf{max}_a\, p)$.

$$
\begin{array}{ll}
 & \mathit{rea}(p) \\
= & \quad \{\!\![\text{ definition }]\!\!\} \\
 & \langle \underline{a}|p + |\underline{a}\rangle p \\
= & \quad \{\!\![\text{ up-boundedness and Lm. 9.3 }]\!\!\} \\
 & \langle \underline{a}|p + |\underline{a}\rangle \mathsf{max}_a\, p \\
= & \quad \{\!\![\text{ Lm. 8.2.3 and disjunctivity of diamond (9) }]\!\!\} \\
 & \langle \underline{a}|\mathsf{max}_a\, p + \langle \underline{a}|(p \cdot |a\rangle p) + |\underline{a}\rangle \mathsf{max}_a\, p \\
= & \quad \{\!\![\text{ definition }]\!\!\} \\
 & \mathit{rea}(\mathsf{max}_a\, p) + \langle \underline{a}|(p \cdot |a\rangle p)\ .
\end{array}
$$

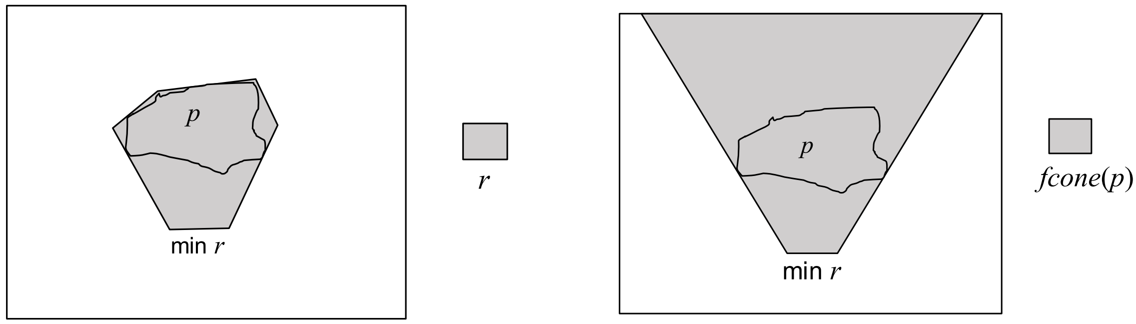
   Now we are done, since by (13) the assumption of strong up-boundedness is equivalent to $\langle \underline{a}|(p \cdot |a\rangle p) \leq \mathit{rea}(\mathsf{max}_a\, p)$.

2.
$$
\begin{array}{ll}
 & \mathit{fbl}(\mathsf{max}_a\, p) \\
= & \quad \{\!\![\text{ (15) }]\!\!\} \\
 & |\underline{a}\rangle \mathsf{max}_a\, p \cdot \mathit{fcone}(\mathsf{max}_a\, p)) \\
= & \quad \{\!\![\text{ boundedness with Lm. 9.3 and Part 1 }]\!\!\} \\
 & |\underline{a}\rangle p \cdot \mathit{fcone}(p) \\
= & \quad \{\!\![\text{ (15) }]\!\!\} \\
 & \mathit{fbl}(p)\ .
\end{array}
$$

□

Lemma 10.2. *Let $r = \mathit{fbl}(p)$ and assume that r is down-bounded.*

1. $r = [\mathsf{min}_a\, r, p]$.
2. $\mathit{fcone}(p) = \langle \underline{a}|\mathsf{min}_a\, r$.



Proof.

1. $(\leq)$

$$
\begin{array}{ll}
 & r \\
= & \quad \{\!\!\{ \text{ since by. (15) } r \leq |\underline{a}\rangle p \text{ } \}\!\!\} \\
 & r \cdot |\underline{a}\rangle p \\
\leq & \quad \{\!\!\{ \text{ down-boundedness of } r \text{ } \}\!\!\} \\
 & \langle a|\mathsf{min}_a\, r \cdot |\underline{a}\rangle p \\
= & \quad \{\!\!\{ \text{ definition of intervals } \}\!\!\} \\
 & [\mathsf{min}_a\, r, p] \ .
\end{array}
$$

$(\geq)$

$$
\begin{array}{ll}
 & [\mathsf{min}_a\, r, p] \\
= & \quad \{\!\!\{ \text{ definition of intervals } \}\!\!\} \\
 & \langle \underline{a}|\mathsf{min}_a\, r \cdot |\underline{a}\rangle p \\
= & \quad \{\!\!\{ \text{ Lm. 9.3 } \}\!\!\} \\
 & \langle \underline{a}|r \cdot |\underline{a}\rangle p \\
= & \quad \{\!\!\{ \text{ definition of } r \text{ } \}\!\!\} \\
 & \langle \underline{a}|(|a\rangle p \cdot \mathit{fcone}(p)) \cdot |\underline{a}\rangle p \\
\leq & \quad \{\!\!\{ \text{ isotony of diamond (10) } \}\!\!\} \\
 & \langle \underline{a}|\mathit{fcone}(p) \cdot |\underline{a}\rangle p \\
\leq & \quad \{\!\!\{ \text{ Lm. 7.3.4 } \}\!\!\} \\
 & \mathit{fcone}(p) \cdot |\underline{a}\rangle p \\
= & \quad \{\!\!\{ \text{ Def. 15 } \}\!\!\} \\
 & r \ .
\end{array}
$$

2. By Lm. 9.3 it suffices to show $\mathit{fcone}(p) = \langle \underline{a}|r$.
$(\geq)$ follows from $\mathit{fbl}(p) \leq \mathit{fcone}(p)$ by Lm. 7.3.4.
$(\leq)$ By Boolean algebra and the definitions

$$\mathit{fcone}(p) = \mathit{fcone}(p) \cdot |\underline{a}\rangle p + \mathit{fcone}(p) \cdot \neg|\underline{a}\rangle p = r + (\mathit{fcone}(p) - |\underline{a}\rangle p) \ .$$

The first summand r is $\leq \langle \underline{a}|r$ by (12). For the second one we reason as follows.

