

A C^0 interior penalty discontinuous Galerkin Method for fourth-order total variation flow. II: Existence and uniqueness

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Funding information

National Science Foundation, DMS-1520886.

We prove the existence and uniqueness of a solution of a C^0 Interior Penalty Discontinuous Galerkin (C^0 IPDG) method for the numerical solution of a fourth-order total variation flow problem that has been developed in part I of the paper. The proof relies on a nonlinear version of the Lax-Milgram Lemma. It requires to establish that the nonlinear operator associated with the C^0 IPDG approximation is Lipschitz continuous and strongly monotone on bounded sets of the underlying finite element space.

KEYWORDS

C^0 interior penalty discontinuous Galerkin method, existence and uniqueness, fourth-order total variation flow

1 | INTRODUCTION

We consider the following fourth-order total variation flow (TVF) problem:

$$\frac{\hat{\partial}w}{\hat{\partial}t} + \beta \hat{\Delta} \hat{\nabla} \cdot \frac{\hat{\nabla}w}{|\hat{\nabla}w|} = 0 \quad \text{in } \hat{Q} := \hat{\Omega} \times (0, \hat{T}), \quad (1.1a)$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta \frac{\hat{\nabla}w}{|\hat{\nabla}w|} = \mathbf{n}_{\hat{\Gamma}} \cdot \hat{\nabla} \hat{\nabla} \cdot \left(\hat{\nabla} \cdot \frac{\hat{\nabla}w}{|\hat{\nabla}w|} \right) = 0 \quad \text{on } \hat{\Sigma} := \hat{\Gamma} \times (0, \hat{T}), \quad (1.1b)$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}. \quad (1.1c)$$

where $\hat{\Omega} \subset \mathbb{R}^2$ is a bounded domain with boundary $\hat{\Gamma} = \partial\hat{\Omega}$, $\hat{T} > 0$ is the final time, $\beta > 0$ is some constant, $\mathbf{n}_{\hat{\Gamma}}$ stands for the exterior unit normal at $\hat{\Gamma}$, and $w^0 \in L^2(\hat{\Omega})$ is some given initial data. The

fourth order Equation (1.1a) has to be understood as the flow problem

$$-\frac{\partial w}{\partial \hat{t}} \in \partial E_{H^{-1}}(w)$$

associated with the total variation- H^{-1} (TV- H^{-1}) minimization of the energy functional

$$E(w) = \beta \int_{\hat{\Omega}} |\hat{\nabla} w| \, dx, \quad \beta > 0, \quad (1.2)$$

where $\partial_{H^{-1}}E(w)$ is the H^{-1} subdifferential of E . In fact, if we introduce an inner product on $H^{-1}(\hat{\Omega})$ according to

$$(w, z)_{-1, \hat{\Omega}} := (\hat{\nabla}(-\hat{\Delta}^{-1}w), \hat{\nabla}(-\hat{\Delta}^{-1}z))_{0, \hat{\Omega}},$$

the subdifferential

$$\partial_{H^{-1}}E(w) = \{v \in H^{-1}(\hat{\Omega}) \mid (v, z - w)_{-1, \hat{\Omega}} \leq E(z) - E(w) \text{ for all } z \in H^{-1}(\hat{\Omega})\}$$

reads as follows (cf., eg, [1]):

$$\partial_{H^{-1}}E(w) = \{\hat{\Delta} \hat{\nabla} \cdot \xi \mid \xi(\hat{x}) \in \partial \Phi(\hat{\nabla} w(\hat{x}))\}.$$

Here, $\Phi(|\eta|)$ and $\partial \Phi(|\eta|)$ are given by

$$\Phi(\eta) = \beta |\eta|, \quad \partial \Phi(\eta) = \begin{cases} \beta \eta / |\eta| & \text{if } \eta \neq \mathbf{0} \\ \{\tau \in \mathbb{R}^2 \mid |\tau| \leq \beta\} & \text{if } \eta = \mathbf{0} \end{cases}. \quad (1.3)$$

The fourth-order TVF problem (Equations (1.1a) to (1.1c) describes surface relaxation below the roughening temperature. We note that similar fourth-order TVF problems occur in image recovery. For more details we refer to Bhandari and coworkers [2] and the references therein.

In the sequel, we consider the regularized fourth-order TVF problem

$$\frac{\partial w}{\partial \hat{t}} + \beta \hat{\Delta} \hat{\nabla} \cdot ((\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w) = 0 \quad \text{in } \hat{Q}, \quad (1.4a)$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta (\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w = 0 \quad \text{on } \hat{\Sigma}, \quad (1.4b)$$

$$\begin{aligned} \mathbf{n}_{\hat{\Gamma}} \cdot \beta \hat{\nabla} (\hat{\nabla} \cdot (\delta^2 + |\hat{\nabla} w|^2)^{-1/2} \hat{\nabla} w) &= 0 \quad \text{on } \hat{\Sigma}, \\ w(\cdot, 0) &= w^0 \quad \text{in } \hat{\Omega}, \end{aligned} \quad (1.4c)$$

where $\delta > 0$ is a regularization parameter. We further consider a scaling in both the time variable and the spatial variables according to

$$t = \delta \hat{t}, \quad x_i = \delta \hat{x}_i, \quad 1 \leq i \leq 2. \quad (1.5)$$

Setting $T := \delta \hat{T}$, $\Omega := \delta \hat{\Omega}$, $\Gamma := \partial \Omega$, $Q := \Omega \times (0, T)$, $\Sigma := \Gamma \times (0, T)$, and $u^0(x) = w^0(\delta^{-1}x)$, as well as

$$\omega(\nabla u) := 1 + |\nabla u|^2, \quad (1.6)$$

the scaled and regularized fourth order TVF problem reads as follows

$$\frac{\partial u}{\partial t} + \beta \delta^2 \Delta \nabla \cdot (\omega(\nabla u)^{-1/2} \nabla u) = 0 \quad \text{in } Q, \quad (1.7a)$$

$$\mathbf{n}_{\Gamma} \cdot \beta \delta^2 (\omega(\nabla u)^{-1/2} \nabla u) = \mathbf{n}_{\Gamma} \cdot \beta \delta^2 \nabla \cdot (\omega(\nabla u)^{-1/2} \nabla u) = 0 \quad \text{on } \Sigma, \quad (1.7b)$$

$$u(\cdot, 0) = u^0 \quad \text{in } \Omega. \quad (1.7c)$$

The numerical solution of the regularized fourth-order TVF problem with periodic boundary conditions has been considered in Kohn and Versieux [3] based on a mixed formulation of the implicitly

in time discretized problem. At each time-step, this amounts to the solution of 2 second-order elliptic PDEs by standard Lagrangian finite elements with respect to a triangulation of the computational domain Ω . On the other hand, a C^0 Interior Penalty Discontinuous Galerkin (C^0 IPDG) method has been developed and implemented in Bhandari and coworkers [2]. The advantage of the C^0 IPDG approach is that it directly applies to the fourth-order problem and thus only requires the numerical solution of one equation by using the same Lagrangian finite elements as in the mixed method.

The paper is organized as follows: After some basic notations from matrix analysis and Lebesgue and Sobolev spaces presented in Section 2, in Section 3 we recall the C^0 IPDG approximation of the implicit in time discretized, regularized, and scaled fourth-order TVF problem from Bhandari and coworkers [2]. Section 4 is devoted to a proof of the existence and uniqueness of a solution of the C^0 IPDG approximation by an application of the nonlinear version of the Lax-Milgram Lemma. In particular, this requires to show that the nonlinear operator associated with the C^0 IPDG approximation is Lipschitz continuous and strongly monotone on bounded subsets of the underlying function space.