$$
\begin{array}{ll}
 & \mathit{fcone}(p) - |\underline{a}\rangle p \leq \langle \underline{a}|r \\
\Leftrightarrow & \quad \{\!\!\{ \text{ shunting (4) } \}\!\!\}
\end{array}
$$

$$
\begin{array}{ll}
 & fcone(p) \le |\underline{a}\rangle p + \langle \underline{a}| r \\
\Leftarrow & \quad \{\!\![\text{ since } p \le r \text{ by Cor. 7.5 }]\!\!\} \\
 & fcone(p) \le |\underline{a}\rangle p + \langle \underline{a}| p \\
\Leftrightarrow & \quad \{\!\![\text{ definition }]\!\!\} \\
 & fcone(p) \le rea(p) \\
\Leftrightarrow & \quad \{\!\![\text{ by Lm. 7.3.5 }]\!\!\} \\
 & \mathsf{TRUE}\ .
\end{array}
$$

11. Separatedness

Definition 11.1. Call p, q min-*separated* if $\mathsf{min}_a\, p \cdot \langle a|q = 0 = \mathsf{min}_a\, q \cdot \langle a|p$. max-*separatedness* is defined symmetrically.

min-separatedness means that the minimal elements of p, q in no way dominate each other.

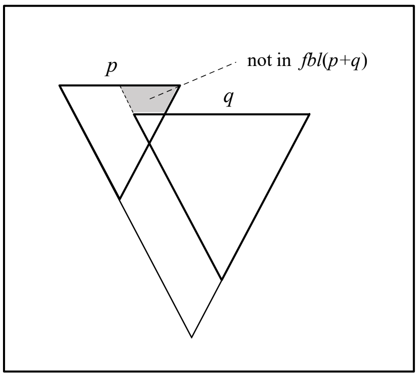
Corollary 11.2. *If p, q are* min*-separated then* $\mathsf{min}_a\,(p+q) = \mathsf{min}_a\, p + \mathsf{min}_a\, q$. *If p, q are* max*-separated then* $\mathsf{max}_a\,(p+q) = \mathsf{max}_a\, p + \mathsf{max}_a\, q$.

Proof. Straightforward by Lm. 8.3 and Boolean algebra. □

Lemma 11.3. *If p, q are* min*-separated and* $\mathsf{min}_a\, p \cdot \langle a|q \le 0$ *then* $fbl(p) \le fbl(p+q)$. *If p, q are* max*-separated and* $\mathsf{max}_a\, p \cdot |a\rangle q \le 0$ *then* $bbl(p) \le bbl(p+q)$.

Proof. By Cor. 11.2 the assumption entails $\mathsf{min}_a\, p \le \mathsf{min}_a\,(p+q)$. Now the claim is immediate from isotony of diamond (10) and *fcone*. The proof for max is symmetric. □

Without the assumptions these properties may fail:

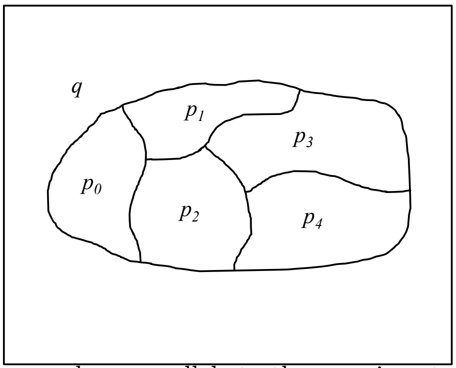


12. Coordinates

For our next results we need more detailed assumptions on the reachability relation a and on regions.

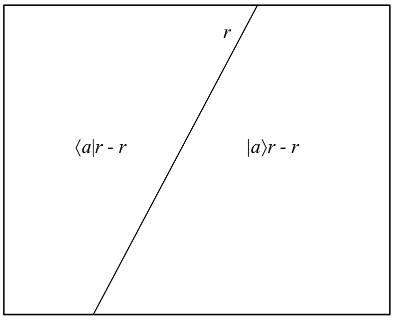
Definition 12.1. A finite family $(p_i)_{i<n}$ of regions ($n \in \mathbb{N}$) is said to *partition* a region q if $q = \sum_{i<n} p_i$ and the p_i are pairwise disjoint, i.e., $i \neq j \Rightarrow p_i \cdot p_j \le 0$.

This is an algebraic formulation of the classical notion of partitions as induced, e.g., by equivalence relations. An example is provided by the following diagram.



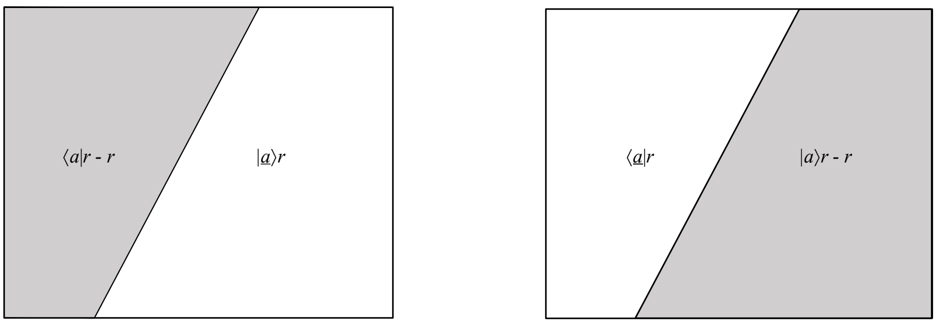
Based on this we define coordinates, such as parallels to the axes in a two-dimensional system. Each coordinate r partitions the plane into three regions: r itself, the points on one side of r and the points on the other side of r. Algebraically this is expressed as follows.

Definition 12.2. A *coordinate* is a region r such that $\langle a|r - r, r$ and $|a\rangle r - r$ partition the overall region 1.



In $\mathbb{R}^2$ straight lines are coordinates, whereas in $\mathbb{R}^3$ their role is played by planes.