2 | BASIC NOTATIONS

For vectors $\underline{\mathbf{x}} = (x_1, \dots, x_n)^T, \underline{\mathbf{y}} = (y_1, \dots, y_n)^T \in \mathbb{R}^n$ and for matrices $\underline{\mathbf{A}} = (a_{ij})_{i,j=1}^n, \underline{\mathbf{B}} = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ we denote by $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ and $\underline{\mathbf{A}} : \underline{\mathbf{B}}$ the Euclidean inner product $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}} = \sum_{i=1}^n x_i y_i$ and the Frobenius inner product $\underline{\mathbf{A}} : \underline{\mathbf{B}} = \sum_{i,j=1}^n a_{ij} b_{ij}$. In particular, $|\underline{\mathbf{x}}| := (\underline{\mathbf{x}} \cdot \underline{\mathbf{x}})^{1/2}$ and $|\underline{\mathbf{A}}| := (\underline{\mathbf{A}} : \underline{\mathbf{A}})^{1/2}$ refer to the Euclidean norm and the Frobenius norm, respectively.

We will further use standard notation from Lebesgue and Sobolev space theory (cf., eg, [4]). In particular, for a bounded domain $D \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(D), 1 \leq p < \infty$, as the Banach space of p th power Lebesgue integrable functions on D with norm $\|\cdot\|_{0,p,D}$ and to $L^\infty(D)$ as the Banach space of essentially bounded functions on D with norm $\|\cdot\|_{0,\infty,D}$. Moreover, we denote by $W^{s,p}(D), s \in \mathbb{R}_+, 1 \leq p \leq \infty$, the Sobolev spaces with norms $\|\cdot\|_{s,p,D}$. We note that for $p=2$ the spaces $L^2(D)$ and $W^{s,2}(D) = H^s(D)$ are Hilbert spaces with inner products $(\cdot, \cdot)_{0,2,D}$ and $(\cdot, \cdot)_{s,2,D}$. In the sequel, we will suppress the subindex 2 and write $(\cdot, \cdot)_{0,D}, (\cdot, \cdot)_{s,D}$ and $\|\cdot\|_{0,D}, \|\cdot\|_{s,D}$ instead of $(\cdot, \cdot)_{0,2,D}, (\cdot, \cdot)_{s,2,D}$ and $\|\cdot\|_{0,2,D}, \|\cdot\|_{s,2,D}$. The space $W_0^{s,p}(D)$ is the closure of C_0^∞ with respect to the $\|\cdot\|_{s,p,D}$ -norm. We refer to $W^{-s,p}(D), s \in \mathbb{R}_+, 1 \leq p \leq \infty$, as the dual of $W_0^{s,q}(D)$, where $1/p + 1/q = 1$. In particular, $H^{-s}(D) = (H_0^s(D))^*$.

3 | C^0 IPDG APPROXIMATION

We perform a discretization in time of Equation (1.7) with respect to a partition of the time interval $[0, T]$ into subintervals $[t_{m-1}, t_m], 1 \leq m \leq M, M \in \mathbb{N}$, of length $\Delta t := t_m - t_{m-1} = T/M$. Denoting by u^m some approximation of u at time t_m , for $1 \leq m \leq M$ we have to solve the problems

$$u^m - u^{m-1} + \Delta t \beta \delta^2 \Delta \nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m) = 0 \text{ in } \Omega, \quad (3.1a)$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 (\omega(\nabla u^m)^{-1/2} \nabla u^m) = 0 \text{ on } \Gamma, \quad (3.1b)$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 \nabla (\nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m)) = 0 \text{ on } \Gamma. \quad (3.1c)$$

We reformulate the second term on the left-hand side of Equation (3.1a) according to

$$\Delta \nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m) = \nabla \cdot \nabla (\nabla \cdot (\omega(\nabla u^m)^{-1/2} \nabla u^m)) = \nabla \cdot \nabla \cdot \nabla (\omega(\nabla u^m)^{-1/2} \nabla u^m). \quad (3.2)$$

As has been shown in Bhandari and coworkers [2], we have

$$\nabla(\omega(\nabla u^m)^{-1/2}\nabla u^m) = \omega(\nabla u^m)^{-3/2}\underline{\underline{\mathbf{M}}}(u^m)D^2u^m, \quad (3.3)$$

where D^2u^m is the 2×2 matrix of second partial derivatives of u^m and the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is given by

$$\underline{\underline{\mathbf{M}}}(u^m) := \begin{pmatrix} 1 + \left(\frac{\partial u^m}{\partial x_2}\right)^2 & -\frac{\partial u^m}{\partial x_1} \frac{\partial u^m}{\partial x_2} \\ -\frac{\partial u^m}{\partial x_1} \frac{\partial u^m}{\partial x_2} & 1 + \left(\frac{\partial u^m}{\partial x_1}\right)^2 \end{pmatrix}. \quad (3.4)$$

We note that the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is symmetric positive definite with the eigenvalues

$$\lambda_{\min}(\underline{\underline{\mathbf{M}}}(u^m)) = 1, \quad \lambda_{\max}(\underline{\underline{\mathbf{M}}}(u^m)) = 1 + |\nabla u^m|^2. \quad (3.5)$$

Setting

$$\underline{\underline{\mathbf{A}}}_1(v) := \omega(\nabla v)^{-3/2}\underline{\underline{\mathbf{M}}}(v), \quad (3.6)$$

the weak formulation of the implicitly in time discretized regularized fourth order TVF problem (3.1a) to (3.1c) reads: Find

$$u^m \in V := \{v \in H^2(\Omega) \mid \mathbf{n}_\Gamma \cdot \beta \delta^2 \omega(\nabla v)^{-1/2} \nabla v = 0 \text{ on } \Gamma\}$$

such that for all $v \in V$ it holds

$$(u^m - u^{m-1}, v)_{0,\Omega} + \Delta t \beta \delta^2 \int_{\Omega} (\underline{\underline{\mathbf{A}}}_1(u^m) D^2 u^m) : D^2 v \, dx = 0. \quad (3.7)$$

For the discretization in space we assume \mathcal{T}_h to be a geometrically conforming, simplicial triangulation of Ω . We denote by $\mathcal{E}_h(\Omega)$ and $\mathcal{E}_h(\Gamma)$ the set of edges of \mathcal{T}_h in the interior of Ω and on the boundary Γ , respectively, and set $\mathcal{E}_h := \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$. For $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_K and h_E the diameter of K and the length of E , and we set $h := \max(h_K \mid K \in \mathcal{T}_h)$. Due to the assumptions on \mathcal{T}_h there exist constants $0 < c_R \leq C_R$, $0 < c_Q \leq C_Q$, and $0 < c_S \leq C_S$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_R h_K \leq h_E \leq C_R h_K, \quad E \in \mathcal{E}_h(\partial K), \quad (3.8a)$$

$$c_Q h \leq h_K \leq C_Q h, \quad (3.8b)$$

$$c_S h_K^2 \leq \text{meas}(K) \leq C_S h_K^2. \quad (3.8c)$$

Denoting by $P_k(T)$, $k \in \mathbb{N}$, the linear space of polynomials of degree $\leq k$ on T , for $k \in \mathbb{N}$ we define

$$V_h := \{v_h \in C^0(\overline{\Omega}) \mid v_h|_T \in P_k(T), \, T \in \mathcal{T}_h\}, \quad (3.9)$$

and note that $V_h \subset H^1(\Omega)$, but $V_h \not\subset H^2(\Omega)$. Further, we introduce

$$\underline{\underline{\mathbf{M}}}_h := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2 \times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2 \times 2}, \, K \in \mathcal{T}_h\} \quad (3.10)$$

as the space of element-wise polynomial moment tensors. For interior edges $E \in \mathcal{E}_h(\Omega)$ such that $E = K_+ \cap K_-$, $K_{\pm} \in \mathcal{T}_h$ and boundary edges on Γ we introduce the average and jump of ∇v_h according to

$$\{\nabla v_h\}_E := \begin{cases} \frac{1}{2}(\nabla v_h|_{E \cap K_+} + \nabla v_h|_{E \cap K_-}) & E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E & E \in \mathcal{E}_h(\Gamma) \end{cases}, \quad (3.11a)$$

$$[\nabla v_h]_E := \begin{cases} \nabla v_h|_{E \cap K_+} - \nabla v_h|_{E \cap K_-} & E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E & E \in \mathcal{E}_h(\Gamma) \end{cases}. \quad (3.11b)$$

The average $\{ \Delta v_h \}_E$ and jump $[\Delta v_h]_E$ are defined analogously. We further denote by \mathbf{n}_E the unit normal vector on E pointing in the direction from K_+ to K_- . In the sequel, for $E \in \mathcal{E}_h$ we will frequently use

$$| \{ v_h w_h \}_E | \leq 2 \{ |v_h| \}_E \{ |w_h| \}_E, \quad (3.12a)$$

$$| [v_h w_h]_E | \leq 4 \{ |v_h| \}_E \{ |w_h| \}_E. \quad (3.12b)$$

In fact, for $E \in \mathcal{E}_h(\Omega)$ Equations (3.12a) and (3.12b) follow from

$$\begin{aligned} | \{ v_h w_h \}_E | &\leq \frac{1}{2} (|v_h|_{E_+} |w_h|_{E_+} + |v_h|_{E_-} |w_h|_{E_-}) \leq 2 \{ |v_h| \}_E \{ |w_h| \}_E, \\ | [v_h w_h]_E | &\leq (|v_h|_{E_+} |w_h|_{E_+} + |v_h|_{E_-} |w_h|_{E_-}) \leq 4 \{ |v_h| \}_E \{ |w_h| \}_E, \end{aligned}$$

whereas it is obvious for $E \in \mathcal{E}_h(\Gamma)$. We will also use

$$\sum_{E \in \mathcal{E}_h} [v_h w_h]_E = \sum_{E \in \mathcal{E}_h} \{ v_h \}_E [w_h]_E + \sum_{E \in \mathcal{E}_h(\Omega)} [v_h]_E \{ w_h \}_E. \quad (3.13)$$

Following the general approach [5] for DG approximations of second-order elliptic boundary value problems, in Bhandari and coworkers [2] we have derived the following C^0 IPDG approximation of Equation (3.7): Find $u_h^m \in V_h$ such that for all $v_h \in V_h$ it holds

$$(u_h^m, v_h)_{0,\Omega} + \Delta t \beta \delta^2 a_h^{IP}(u_h^m, v_h; u_h^m) = (u_h^{m-1}, v_h)_{0,\Omega}, \quad v_h \in V_h. \quad (3.14)$$

Here, for $z_h \in V_h$ the mesh-dependent semilinear C^0 IPDG form $a_h^{IP}(\cdot, \cdot; z_h) : V_h \times V_h \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} a_h^{IP}(u_h, v_h; z_h) &:= \sum_{K \in \mathcal{T}_h} (\underline{\underline{\mathbf{A}}}_1(z_h) D^2 u_h, D^2 v_h)_{0,K} \\ &\quad - \sum_{E \in \mathcal{E}_h} (\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(z_h) D^2 u_h \}_E \mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4} \nabla v_h]_E)_{0,E} \\ &\quad - \sum_{E \in \mathcal{E}_h} (\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(z_h) D^2 v_h \}_E \mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4} \nabla u_h]_E)_{0,E} \\ &\quad + \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} (\mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4} \nabla u_h]_E, \mathbf{n}_E \cdot [\omega(\nabla z_h)^{-1/4} \nabla v_h]_E)_{0,E}, \end{aligned} \quad (3.15)$$

where $\alpha > 0$ is a penalty parameter and

$$\underline{\underline{\mathbf{A}}}_2(z_h) := \omega(\nabla z_h)^{-5/4} \underline{\underline{\mathbf{M}}}(z_h). \quad (3.16)$$

4 | EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE C^0 IPDG APPROXIMATION

The existence and uniqueness of a solution of the C^0 IPDG approximation Equation (3.14) can be shown using the following nonlinear analogue of the Lax-Milgram Lemma.

Theorem 4.1 *Let V be a Hilbert space with inner product $(\cdot, \cdot)_V$ and associated norm $\|\cdot\|_V$ and let V^* be the dual space with norm $\|\cdot\|_{V^*}$. We denote by $\langle \cdot, \cdot \rangle_{V^*,V}$ the dual pairing between V^* and V . Let $A : V \rightarrow V^*$ be a nonlinear operator with $A(0) = 0$ that is Lipschitz continuous on $B(0, R) := \{v \in V \mid \|v\|_V \leq R\}$, $R > 0$, that is, there exists a constant $\Gamma(R) > 0$ such that for all $v, w \in V$ it holds*

$$\|A(v) - A(w)\|_{V^*} \leq \Gamma(R) \|v - w\|_V. \quad (4.1)$$

Moreover, assume that $A : V \rightarrow V^*$ is strongly monotone on $B(0, R)$, that is, there exists a constant $\gamma(R) > 0$ such that for all $v, w \in B(0, R)$ it holds

$$\langle A(v) - A(w), v - w \rangle_{V^*, V} \geq \gamma(R) \|v - w\|_V^2. \quad (4.2)$$

Then, for any $\ell \in V^*$ with

$$\|\ell\|_{V^*} \leq \frac{\Gamma(R)^2}{\gamma(R)} \left(1 - \sqrt{1 - \frac{\gamma(R)^2}{\Gamma(R)^2}} \right) R, \quad (4.3)$$

the nonlinear equation

$$Au = \ell \quad (4.4)$$

has a unique solution $u \in B(0, R)$.

Proof. We refer to $\tau : V^* \rightarrow V$ as the Riesz mapping, that is,

$$\langle \ell, v \rangle_{V^*, V} = (\tau \ell, v)_V, \quad \ell \in V^*, \quad v \in V. \quad (4.5)$$

Then, $u \in B(0, R)$ is a solution of Equation (4.4) if and only if u is a fixed point of the nonlinear map $T : V \rightarrow V$ defined by means of

$$T(v) := v - \rho(\tau A(v) - \tau \ell), \quad v \in V, \quad \rho > 0.$$

Due to Equation (4.5) we have

$$\|T(v) - T(w)\|_V^2 = \|v - w\|_V^2 - 2\rho \langle A(v) - A(w), v - w \rangle_{V^*, V} + \rho^2 \|A(v) - A(w)\|_{V^*}^2. \quad (4.6)$$

Now, using Equations (4.1) and (4.2) it follows that

$$\|T(v) - T(w)\|_V^2 \leq q \|v - w\|_V^2, \quad q := 1 - 2\rho\gamma(R) + \rho^2\Gamma(R)^2.$$

For $\rho \in (0, 2\gamma(R)/\Gamma(R)^2)$ we have $q < 1$ and hence, T is a contraction on $B(0, R)$. We note that q attains its minimum $q_{\min} = 1 - \gamma(R)^2/\Gamma(R)^2$ for $\rho_{\min} = \gamma(R)/\Gamma(R)^2$. Moreover, choosing $w = 0$ in Equation (4.6) and observing $A(0) = 0$, we have

$$\|T(v) - T(0)\|_V^2 \leq q_{\min} \|v\|_V^2,$$

and hence, for $v \in B(0, R)$ it holds

$$\|T(v)\|_V \leq \|T(v) - T(0)\|_V + \|T(0)\|_V \leq \sqrt{q_{\min}} R + \rho \|\ell\|_{V^*}. \quad \blacksquare$$

Consequently, we have

$$\|T(v)\|_V \leq R, \quad (4.7)$$

if $\ell \in V^*$ satisfies Equation (4.3). We deduce from Equation (4.7) that $T(B(0, R)) \subset B(0, R)$. The Banach fixed point theorem asserts the existence and uniqueness of a fixed point in $B(0, R)$.