Corollary 12.3. *A coordinate r satisfies $\langle a|r + r + |a\rangle r = 1$. Moreover, $\langle a|r - r$ and $|\underline{a}\rangle r$ partition 1 as do $\langle \underline{a}|r$ and $|a\rangle r - r$. As a consequence, $\neg|\underline{a}\rangle r = \langle a|r - r$ and $\neg\langle \underline{a}|r = |a\rangle r - r$.*

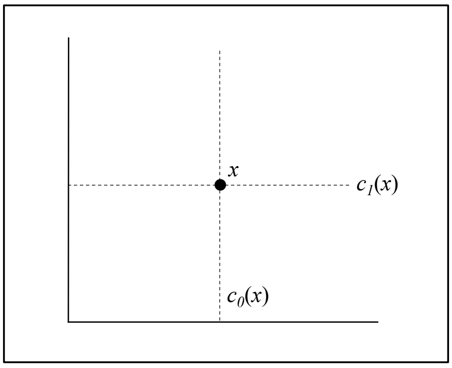


Proof. By definition we have $(\langle a|r - r) + r + (|a\rangle r - r) = 1$, which by Boolean algebra simplifies to the formula of the first claim. The second claim is shown similarly. ☐

In Euclidean geometry the coordinate values of a point are determined by parallels to the axes through that point. We mimic this algebraically.

Definition 12.4. Remember that Reg is the set of all regions. We say that the set Pt of points has n *dimensions* if there is a family $(c_i)_{i<n}$ of *coordinate functions* $c_i : \mathsf{Pt} \to \mathsf{Reg}$ such that the following conditions hold.

1. All regions $c_i(x)$ are coordinates with $c_i(x) \leq rea(x)$. This means that all points in x's i-th coordinate are reachable from x.
2. All $x \in \mathsf{Pt}$ satisfy $x = \prod_{i<n} c_i(x)$., i.e., each point is uniquely represented as the intersection of all its coordinates.



3. For every i the coordinates delivered by c_i satisfy $y \leq c_i(x) \Leftrightarrow c_i(y) = c_i(x)$.
4. Every coordinate c_i is compatible with the transition relation: $y \leq \neg|\underline{a}\rangle c_i(x) \Rightarrow \langle\underline{a}|c_i(y) \leq \neg|\underline{a}\rangle c_i(x)$ and $y \leq \neg\langle\underline{a}|c_i(x) \Rightarrow |\underline{a}\rangle c_i(y) \leq \langle\underline{a}|c_i(x)$.

Part 3 means that if a point y lies in the i-coordinate of point x then their i-coordinates coincide and vice versa. Therefore $c_i(x)$ is the equivalence class of x w.r.t. the relation $y \equiv_i z \Leftrightarrow_{df} c_i(y) = c_i(z)$. By contraposition, $y \not\leq c_i(x) \Rightarrow c_i(y) \cdot c_i(x) \leq 0$, which means that if y is not in $c_i(x)$ then $c_i(y)$ and $c_i(x)$ have empty intersection. This can be interpreted as saying that $c_i(y)$ is a (unique) "parallel" to $c_i(x)$ through y. Part 4 means that if a point y does not lie above/below a coordinate the same is true for the whole region below/above y's coordinate of the same type.

The requirements above may be viewed as weak versions of the classical postulates of Euclidean geometry: we do not consider arbitrary lines but just the very limited class of coordinates. We still require that two non-parallel coordinates intersect in one point and that to every coordinate and a point outside it there is a unique parallel of the same coordinate type through the point.

One may wonder whether the points in a coordinate should be linearly ordered w.r.t. reachability. But to obtain the essential partition properties one would, for instance, in the 3D case need to use planes as coordinates, and these cannot sensibly be linearly ordered.

Corollary 12.5. *All $x \in \mathsf{Pt}$ satisfy $|a\rangle x \leq \prod_{i<n} |a\rangle c_i(x)$ and $\langle a|x \leq \prod_{i<n} \langle a|c_i(x)$. This means that the cones of x are contained in the intersections of the cones of x's coordinates.*

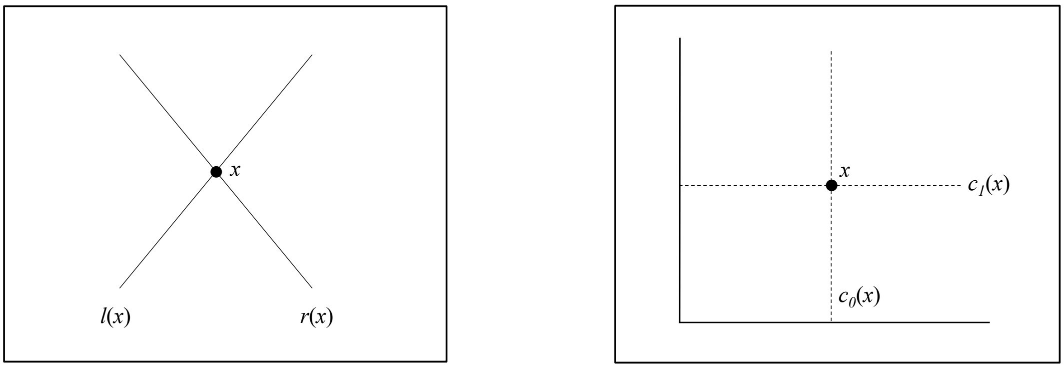
Proof. This follows from $x = \prod_{i<n} c_i(x)$ and isotony of the diamonds (10). □

In the next section we use coordinates to derive formulas for the region blocked by a sum of regions.

13. Blocks of Sums

While all our developments in Sects. 4–12 were completely independent of concrete dimensionalities of space and time, to obtain further results we restict ourselves to one spatial dimension; some aspects of the case of two-dimensional space are discussed in Sect. 14.

In the concrete wayfinder model for 1D space the lines having the directions of the left and right cone boundaries form a system of coordinates. This is similar to the situation depicted in Def. 12.4.2. For easier memorability we write l instead of c_0 and r instead of c_1.

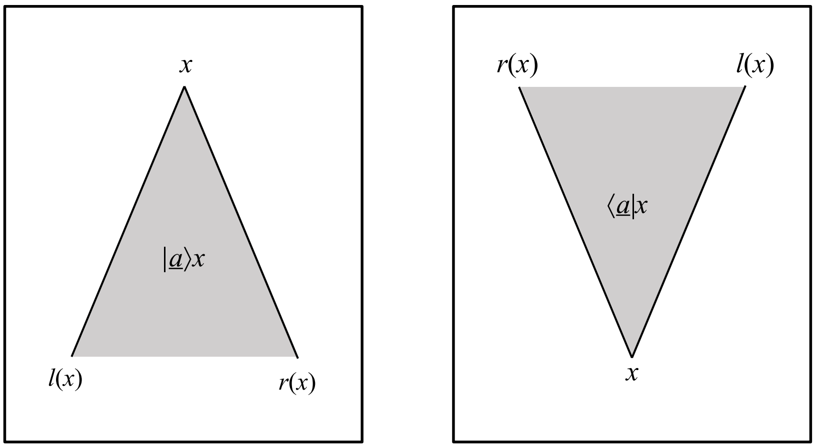


Next we move the algebra a bit closer to that model by requiring additional properties of coordinates.

Definition 13.1.

1. The coordinates l and r define a *grid* if for all points x, y with $x \neq y$ the intersection $l(x) \cdot r(y)$ is a point again.
2. l and r are *cone generators* if the inequations in Cor. 12.5 strengthen to equalities, i.e.,

$$|\underline{a}\rangle x = |\underline{a}\rangle l(x) \cdot |\underline{a}\rangle r(x) \ .$$



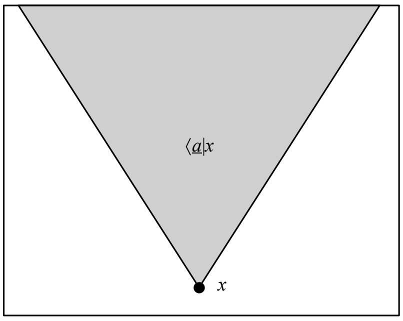
In Sect. 14 it becomes clear that in the case of 2D space coordinates will not be cone generators.

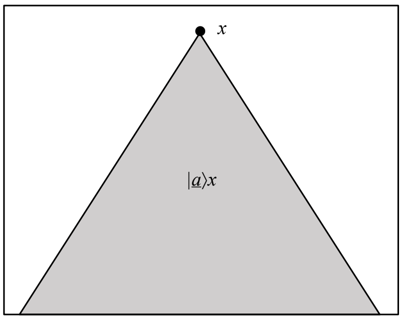
One might conjecture that if l and r are cone generators then $fcone(x) = \langle\underline{a}|x$ and $bcone(x) = |\underline{a}\rangle x$. However, this need not be the case. If a is a linear strict-order then for all points x one has $fcone(x) = 1 = bcone(x)$, which generally differs from $\langle\underline{a}|x$ and $|\underline{a}\rangle x$.

Besides the above "global" requirements we also give further properties of regions.

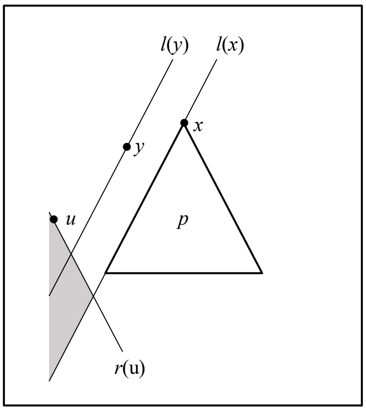
Definition 13.2.

1. Region p is *up-pointed* if p is up-bounded and $\mathsf{max}_a\, p$ is a point. A *proper down-cone* is a region of the form $p = |\underline{a}\rangle x$ for a point x; its *tip* is x. *Down-pointedness* and *proper up-cones* are defined dually.





2. An up-pointed region p with tip x is *left-down-limited* if there is a point u such that $p \leq \langle\underline{a}|r(u)$ and for all points $y \leq \neg|\underline{a}\rangle l(x) \cdot \neg|\underline{a}\rangle r(u)$ we have $l(y) \cdot |\underline{a}\rangle r(u) \neq 0$.



The first condition means that all of p is above the r-coordinate of u, which makes $r(u)$ a "left lower bound" for p; in particular, p does not extend indefinitely along its left bound $l(x)$. The second condition means that through every point y properly left of $l(x)$ we can draw a parallel to $l(x)$ that meets the area below $r(u)$, which is guaranteed to "stay away" from the region blocked by p. This is made precise in Lem. 13.4. *Right-down-limitedness* is defined symmetrically. On can define analogous notions of *up-limitedness* for down-pointed regions; we will deal only with up-pointed ones here, though.

As depicted, the coordinate $r(u)$ does not need to actually meet p. However, in subsequent pictures we will draw it as touching the respective region.

The tips can be retrieved from proper cones using Lm. 8.2.4:

Corollary 13.3. *For point x we have $\mathsf{max}_a\, |\underline{a}\rangle x = x = \mathsf{min}_a\, \langle\underline{a}|x$. Hence every proper down/up-cone is up/down-pointed.*

Limitedness admits an auxiliary result about cones.

Lemma 13.4. *Assume that l and r are cone generators and consider an up-pointed region p with tip x. If p is left-down-limited then $\neg|\underline{a}\rangle l(x) \leq \neg bcone(p)$. Symmetrically, if p is right-down-limited then $\neg|\underline{a}\rangle r(x) \leq \neg bcone(p)$.*

Proof. We only show the first claim. Assume $p \leq \langle\underline{a}|r(u)$ for some u with the further property stated in Def. 13.2. By isotony of diamonds (10) and transitivity of a we obtain

$$\langle\underline{a}|p \leq \langle\underline{a}|\langle\underline{a}|r(u) = \langle\underline{a}|r(u)$$

and hence, by Boolean algebra, $\neg\langle\underline{a}|r(u) \leq \neg\langle\underline{a}|p$. Next, since l and r are cone generators, we have $p \leq |\underline{a}\rangle l(x)$, and obtain, again by isotony of diamonds (10) and transitivity of a,

$$|\underline{a}\rangle p \leq |\underline{a}\rangle|\underline{a}\rangle l(x) \leq |\underline{a}\rangle l(x)$$

and hence, by Boolean algebra, $\neg|\underline{a}\rangle l(x) \leq \neg|\underline{a}\rangle p$.