In order to apply the previous result to the C^0 IPDG method Equation (3.14), we introduce a mesh-dependent semi-norm $|\cdot|_{2, h, \Omega}$ and weighted norm $\|\cdot\|_{2, h, \Omega}$ on V_h according to

$$|v_h|_{2, h, \Omega} := \left(\sum_{K \in \mathcal{T}_h} \int_K D^2 v_h : D^2 v_h \, dx + \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_{E \in \mathcal{E}_h} |\mathbf{n}_E \cdot [\nabla v_h]_E|^2 \, ds \right)^{1/2}, \quad (4.8a)$$

$$\|v_h\|_{2, h, \Omega} := (\|v_h\|_{0, \Omega}^2 + |v_h|_{2, h, \Omega}^2)^{1/2}. \quad (4.8b)$$

We further note that (3.14) can be written as the nonlinear equation

$$A_h^{DG} u_h^m = \ell_h, \quad (4.9)$$

where the nonlinear operator $A_h^{DG} : V_h \rightarrow V_h^*$ and the linear functional $\ell_h \in V_h^*$ are given by

$$\langle A_h^{DG} v_h, w_h \rangle_{V_h^*, V_h} := (v_h, w_h)_{0, \Omega} + \Delta t \beta \delta^2 a_h^{DG}(v_h, w_h; v_h), \quad v_h, w_h \in V_h, \quad (4.10)$$

$$\ell_h(v_h) := (u_h^{m-1}, v_h)_{0, \Omega}, \quad v_h \in V_h. \quad (4.11)$$

For the proof of Lipschitz continuity on bounded sets and strong monotonicity of the nonlinear operator A_h^{DG} we need the inverse estimates (cf., eg, [6, 7]): For $p \in [1, \infty]$ and $\ell, m \in \mathbb{N}_0$ it holds

$$\|v_h\|_{m,p,K} \leq \frac{C_{inv}}{\text{meas}(K)^{\max(0, \frac{1}{2} - \frac{1}{p})} h_K^{m-\ell}} \|v_h\|_{\ell,K}, \quad v_h \in V_h, \quad (4.12)$$

where C_{inv} is a positive constant that only depends on k, ℓ, m, p , and the shape regularity of the triangulation. We further need the trace inequalities (cf., eg, [8, 9]): For $p \in [1, \infty]$, $m \in \mathbb{N}_0$, and $K \in \mathcal{T}_h$ it holds

$$\|\nabla v_h\|_{m,p,\partial K} \leq C_T h_K^{-1/p} \|\nabla v_h\|_{m,p,K}, \quad v_h \in V_h, \quad (4.13a)$$

$$\|D^2 v_h\|_{m,p,\partial K} \leq C_T h_K^{-1/p} \|D^2 v_h\|_{m,p,K}, \quad v_h \in V_h, \quad (4.13b)$$

where C_T is a positive constant that only depends on k, m, p , and the shape regularity of the triangulation. Moreover, we will frequently use the following Poincaré-Friedrichs inequality for piecewise H^2 -functions (cf., eg, [10])

$$\|\nabla v_h\|_{0,\Omega} \leq C_{PF} |v_h|_{2,h,\Omega}, \quad v_h \in V_h, \quad (4.14)$$

where $C_{PF} > 0$ is a constant that only depends on Ω and the shape regularity of the triangulation.

In the sequel, we will frequently use some basic estimates for the weight function $\omega(\nabla v_h)$. In particular, for $\beta > 0$ and $v \in V_h$ it holds

$$\omega(\nabla v)^{-\beta} = (1 + |\nabla v|^2)^{-\beta} \leq 1, \quad (4.15a)$$

$$\begin{aligned} \omega(\nabla v)^{-(\beta+1)} |\nabla v| &\leq \omega(\nabla v)^{-(\beta+1)} (1 + |\nabla v|^2)^{1/2} \\ &\leq \omega(\nabla v)^{-(\beta+1/2)} \leq 1. \end{aligned} \quad (4.15b)$$

Moreover, for $v, w \in V_h$ and $\xi(s) := w + s(v - w)$, $s \in [0, 1]$, it holds

$$\omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta} = -2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla(v - w) \, ds, \quad (4.16a)$$

$$\begin{aligned} \omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) &= \int_0^1 \omega(\nabla \xi(s))^{-\beta} \underline{\underline{\mathbf{F}}}(\xi(s); v - w) \, ds \\ &\quad - 2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla(v - w) \underline{\underline{\mathbf{M}}}(\xi(s)) \, ds, \end{aligned} \quad (4.16b)$$

where the matrix $\underline{\underline{\mathbf{F}}}(v; w)$, $v, w \in V_h$ is given by

$$\underline{\underline{\mathbf{F}}}(v; w) := \begin{pmatrix} 2 \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_2} & \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} \\ \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} & 2 \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1} \end{pmatrix}, \quad v, w \in V_h. \quad (4.17)$$

An easy computation yields

$$|\underline{\underline{\mathbf{F}}}(v; w)|^2 \leq 5 |\nabla v|^2 |\nabla w|^2. \quad (4.18)$$

It follows from Equations (4.15b) and (4.16a) that

$$| \omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta} | \leq 2\beta | \nabla(v - w) |, \quad (4.19a)$$

whereas in view of Equations (3.5), (4.15b), (4.16b), and (4.18) we have

$$| \omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) | \leq (2\beta + \sqrt{5}) | \nabla(v - w) |, \quad (4.19b)$$

We will first show that the nonlinear operator A_h^{DG} is Lipschitz continuous on the ball

$$B_h(0, R) := \{v_h \in V_h \mid \|v_h\|_{2,h,\Omega} \leq R\}. \quad (4.20)$$

Theorem 4.2 *The nonlinear operator A_h^{DG} is Lipschitz continuous on the ball $B_h(0, R)$. In particular, there exists $\Gamma(h, R) > 0$ such that*

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} \leq \Gamma(h, R) \|v_h - w_h\|_{2,h,\Omega}, \quad v_h, w_h \in B_h(0, R). \quad (4.21)$$

Proof. For $v_h, w_h \in B_h(0, R)$ we set $\xi_h := v_h - w_h$. In view of the definition (4.10) of the nonlinear operator A_h^{DG} we have

$$\begin{aligned} \|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} &= \sup_{\|z_h\|_{2,h,\Omega} \leq 1} | \langle A_h^{DG} v_h - A_h^{DG} w_h, z_h \rangle_{V_h^*, V_h} | = \\ &= \sup_{\|z_h\|_{2,h,\Omega} \leq 1} | (\xi_h, z_h)_{0,\Omega} + \Delta t \beta \delta^2 (a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h)) |. \end{aligned} \quad (4.22)$$

According to the definition (3.15) of the semilinear form $a_h^{DG}(\cdot, \cdot; \cdot)$ we find

$$a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h) = \quad (4.23)$$

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h) : D^2 z_h \, dx \\ & - \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h \}_E \, \mathbf{n}_E \, \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\ & \quad - \mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h \}_E \, \mathbf{n}_E \, \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \, ds \\ & - \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(v_h) D^2 z_h \}_E \, \mathbf{n}_E \, \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \\ & \quad - \mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(w_h) D^2 z_h \}_E \, \mathbf{n}_E \, \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) \, ds \\ & + \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \, \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\ & \quad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \, \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \, ds. \end{aligned}$$

We will estimate the four terms on the right-hand side of Equation (4.23) separately.