Consider now an arbitrary point $y \leq \neg|\underline{a}\rangle l(x)$ and choose some point $z \leq l(y) \cdot |\underline{a}\rangle r(u)$, which exists by the assumption of left-down-limitedness. Then, by definition of z, the assumption on y with Def. 12.4.4, isotony of diamond (10), Boolean algebra and the definition of *rea*,

$$z \leq l(y) \cdot |\underline{a}\rangle r(u) \leq \neg|\underline{a}\rangle l(x) \cdot |\underline{a}\rangle r(u) \leq \neg|\underline{a}\rangle p \cdot \neg\langle\underline{a}|p = \neg(\langle\underline{a}|p + |\underline{a}\rangle p) = \neg rea(p)\ .$$

Moreover, by Def. 12.4.1 and Cor. 12.3 we infer $z \leq |\underline{a}\rangle y$ and hence $|\underline{a}\rangle y \cdot \neg rea(p) \neq 0$, equivalently $|\underline{a}\rangle y \not\leq rea(p)$. By the dual of (13) therefore $y \not\leq bcone(p)$ and hence, by Lm. 4.4, $y \leq \neg bcone(p)$.
Since $y \leq \neg|\underline{a}\rangle l(x)$ was arbitrary, we infer $\neg|\underline{a}\rangle l(x) \leq \neg bcone(p)$. □

Now we can state our first block-of-sum result.

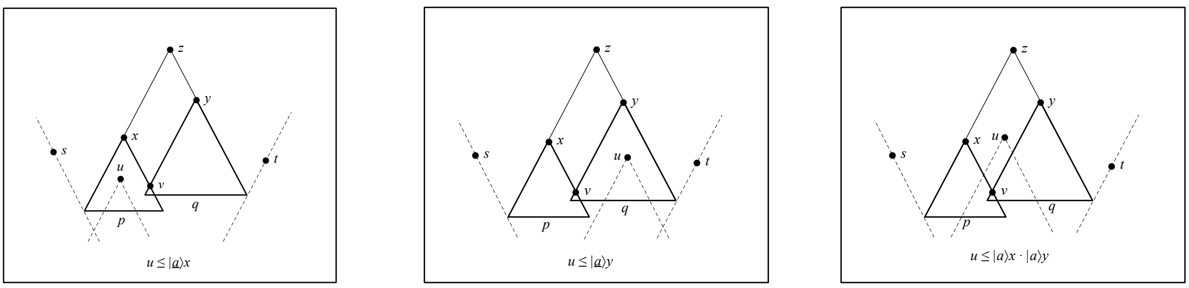
Theorem 13.5. *Let l and r induce a grid and be cone generators. Consider two regions p and q such that $\hat{p} =_{df} bbl(p)$ and $\hat{q} =_{df} bbl(q)$ are up-pointed with tips x and y. Assume moreover that $x \leq |\underline{a}\rangle r(y)$ and $y \leq |\underline{a}\rangle l(x)$ and that $p + q$ is left-down-limited and right-down-limited. Assume further that $\hat{p} \cdot \hat{q} \neq 0$, i.e., that the blocks of p and q intersect. Then with $z =_{df} l(x) \cdot r(y)$ one has $bcone(p + q) = |\underline{a}\rangle z$ and therefore $bbl(p + q) = |\underline{a}\rangle z \cdot \langle\underline{a}|(p + q)$.*

In the diagram, s and t are points that define left and right bounding coordinates for $p+q$. The assumptions $x \leq |\underline{a}\rangle r(y)$ and $y \leq |\underline{a}\rangle l(x)$ say that x is below $r(y)$ and y is below $l(x)$. The intersection point z of $l(x)$ and $r(y)$ then serves as kind of a supremum of x and y and its lower cone encompasses the region blocked by p and q; the exact extent of that region is obtained by intersecting the lower cone of z with the upper cones of p and q.

Proof. We first prove that $|\underline{a}\rangle z \leq bcone(p + q)$. To this end we show for all $u \leq |\underline{a}\rangle z$ that $|\underline{a}\rangle u \leq rea(p + q)$.

Since $z = l(x) \cdot r(y)$ we know by Def. 12.4.3 that $l(z) = l(x)$ and $r(z) = r(y)$. By Cor. 12.5 we have $|\underline{a}\rangle z \leq |\underline{a}\rangle l(z) \cdot |\underline{a}\rangle r(z) = |\underline{a}\rangle l(x) \cdot |\underline{a}\rangle r(y)$ and hence $u \leq |\underline{a}\rangle l(x) \cdot |\underline{a}\rangle r(y)$.

By the partition properties of coordinates (Cor. 12.5), u therefore lies in $|\underline{a}\rangle x$ or in $|\underline{a}\rangle y$ or in $\langle a|r(x) \cdot \langle a|l(y)$. These three cases are shown in the diagrams below.



In the first case $|\underline{a}\rangle u \leq |\underline{a}\rangle x \leq rea(p) \leq rea(p + q)$.

In the second case $|\underline{a}\rangle u \leq |\underline{a}\rangle y \leq \mathit{rea}(q) \leq \mathit{rea}(p+q)$.

In the third case we set $v =_{df} r(x) \cdot l(y)$. The parallelogram spanned by v, x, y, z is the "emerging" blocked region mentioned immediately after Cor. 7.7. By pointedness of $\hat{p}$ and $\hat{q}$ then $v \leq \hat{p} \cdot \hat{q}$ and hence $v \leq \langle \underline{a}|p \cdot \langle \underline{a}|q \leq \langle \underline{a}|p$. By construction $u \leq \langle a|v$ and hence $u \leq \mathit{rea}(p) \leq \mathit{rea}(p+q)$ as well. Consider now an arbitrary point $w \leq |\underline{a}\rangle u$. We can do the same case analysis for w again and obtain in this way that $w \leq \mathit{rea}(p+q)$ and are done.