(1) For the first term on the right-hand side of Equation (4.23) we obtain

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h) : D^2 z_h \, dx \\ & = \underbrace{\sum_{K \in \mathcal{T}_h} \int_K \underline{\underline{\mathbf{A}}}_1(v_h) D^2 \xi_h : D^2 z_h \, dx}_{=I_1} + \underbrace{\sum_{K \in \mathcal{T}_h} \int_K (\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)) D^2 w_h : D^2 z_h \, dx}_{=I_2}. \end{aligned}$$

In view of Equations (3.5), (3.6), and (4.15a) and using Hölder's inequality as well as the Cauchy-Schwarz inequality, we get the following upper bound for I_1 :

$$\begin{aligned}
|I_1| &\leq \sum_{K \in \mathcal{T}_h} \int_K |D^2 \xi_h| |D^2 z_h| \, dx \\
&\leq \sum_{K \in \mathcal{T}_h} \left(\int_K |D^2 \xi_h|^2 \, dx \right)^{1/2} \left(\int_K |D^2 z_h|^2 \, dx \right)^{1/2} \\
&\leq \left(\sum_{K \in \mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \, dx \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \, dx \right)^{1/2}. \tag{4.24}
\end{aligned}$$

Likewise, using Equations (3.8b), (3.8c), (4.16b), the inverse inequality Equation (4.12), the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), and observing $\|D^2 w_h\|_{0,K} \leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$, we can estimate I_2 from above as follows:

$$\begin{aligned}
|I_2| &\leq \sum_{K \in \mathcal{T}_h} \int_K | \underline{\mathbf{A}}_1(v_h) - \underline{\mathbf{A}}_1(w_h) | |D^2 w_h| |D^2 z_h| \, dx \\
&\leq (3 + \sqrt{5}) \sum_{K \in \mathcal{T}_h} \int_K | \nabla \xi_h | |D^2 w_h| |D^2 z_h| \, dx \\
&\leq (3 + \sqrt{5}) \sum_{K \in \mathcal{T}_h} \| \nabla \xi_h \|_{0,\infty,K} \left(\int_K |D^2 w_h|^2 \, dx \right)^{1/2} \left(\int_K |D^2 z_h|^2 \, dx \right)^{1/2} \\
&\leq (3 + \sqrt{5}) c_S^{-1/2} C_{inv} R \sum_{K \in \mathcal{T}_h} h_K^{-1} \| \nabla \xi_h \|_{0,K} \| D^2 z_h \|_{0,K} \\
&\leq (3 + \sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} R h^{-1} \left(\sum_{K \in \mathcal{T}_h} \| \nabla \xi_h \|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \| D^2 z_h \|_{0,K}^2 \right)^{1/2} \\
&\leq (3 + \sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R h^{-1} | \xi_h |_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \| D^2 z_h \|_{0,K}^2 \right)^{1/2}.
\end{aligned}$$

Hence, setting $C_A^{(1)} := (3 + \sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R$, we thus have

$$|I_2| \leq C_A^{(1)} h^{-1} | \xi_h |_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \| D^2 z_h \|_{0,K}^2 \right)^{1/2}. \tag{4.25}$$

(2) Setting $\tilde{\omega}(\nabla v_h, \nabla w_h) := \omega(\nabla v_h)^{-1/4} - \omega(\nabla w_h)^{-1/4}$, the second term on the right-hand side of Equation (4.23) can be written as

$$\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(v_h) D^2 v_h \}_E - \mathbf{n}_E \cdot \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\
&\quad - \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 w_h \}_E - \mathbf{n}_E \cdot \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) \, ds \\
&= \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(v_h) D^2 \xi_h \}_E - \mathbf{n}_E \cdot \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \, ds}_{=II_1}
\end{aligned}$$

$$\begin{aligned}
& + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\mathbf{A}}_2(v_h) - \underline{\mathbf{A}}_2(w_h))D^2 w_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E ds}_{=II_2} \\
& + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(w_h)D^2 w_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla z_h]_E ds}_{=II_3}.
\end{aligned}$$

Setting $E_1 := E_+$, $E_2 := E_-$, for $E \in \mathcal{E}_h(\Omega)$, and using Equations (3.5), (3.8a), (3.16), (4.15a), and the trace inequality Equation (4.13b), for the first term II_1 we find

$$\begin{aligned}
|II_1| & \leq \sum_{E \in \mathcal{E}_h} \int_E |D^2 \xi_h|_E |\mathbf{n}_E \cdot [\nabla z_h]_E| ds \\
& \leq \frac{1}{2} \sum_{E \in \mathcal{E}_h(\Omega)} \int_E \sum_{i=1}^2 |D^2 \xi_h|_{E_i} |\mathbf{n}_E \cdot [\nabla z_h]_E| ds + \sum_{E \in \mathcal{E}_h(\Gamma)} \int_E |D^2 \xi_h| |\mathbf{n}_E \cdot [\nabla z_h]_E| ds \\
& \leq \sum_{E \in \mathcal{E}_h(\Omega)} \sum_{i=1}^2 h_E^{1/2} \left(\int_E |D^2 \xi_h|_{E_i}^2 ds \right)^{1/2} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& + \sum_{E \in \mathcal{E}_h(\Gamma)} \left(h_E^{1/2} \int_E |D^2 \xi_h|^2 ds \right)^{1/2} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& \leq C_R^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K \|D^2 \xi_h\|_{0,\partial K}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& \leq C_R^{1/2} C_T \left(\sum_{K \in \mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}.
\end{aligned}$$

We thus have

$$|II_1| \leq C_A^{(2)} \left(\sum_{K \in \mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}, \quad (4.26)$$

where $C_A^{(2)} := C_R^{1/2} C_T$. In a similar way, using (3.5), (3.8a)-(3.8c), (3.16), (4.16b), the inverse inequality (4.12), the trace inequality (4.13a), the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), and observing $\|D^2 w_h\|_{0,K} \leq \|w_h\|_{2,h,\Omega} \leq R$, $K \in \mathcal{T}_h$, the second term II_2 can be estimated from above according to

$$\begin{aligned}
|II_2| & \leq \left(\frac{5}{2} + \sqrt{5} \right) C_R^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K}^2 h_K \int_{\partial K} |D^2 w_h|^2 ds \right)^{1/2} \\
& \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& \leq \left(\frac{5}{2} + \sqrt{5} \right) C_R^{1/2} C_T \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K}^2 \int_K |D^2 w_h|^2 ds \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& \leq \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-1} c_S^{-1} C_{inv} C_R^{1/2} C_T R h^{-1} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^2 \right)^{1/2} \\
& \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\
& \leq C_A^{(3)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}, \tag{4.27}
\end{aligned}$$

where $C_A^{(3)} := \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$. In a similar way, for II_3 we obtain

$$\begin{aligned}
|II_3| & \leq C_R^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla z_h\|_{0,\infty,K}^2 h_K \int_{\partial K} |D^2 w_h|^2 ds \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} \\
& \leq C_R^{1/2} C_T \left(\sum_{K \in \mathcal{T}_h} \|\nabla z_h\|_{0,\infty,K}^2 \int_K |D^2 w_h|^2 dx \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} \\
& \leq c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R h^{-1} |z_h|_{2,h,\Omega} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2}.
\end{aligned}$$

and hence,

$$|II_3| \leq C_A^{(4)} h^{-1} |z_h|_{2,h,\Omega} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2}, \tag{4.28}$$

where $C_A^{(4)} := c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$.

(3) For the third term on the right-hand side of Equation (4.23) we have

$$\begin{aligned}
& \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(v_h) D^2 z_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \\
& - \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 z_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) ds \\
& = \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ (\underline{\mathbf{A}}_2(v_h) - \underline{\mathbf{A}}_2(w_h)) D^2 z_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E ds}_{=III_1} \\
& + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 z_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla v_h]_E ds}_{=III_2} \\
& + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 z_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E ds}_{=III_3}.
\end{aligned}$$

The terms III_1 , III_2 , and III_3 can be estimated from above in much the same way as the corresponding terms for II . We obtain

$$|III_1| \leq C_A^{(5)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 ds \right)^{1/2}, \quad (4.29)$$

where $C_A^{(5)} := \left(\frac{5}{2} + \sqrt{5}\right) c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_{TR}$, and

$$|III_2| \leq C_A^{(6)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 ds \right)^{1/2}, \quad (4.30a)$$

$$|III_3| \leq C_A^{(7)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 ds \right)^{1/2}, \quad (4.30b)$$

where $C_A^{(6)} = C_A^{(7)} := c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_{TR}$.

(4) Finally, for the fourth term on the right-hand side of Equation (4.23) we get

$$\begin{aligned} & \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \\ & \quad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E) ds \\ & = \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E ds}_{=IV_1} \\ & + \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla z_h]_E ds}_{=IV_2} \\ & + \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla w_h]_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E ds}_{=IV_3}. \end{aligned} \quad (4.31)$$

Using Equation (3.8a), (4.15a), the trace inequality (4.13a), and the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), for IV_1 we obtain

$$\begin{aligned} |IV_1| & \leq \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E| |\mathbf{n}_E \cdot [\nabla z_h]_E| ds \\ & \leq \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2} \\ & \leq C_A^{(8)} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \right)^{1/2} \left(\sum_{E \in \mathcal{E}_h} \int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}, \end{aligned}$$

where $C_A^{(8)} := \alpha$. Setting $K_1 := K_+$ and $K_2 := K_-$ for $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, and $K_1 = K_2 = K$ for $E \in \mathcal{E}_h(\Gamma)$, $E \in \mathcal{E}_h(K \cap \Gamma)$, the term IV_2 can be estimated from above as

follows:

$$|IV_2| \leq \alpha \sum_{E \in \mathcal{E}_h} \sum_{i=1}^2 \|\nabla \xi_h\|_{0,\infty,K_i} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}.$$

Using Equations (3.8b), (3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), for IV_1 , we have

$$\begin{aligned} \sum_{i=1}^2 \|\nabla \xi_h\|_{0,\infty,K_i} &\leq c_R^{-1} c_S^{-1} C_{inv} h^{-1} \sum_{i=1}^2 \|\nabla \xi_h\|_{0,K_i} \\ &\leq 2c_R^{-1} c_S^{-1} C_{inv} h^{-1} \|\nabla \xi_h\|_{0,\Omega} \leq 2c_R^{-1} c_S^{-1} C_{inv} C_{PF} h^{-1} |\xi_h|_{2,h,\Omega}. \end{aligned}$$

Hence, observing $\left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \leq \|w_h\|_{2,h,\Omega} \leq R$, we obtain

$$|IV_2| \leq C_A^{(9)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}. \quad (4.32)$$

where $C_A^{(9)} := 2\alpha c_R^{-1} c_S^{-1} C_{inv} C_{PF} R$. In the same way we get

$$|IV_3| \leq C_A^{(10)} h^{-1} |\xi_h|_{2,h,\Omega} \left(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\nabla z_h]_E|^2 ds \right)^{1/2}. \quad (4.33)$$

where $C_A^{(10)} := C_A^{(9)}$.

Setting $C_A := \sum_{i=1}^{10} C_A^{(i)}$, it follows from Equations (4.22) and (4.33) that

$$|\langle A_h^{DG} v_h - A_h^{DG} w_h, z_h \rangle_{V_h^*, V_h}| \leq \max(1, \beta \Delta t \delta^2 C_A h^{-1}) \|\xi_h\|_{2,h,\Omega} \|z_h\|_{2,h,\Omega},$$

which implies Equation (4.21) with $\Gamma(h, R) := \max(1, \beta \Delta t \delta^2 C_A h^{-1})$. \blacksquare

Theorem 4.3 *Under the assumption that there exist constants $0 < \kappa \ll 1$ and $C_\Delta > 0$ such that*

$$\beta \Delta t \delta^2 \leq C_\Delta h^{4+\kappa}, \quad (4.34)$$

for sufficiently small $0 < h < 1$ there exists $\gamma(h, R) > 0$ such that for $v_h, w_h \in B_h(0, R)$ it holds

$$\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \geq \gamma(h, R) \|v_h - w_h\|_{2,h,\Omega}^2. \quad (4.35)$$

Proof. For $v_h, w_h \in B_h(0, R)$ we set $\xi_h := v_h - w_h$. Taking the definition (4.10) of the nonlinear operator A_h^{DG} into account, we have

$$\langle A_H^{DG} v_h - A_h^{DG} w_h, \xi_h \rangle_{V_h^*, V_h} = \|\xi_h\|_{0,\Omega}^2 + \beta \Delta t \delta^2 (a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h)). \quad (4.36)$$

Recalling the definitions (3.6), (3.16) of $\underline{\mathbf{A}}_1$ and $\underline{\mathbf{A}}_2$, for the second term on the right-hand side of Equation (4.36) it follows that

$$\begin{aligned} a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h) &= \sum_{K \in \mathcal{T}_h} \int_K (\underline{\mathbf{A}}_1(v_h) D^2 v_h - \underline{\mathbf{A}}_1(w_h)) D^2 w_h : D^2 \xi_h \, dx \\ &\quad - \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(v_h) D^2 v_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \end{aligned}$$

$$\begin{aligned}
& - \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 w_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E \, ds \\
& - \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(v_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \\
& \quad - \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) \, ds \\
& + \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \\
& \quad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E) \, ds. \tag{4.37}
\end{aligned}$$

As in the previous theorem, we will estimate the four terms on the right-hand side of Equation (4.37) separately.

(1) For the first term we obtain

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \int_K (\underline{\mathbf{A}}_1(v_h) D^2 v_h - \underline{\mathbf{A}}_1(w_h) D^2 w_h) : D^2 \xi_h \, dx \\
& = \underbrace{\sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{A}}_1(v_h) D^2 \xi_h : D^2 \xi_h \, dx}_{=I_1} + \underbrace{\sum_{K \in \mathcal{T}_h} \int_K (\underline{\mathbf{A}}_1(v_h) - \underline{\mathbf{A}}_1(w_h)) D^2 w_h : D^2 \xi_h \, dx}_{=I_2}.
\end{aligned}$$

As far as I_1 is concerned, due to Equations (3.5) and (3.6) we have

$$\int_K \underline{\mathbf{A}}_1(v_h) D^2 \xi_h : D^2 \xi_h \, dx \geq (1 + \|\nabla v_h\|_{0,\infty,K}^2)^{-3/2} \|D^2 \xi_h\|_{0,K}^2.$$

Using Equations (3.8b), (3.8c), the inverse inequality (4.12), the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), and observing $\|v_h\|_{2,h,\Omega} \leq R$, we get

$$\begin{aligned}
\|\nabla v_h\|_{0,\infty,K}^2 & \leq c_S^{-2} C_{inv}^2 h_K^{-2} \|\nabla v_h\|_{0,K}^2 \leq c_Q^{-2} c_S^{-2} C_{inv}^2 h^{-2} \|\nabla v_h\|_{0,\Omega}^2 \\
& \leq c_Q^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2 h^{-2} \|v_h\|_{2,h,\Omega}^2 \leq \gamma_M^{(0)} h^{-2},
\end{aligned}$$

where $\gamma_M^{(0)} := c_Q^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2 R^2$. Observing $h \leq 1$, it follows that

$$(1 + \|\nabla v_h\|_{0,\infty,K}^2)^{-3/2} \geq h^3 (h^2 + \gamma_M^{(0)})^{-3/2} \geq h^3 (1 + \gamma_M^{(0)})^{-3/2} = \gamma_M^{(1)} h^3,$$

where $\gamma_M^{(1)} := (1 + \gamma_M^{(0)})^{-3/2}$. Hence, we obtain the following lower bound for I_1 :

$$I_1 \geq \gamma_M^{(1)} h^3 \sum_{K \in \mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2. \tag{4.38}$$