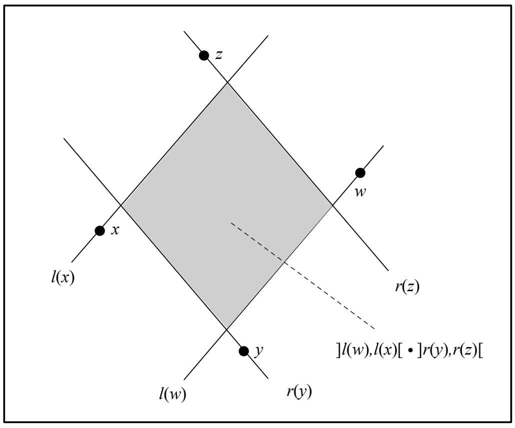
It remains to prove that $\neg|\underline{a}\rangle z \leq \neg \mathit{bcone}(p+q)$. Since l and r are cone generators, Boolean algebra shows

$$\neg|\underline{a}\rangle z \;=\; \neg(|\underline{a}\rangle l(z) \cdot |\underline{a}\rangle r(z)) \;=\; \neg|\underline{a}\rangle l(z) + \neg|\underline{a}\rangle r(z)\ .$$

Now we exploit that $p+q$ is left-down-limited and right-down-limited. Using Lm. 13.4 we infer $\neg|\underline{a}\rangle l(z) = \neg|\underline{a}\rangle l(x) \leq \neg \mathit{bcone}(p+q)$ and $\neg|\underline{a}\rangle r(z) = \neg|\underline{a}\rangle r(y) \leq \neg \mathit{bcone}(p+q)$, and are done. $\square$

For the next theorem we need an additional notion.

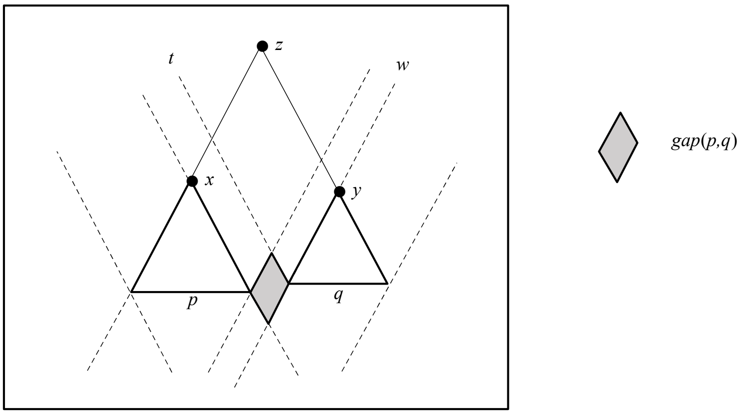
Definition 13.6. The coordinate functions l and ry satisfy the *parallelogram property* if for points w, x, y, z with $w \leq |a\rangle l(x)$ and $y \leq |a\rangle r(z)$ we have $]l(w), l(x)[\, \cdot\,]r(y), r(z)[\, \neq 0$.



The conditions $w \leq |a\rangle l(x)$ and $y \leq |a\rangle r(z)$ mean that w lies below the l-coordinate of x and y lies below the r-coordinate of z. Since by Def. 12.4.2 all l-coordinates are "parallel", and likewise all r-coordinates, $l(w), l(x), r(y), r(z)$ span a "parallelogram" whose interior is the intersection $(]l(w), l(x)[) \cdot (]r(y), r(z)[)$ of the open intervals between $l(w), l(x)$ and $r(y), r(z)$. The parallelogram property requires this interior to be non-empty under the given assumptions.

Now we can formulate our second block-of-sum result, for the case of non-overlapping blocks.

Theorem 13.7. *Let l and r induce a grid, be cone generators and satisfy the parallelogram property. Consider two regions p and q such that $\mathit{bbl}(p)$ and $\mathit{bbl}(p)$ are up-pointed with tips x and y. Assume moreover that $x \leq |\underline{a}\rangle r(y)$ and $y \leq |\underline{a}\rangle l(x)$ and that both p and q are left-down-limited and right-down-limited. Assume finally that p and q are* min*-separated. Then one has $\mathit{bcone}(p+q) = \mathit{bcone}(p) + \mathit{bcone}(q)$ and therefore $\mathit{bbl}(p+q) = \mathit{bbl}(p) + \mathit{bbl}(q)$.*



Proof. As in the proof of Th. 13.5 one shows that $\neg|\underline{a}\rangle z \le \neg bcone(p+q)$, where $z =_{df} l(x) \cdot r(y)$.

But contrary to Th. 13.5 we can show that min-sparatedness implies $|\underline{a}\rangle z \not\le bcone(p+q)$. More precisely, the region $gap(p,q)$ hatched in the above picture satisfies $gap(p,q) \le \neg rea(p+q)$ and $gap(p,q) \le |\underline{a}\rangle z = |\underline{a}\rangle|\underline{a}\rangle z$ and hence $|\underline{a}\rangle z \not\le bcone(p+q)$. The details follow.

The parallelogram spanned by z, x, y is $r =_{df} |a\rangle z \cdot \langle\underline{a}|r(x) \cdot \langle\underline{a}|l(y)$. Since p is right-down-limited and q is left-down-limited, there are coordinates t and w with properties as in Def. 13.2. By min-sparatedness, $t, w, r(x)$ and $l(y)$ satisfy the assumptions of Def. 13.6 and therefore the parallelogram

$$gap(p,q) =_{df} (]l(x), t[) \cdot (]r(y), w[)$$

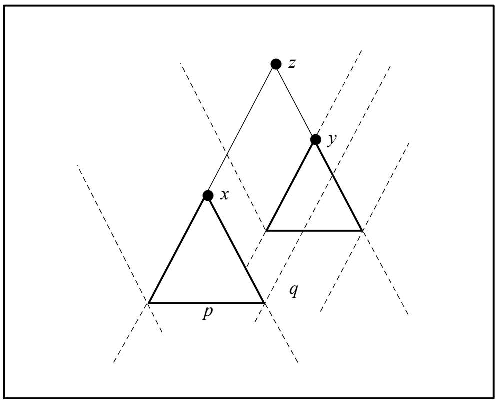
is non-empty. By construction and Boolean algebra, $gap(p,q) \le \neg rea(p) \cdot \neg rea(q) = \neg rea(p+q)$.

We show now that for all points $u \le r$ the intersection of $|\underline{a}\rangle u$ and $gap(p,q)$ is non-empty and hence $u \not\le bcone(p+q)$. If u lies strictly above $gap(p,q)$ this is clear. If $u \le gap(p,q)$ then $l(u)$ and $r(u)$ lie strictly between $l(x)$ and t or $r(y)$ and s, respectively. This means they satisfy again the assumption of Def. 13.6, and hence the parallelogram $(]l(x), l(u)[) \cdot (]r(y), r(u)[)$ is non-empty again. Spelling out the definitions and using that l, r are cone generators yields

$$\begin{array}{rcl}]l(x), l(u)[\cdot]r(y), r(u)[& = & \langle a|l(x) \cdot |a\rangle l(u) \cdot \langle a|r(y) \cdot |a\rangle r(u) \\ & = & |a\rangle l(u) \cdot |a\rangle r(u) \cdot \langle a|l(x) \cdot \langle a|r(y) \le |\underline{a}\rangle l(u) \cdot |\underline{a}\rangle r(u) \;=\; |\underline{a}\rangle u \;. \end{array}$$

In sum, only the part of $|\underline{a}\rangle z$ consisting of $bcone(p) + bcone(q)$ belongs to $bcone(p+q)$, which shows the claim. $\square$

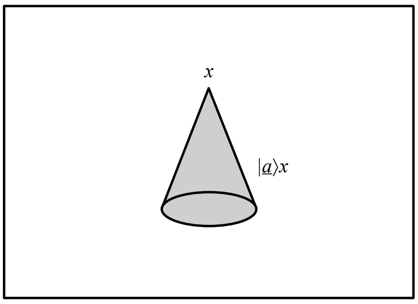
Note that his result crucially depends on min-sparatedness, as the following picture shows:



not min-separated!

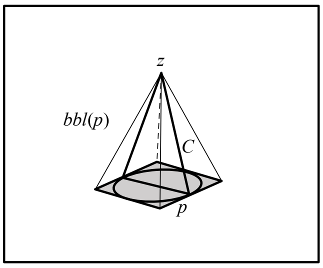
14. On 2D Space

As shown at the beginning of Sect. 3, in the case of 2D space the inverse image $|a\rangle x$ of a point x takes the form of a proper circular cone:



For all points the opening angles of their cones have the same value, say α. This can be used to find the backward blocked region *bbl* (cf. Def. 7.6) of a square, whose shape may come as a kind of surprise, since it does not involve any "curvature":

Theorem 14.1. *Consider a square p with side length l in 2D space. Place a cone C with opening angle α and base diameter l onto p. Connect the tip z of C with the four corners of p. The resulting pyramid forms $bbl(p)$.*



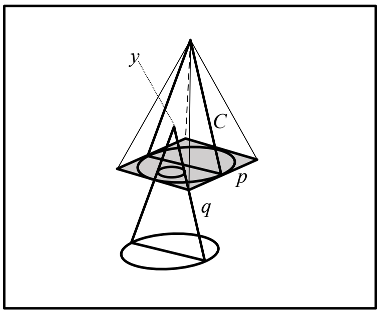
Proof. The proof is not given algebraically, since this would require a considerable amount of further concepts. However, both the theorem and its proof are very much inspired by the techniques of the previous sections.

The pyramid in the claim can be described as the interval $r =_{df} [p, z]$.

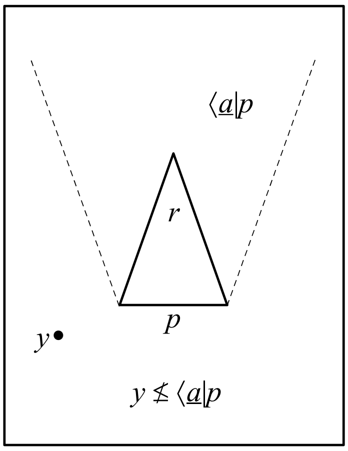
1. We first show $r \leq bbl(p)$. By the dual of Cor. 7.5 this is equivalent to

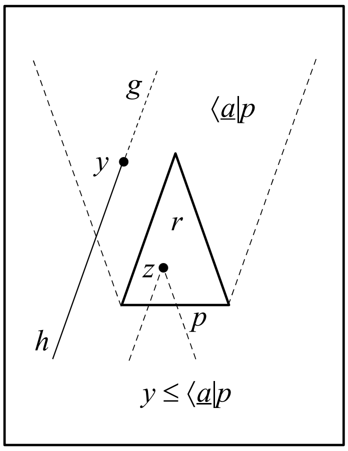
$$r \leq \langle \underline{a}|p \wedge |\underline{a}\rangle r \leq rea(p)\ . \tag{$*$}$$

   The first conjunct is immediate from the definition of intervals.
   The second conjunct holds, by additivity of diamond, if $|\underline{a}\rangle y \leq rea(p)$ for every point $y \in r$. Since the cones $q =_{df} |\underline{a}\rangle y$ and C have the same opening angle, q can be viewed as resulting from shifting a copy of C to y. By construction, the lateral faces of r are tangential to the cone C, and so during the shifting the upper part of the copy cannot leave the pyramid, even if y lies on one of the lateral faces.