In order to estimate I_2 from above, we use Equations (3.8b), (3.8c), (4.19b), Hölder's inequality, the inverse inequality (4.12), the Cauchy-Schwarz inequality, and observe $\|D^2 w_h\|_{0,K} \leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$:

$$\begin{aligned}
|I_2| & \leq (3 + \sqrt{5}) \sum_{K \in \mathcal{T}_h} \int_K |\nabla \xi_h| |D^2 w_h| |D^2 \xi_h| \, dx \\
& \leq (3 + \sqrt{5}) \sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K} \left(\int_K |D^2 w_h|^2 \, dx \right)^{1/2} \left(\int_K |D^2 \xi_h|^2 \, dx \right)^{1/2} \\
& \leq (3 + \sqrt{5}) c_S^{-1} C_{inv} \sum_{K \in \mathcal{T}_h} h_K^{-1} \|\xi_h\|_{0,K} \|D^2 w_h\|_{0,K} \|D^2 \xi_h\|_{0,K}
\end{aligned}$$

$$\begin{aligned}
&\leq (3 + \sqrt{5})c_S^{-1}C_{inv}^2R \sum_{K \in \mathcal{T}_h} h_K^{-1} \|\xi_h\|_{0,K} h_K^{-2} \|\xi_h\|_{0,K} \\
&\leq (3 + \sqrt{5})c_Q^{-3}c_S^{-1}C_{inv}^2R h^{-3} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2.
\end{aligned}$$

Hence, it follows that

$$|I_2| \leq C_B^{(1)} h^{-3} \|\xi_h\|_{0,\Omega}^2, \quad (4.39)$$

where $C_B^{(1)} := (3 + \sqrt{5})c_Q^{-3}c_S^{-1}C_{inv}^2R$.

(2) We now deal with the second term on the right-hand side of Equation (4.37) which we rewrite as follows:

$$\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(v_h)D^2v_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \\
&\quad - \mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(w_h)D^2w_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E) ds \\
&= \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(v_h)D^2\xi_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E ds}_{=II_1} \\
&+ \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\mathbf{A}}_2(v_h) - \underline{\mathbf{A}}_2(w_h))D^2w_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E ds}_{=II_2} \\
&+ \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\mathbf{A}}_2(w_h)D^2w_h\}_E \mathbf{n}_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla \xi_h]_E ds}_{=II_3},
\end{aligned}$$

where $\tilde{\omega}(\nabla v_h, \nabla w_h) := \omega(\nabla v_h)^{-1/4} - \omega(\nabla w_h)^{-1/4}$. In view of Equations (3.5), (3.8b), (3.12), (3.16), (4.15), Hölder's inequality, the Cauchy-Schwarz inequality, the inverse inequality (4.12), and the trace inequality (4.13b) we can estimate II_1 from above as follows:

$$\begin{aligned}
|II_1| &\leq 8 \sum_{E \in \mathcal{E}_h} \int_E \{|D^2\xi_h\}_E \{|\nabla \xi_h\}_E ds \\
&\leq 4 \sum_{E \in \mathcal{E}_h} \left(\int_E \{|D^2\xi_h|^2\}_E ds \right)^{1/2} \left(\int_E \{|\nabla \xi_h|^2\}_E ds \right)^{1/2} \\
&\leq 4 \sum_{K \in \mathcal{T}_h} \left(\int_{\partial K} |D^2\xi_h|^2 ds \right)^{1/2} \left(\int_{\partial K} |\nabla \xi_h|^2 ds \right)^{1/2} \\
&\leq 4c_Q^{-1}h^{-1} \sum_{K \in \mathcal{T}_h} h_K^{1/2} \|D^2\xi_h\|_{0,\partial K} h_K^{1/2} \|\nabla \xi_h\|_{0,\partial K} \\
&\leq 4c_Q^{-1}C_T^2h^{-1} \left(\sum_{K \in \mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^2 \right)^{1/2} \\
&\leq 4c_Q^{-4}C_{inv}^2C_T^2h^{-4} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2 \\
&\leq 4c_Q^{-4}C_{inv}^2C_T^2h^{-4} \|\xi_h\|_{0,\Omega}^2.
\end{aligned}$$

Hence, we obtain

$$|II_1| \leq C_B^{(2)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad (4.40)$$

where $C_B^{(2)} := 4c_Q^{-4} C_{inv}^2 C_T^2$. Likewise, for II_2 we have

$$\begin{aligned} |II_2| &\leq 4 \left(\frac{5}{2} + \sqrt{5} \right) \sum_{E \in \mathcal{E}_h} \int_E \{ |\nabla \xi_h|^2 \}_E \{ |D^2 w_h| \}_E ds \\ &\leq 2 \left(\frac{5}{2} + \sqrt{5} \right) \sum_{E \in \mathcal{E}_h} \left(\int_E \{ |\nabla \xi_h|^4 \}_E ds \right)^{1/2} \left(\int_E \{ |D^2 w_h|^2 \}_E ds \right)^{1/2} \\ &\leq 2 \left(\frac{5}{2} + \sqrt{5} \right) \sum_{K \in \mathcal{T}_h} \left(\int_{\partial K} |\nabla \xi_h|^4 ds \right)^{1/2} \left(\int_{\partial K} |D^2 w_h|^2 ds \right)^{1/2} \\ &= 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-1} h^{-1} \sum_{K \in \mathcal{T}_h} h_K^{1/2} \|\nabla \xi_h\|_{0,4,\partial K}^2 h_K^{1/2} \|D^2 w_h\|_{0,\partial K} \\ &\leq 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-1} C_T^2 h^{-1} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,4,K}^4 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|D^2 w_h\|_{0,K}^2 \right)^{1/2} \\ &\leq 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-1} c_S^{-1/2} C_{inv} C_T^2 R h^{-2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^4 \right)^{1/2} \\ &\leq 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R h^{-4} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2. \end{aligned}$$

It follows that

$$|II_2| \leq C_B^{(3)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad (4.41)$$

where $C_B^{(3)} := 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$. Finally, II_3 can be bounded from above in much the same way as II_2 . We get

$$|II_3| \leq C_B^{(4)} h^{-4} \|\xi_h\|^2, \quad (4.42)$$

where $C_B^{(4)} := 2c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$.

(3) For the third term on the right-hand side of Equation (4.37) we have

$$\begin{aligned} &\sum_{E \in \mathcal{E}_h} \int_E (\mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(v_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \\ &\quad - \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E) ds \\ &= \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ (\underline{\mathbf{A}}_2(v_h) - \underline{\mathbf{A}}_2(w_h)) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E ds}_{=III_1} \\ &\quad + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla v_h]_E ds}_{=III_2} \\ &\quad + \underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{ \underline{\mathbf{A}}_2(w_h) D^2 \xi_h \}_E \mathbf{n}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E ds}_{=III_3}. \end{aligned}$$

The three terms can be estimated from above in a similar way as the corresponding terms in II. We obtain

$$|III_1| \leq C_B^{(5)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |III_2| \leq C_B^{(6)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |III_3| \leq C_B^{(7)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad (4.43)$$

where $C_B^{(5)} := 2 \left(\frac{5}{2} + \sqrt{5} \right) c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$, $C_B^{(6)} := 2c_Q^{-3} c_S^{-1/2} C_{inv}^3 C_T^2 R$, and $C_B^{(7)} := C_B^{(6)}$.