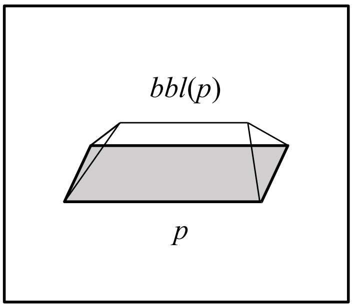
   Therefore that upper part is above p and hence in $\langle \underline{a}|p \leq rea(p)$. Likewise, the lower part of the copy cannot leave $|\underline{a}\rangle p$, and by $|\underline{a}\rangle p \leq rea(p)$ and we are done.
2. It remains to show that all points outside r do not belong to $bbl(p)$. Consider such a point y. We illustrate the arising situations with cross-sectional views.





   If $y \notin \langle \underline{a}|p$ then $y \notin bbl(p)$ by the first conjunct of $(*)$. Otherwise, since y has a positive distance from r, we can draw through y a parallel g to the lateral surface of C. Now we look at the part of g that lies within the cone $|\underline{a}\rangle y$, i.e., $h =_{df} g \cdot |\underline{a}\rangle y$. All cones emanating from points z in p have a positive distance from h. Now, since a pyramid is left-down-limited and right-down-limited, the points on h below the respective bounding coordinate of p cannot be reached from points in p. Therefore $|\underline{a}\rangle y \not\leq rea(p)$ and hence, by the second conjunct of $(*)$, $y \notin bbl(p)$. $\square$

For a general rectangle an analogous construction results in a roof-like shape of the bbl:



15. Related Work

Although there is a vast amount of papers on spatial and spatio-temporal logics, in particular, modal ones, (e.g. [1, 3, 12, 26]), the particular problem of characterising and avoiding obstacles in a modal/algebraic way does not seem to have been tackled in the literature. Of course, there are non-modal/non-algebraic mathematically-based treatments of these phenomena, e.g. in robot control [2].

There exist some attempts at formalising the particular problem area of obstacles for wayfinders (e.g. [24]), but not in a very deep or systematic way. A real breakthrough occurred in the master's thesis [20] which presented a formalisation within the functional programming language HASKELL together with a simple visualisation tool for obstacle analysis.

Concerning technicalities, our description of cones bears resemblance to Vulikh's treatment in general vector spaces [28]. However, there no modal operators are used.

Acknowledgements I am grateful to Sabine Timpf for drawing my attention to the subject of this paper. That also motivated the excellent thesis of David Pätzel [20], which was the basis for testing and extending my earlier ideas on the algebraic abstraction of issues around wayfinders and putting them to actual non-trivial work. David also provided many helpful remarks and suggestions, as did the editors and the anonymous referees.

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16. Appendix: Additional Lemmata

16.1. Transitivity

Lemma 16.1. *The swapping rules* (8) *imply*

$$p \leq |a]\langle a|p \ , \qquad \langle a||a]q \leq q \ , \qquad p \leq [a||a\rangle p \ , \qquad |a\rangle[a|q \leq q \ .$$

Proof. For the first two, set $q = \langle a|p$ and $p = |a]q$, resp. The remaining ones are symmetric. □

Lemma 16.2. *The following characterisations of transitivity of a are equivalent:*

$$\forall\, p : |a\rangle|a\rangle p \leq |a\rangle p \ , \qquad \forall\, p : [a|p \leq [a|[a|p \ , \qquad \forall\, p : |a]p \leq |a]|a]p \ , \qquad \forall\, p : \langle a|\langle a|p \leq \langle a|p \ .$$

Proof. Assume $|a\rangle|a\rangle p \leq |a\rangle p$ for arbitrary p. Then

$$
\begin{array}{ll}
 & \mathsf{TRUE} \\
\Leftrightarrow & \quad \{\!\![\text{ by Lm. 16.1 }]\!\!\} \\
 & |a\rangle[a|p \leq p \\
\Rightarrow & \quad \{\!\![\text{ assumption and isotony of diamond (10) }]\!\!\}
\end{array}
$$

$$
\begin{array}{ll}
& |a\rangle|a\rangle[a|p \le p \\
\Leftrightarrow & \quad \{\!\![\text{ by (8) twice }]\!\!\} \\
& [a|p \le [a|[a|p\ .
\end{array}
$$

Next,

$$
\begin{array}{ll}
& \forall\, p : [a|p \le [a|[a|p \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\
& \forall\, p : \neg[a|[a|p \le \neg[a|p \\
\Leftrightarrow & \quad \{\!\![\text{ definition of box and Boolean algebra }]\!\!\} \\
& \forall\, p : \langle a|\langle a|\neg p \le \langle a|\neg p \\
\Leftrightarrow & \quad \{\!\![\text{ logic }]\!\!\} \\
& \forall\, q : \langle a|\langle a|\neg q \le \langle a|\neg q\ .
\end{array}
$$

Finally, by a derivation symmetric to the first one, $\langle a|\langle a|p \le \langle a|p$ implies $|a]p \le |a]|a]p$, which completes the cycle. □

16.2. Trichotomic Elements

Definition 16.3. Element a is *trichotomic* iff for all points x we have $\langle a|x \cdot |a\rangle x \le 0$.

This means that the points "before" p are strictly separate from the points "after" p. The standard linear orders on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are trichotomic.

Lemma 16.4.

1. *If a is trichotomic then it is irreflexive.*
2. *If a is irreflexive and transitive then it is trichotomic.*

Proof.

1. Suppose $x \cdot a \cdot x \ne 0$ for some point x. Then by Lm. 4.6 we have $x \le \langle a|x$ and $x \le |a\rangle x$, hence $x \le \langle a|x \cdot |a\rangle x$ and therefore $\langle a|x \cdot |a\rangle x \ne 0$, a contradiction to a being trichotomic.
2. Consider a point x and assume there is a point $y \le \langle a|x \cdot |a\rangle x$, i.e., $y \le \langle a|x$ and $y \le |a\rangle x$. Then by Lm. 4.6 also $x \le \langle a|y$. Therefore, by isotony of diamond (10) and transitivity, $x \le \langle a|\langle a|x \le \langle a|x$ and hence by Cor. 4.7 $x \cdot a \cdot x \ne 0$, contradicting irreflexivity of a.

□