(4) For the fourth term on the right-hand side of Equation (4.37) we obtain

$$\begin{aligned} & \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E (\mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla v_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \\ & \quad - \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E) ds \\ &= \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla \xi_h]_E ds}_{=IV_1} \\ & \quad + \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla w_h]_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla \xi_h]_E ds}_{=IV_2} \\ & \quad + \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\tilde{\omega}(\nabla v_h, \nabla w_h) \nabla w_h]_E \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla \xi_h]_E ds}_{=IV_3}. \end{aligned} \quad (4.44)$$

In view of Equation (3.13), the first term IV_1 can be further split according to

$$\begin{aligned} IV_1 &= \alpha \underbrace{\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E ds}_{=IV_{11}} \\ & \quad + \alpha \underbrace{\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E ds}_{=IV_{12}} \\ & \quad + \alpha \underbrace{\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E ds}_{=IV_{13}} \\ & \quad + \alpha \underbrace{\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\nabla v_h)^{-1/4}\}_E [\nabla \xi_h]_E \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4}]_E \{\nabla \xi_h\}_E ds}_{=IV_{14}}. \end{aligned}$$

For IV_{11} , setting $E_1 := E_+$ and $E_2 := E_-$ for $E \in \mathcal{E}_h(\Omega)$, we have

$$\begin{aligned} IV_{11} &\geq \alpha \sum_{E \in \mathcal{E}_h(\Omega)} \left(1 + \frac{1}{2} \sum_{i=1}^2 \|\nabla v_h\|_{0,\infty,E_i}^2 \right)^{-1/2} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds \\ & \quad + \alpha \sum_{E \in \mathcal{E}_h(\Gamma)} (1 + \|\nabla v_h\|_{0,\infty,E}^2)^{-1/2} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds. \end{aligned}$$

Taking advantage of Equations (3.8b), (3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2 -functions (4.14), it follows that for $E \in \mathcal{E}_h(\partial K)$ it holds

$$\begin{aligned} \|\nabla v_h\|_{0,\infty,E} &\leq \|\nabla v_h\|_{0,\infty,K} \leq c_S^{-1/2} C_{inv} h_K^{-1} \|\nabla v_h\|_{0,K} \\ &\leq c_Q^{-1} c_S^{-1/2} C_{inv} h^{-1} \|\nabla v_h\|_{0,\Omega} \leq c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} h^{-1} \|v_h\|_{2,h,\Omega} \leq c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} h^{-1} R, \end{aligned}$$

and hence, observing $h < 1$, we get

$$\begin{aligned} (1 + \|\nabla v_h\|_{0,\infty,E}^2)^{-1/2} &\geq (1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2 h^{-2})^{-1/2} \\ &= (h^2 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2} h \geq (1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2} h. \end{aligned}$$

Consequently, we obtain

$$IV_{11} \geq \alpha \gamma_M^{(2)} h \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla \xi_h]_E|^2 ds, \quad (4.45)$$

where $\gamma_M^{(2)} := \alpha(1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2}$. The remaining terms IV_{1i} , $2 \leq i \leq 4$, can be estimated from above similarly as the corresponding terms in Theorem 4.2:

$$|IV_{12}| \leq C_B^{(8)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |IV_{13}| \leq C_B^{(9)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |IV_{14}| \leq C_B^{(10)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad (4.46)$$

where $C_B^{(8)} := 2\alpha c_Q^{-4} c_R^{-1} C_{inv}^2 C_T^2$ and $C_B^{(9)} = C_B^{(10)} := 2C_B^{(8)}$. The remaining two terms IV_2 and IV_3 can be estimated from above in the same way. Using Equations (3.8a), (3.8b), (4.19a), the inverse inequality (4.12), the trace inequality (4.13a), the Cauchy-Schwarz inequality, and observing

$$\left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \leq \|w_h\|_{2,h,\Omega} \leq R,$$

we obtain

$$\begin{aligned} |IV_2| &\leq 4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E| \{|\nabla \xi_h\|_E^2\} ds \\ &\leq 4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E \in \mathcal{E}_h} h_E^{-1/2} \left(\int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \left(\int_E \{|\nabla \xi_h\|_E^4\} ds \right)^{1/2} \\ &\leq 2\alpha c_Q^{-3/2} c_R^{-1/2} h^{-3/2} \left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\nabla w_h]_E|^2 ds \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \xi_h\|_{0,4,\partial K}^4 \right)^{1/2} \\ &\leq 2\alpha c_Q^{-3/2} c_R^{-1/2} C_T R h^{-3/2} \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,4,K}^4 \right)^{1/2} \\ &\leq 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \left(\sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^4 \right)^{1/2} \\ &\leq 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2. \end{aligned}$$

Hence, it follows that

$$|IV_2| \leq C_B^{(11)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \quad (4.47)$$

where $C_B^{(11)} := 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R$. Moreover, we get

$$|IV_3| \leq C_B^{(12)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \quad (4.48)$$

where $C_B^{(12)} := C_B^{(11)}$. Setting $C_B := \sum_{i=1}^{12} C_B^{(i)}$ and observing (4.34) as well as $h < 1$, it follows from Equations (4.36) to (4.48) that

$$\begin{aligned} & \langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \\ & \geq (1 - C_\Delta C_B h^k) \|\xi_h\|_{0,\Omega}^2 + \min(\gamma_M^{(1)}, \alpha \gamma_M^{(2)}) h^3 \|\xi_h\|_{2,h,\Omega}^2. \end{aligned} \quad (4.49)$$

We choose $h_{min} > 0$ such that

$$q := C_\Delta C_B h_{min}^k < 1 \quad \text{and} \quad \min(\gamma_M^{(1)}, \alpha \gamma_M^{(2)}) h_{min}^3 < 1 - q. \quad (4.50)$$

Then, for $h \leq h_{min}$ (4.35) follows from Equations (4.49) and (4.50) with

$$\gamma(h, R) := \min(\gamma_M^{(1)}, \alpha \gamma_M^{(2)}) h^3. \quad \blacksquare \quad (4.51)$$

Corollary 4.1 *Assume that u_h^{m-1} satisfies*

$$\|u_h^{m-1}\|_{0,\Omega} \leq \frac{\Gamma(R)^2}{\gamma(R)} \left(1 - \sqrt{1 - \frac{\gamma(R)^2}{\Gamma(R)^2}} \right) R$$

for some $R > 0$ and that (4.34) holds true. Then, for sufficiently small grid size h , the C^0 IPDG approximation (3.14) has a unique solution $u_h^m \in B_h(0, R)$.

Proof. Using the Lipschitz continuity (4.22) and the strong monotonicity (4.35) of the nonlinear operator A_h^{DG} , the result follows from the nonlinear analogue of the Lax-Milgram Lemma (Theorem 4.1). \blacksquare

Remark 4.1 If we choose $h_{min} > 0$ such that Equation (4.49) is satisfied as well as $h_{min} < \beta C_\Delta C_A$, for $h \leq h_{min}$ we have $\Gamma(h, R) = \beta C_\Delta C_A h^{-1}$ in Theorem 4.2 and the application of Theorem 4.1 for $V = V_h$ and $A = A_h^{DG}$ implies that the fixed point operator T is a contraction as long as

$$\rho < 2 \frac{\gamma(h, R)}{\Gamma(h, R)^2} = 2 \frac{\min(\gamma_M^{(1)}, \alpha \gamma_M^{(2)})}{C_\Delta^2 C_A^2} h^5. \quad (4.52)$$

In other words, the contraction property degenerates for $h \rightarrow 0$. This reflects the very singular character of the fourth-order TVF.

ACKNOWLEDGMENTS

The work of the authors has been supported by the National Science Foundation grant DMS-1520886.

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