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Characterizing Determinacy in Kleene Algebras

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Abstract

Elements of Kleene algebras can be used, among other ways, as abstractions of the input-output semantics of nondeterministic programs or as models for the association of pointers with their target objects. One is interested in a notion of determinacy, in the first case, to distinguish deterministic programs and, in the second case, since it does not make sense for a pointer to point to two different objects. We discuss several candidate notions of determinacy and clarify their relationship. Some characterizations that are equivalent when the underlying Kleene algebra is an (abstract) relation algebra are not equivalent for general Kleene algebras.

1 Introduction

Elements of Kleene algebras can be used, among other ways, as abstractions of the input-output semantics of nondeterministic programs [5] or as models for the association of pointers with their target objects in the style of [11]. One is interested in a notion of determinacy, in the first case, to distinguish deterministic programs and, in the second case, since it does not make sense for a pointer to point to two different objects. We discuss several candidate notions of determinacy and clarify their relationship. Some characterizations that are equivalent when the underlying Kleene algebra is an (abstract) relation algebra are not equivalent for general Kleene algebras.

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2 Kleene Algebras

In our definitions we follow [4], since we want to admit general recursive definitions, not just the Kleene star. We are well aware that there are different definitions (see e.g. [8]).

Definition 1 *A* (standard) Kleene algebra (KA) *or* S-algebra *is a sixtuple* $(K, \leq, 0, \top, \cdot, 1)$ *satisfying the following properties:*

(a) $(K, \leq)$ *is a complete lattice with least element* 0 *and greatest element* $\top$. *The supremum of a subset* $L \subseteq K$ *is denoted by* $\sqcup L$. *The supremum of two elements* $x, y \in K$ *is denoted by* $x + y$.
(b) $(K, \cdot, 1)$ *is a monoid.*
(c) The operation $\cdot$ *is universally disjunctive (i.e. distributes through arbitrary suprema) in both arguments.*

This kind of structure is also known as a *unital quantale* (see e.g. [3]).

Example 2 Consider an alphabet A. Then A^* is the set of all finite words over A, $\bullet$ denotes concatenation, extended pointwise to sets of words, and ε the empty word. As customary in formal language theory, we identify a singleton language with its only element. Then perhaps the best-known example of a KA is LAN $\triangleq$ $(\mathcal{P}(A^*), \subseteq, \emptyset, A^*, \bullet, \varepsilon)$, the algebra of formal languages. A related KA is the algebra PAT of path sets in a directed graph under path concatenation (see e.g. [10] for a precise definition).

Another important KA is REL $\triangleq$ $(\mathcal{P}(M \times M), \subseteq, \emptyset, M \times M, ;, I)$, the algebra of homogeneous binary relations over some set M under relational composition $;$.

Definition 3 *(a) A* Boolean algebra *is a distributive and complemented lattice. The complement of an element* x *is denoted by* $\overline{x}$.
(b) A Boolean algebra is complete *if its underlying lattice is complete.*
(c) An atom *of a lattice with least element* 0 *is a minimal element in the set of elements different from* 0. *The set of all atoms of a lattice with* 0 *is denoted by* At. *For an element* x *of such a lattice,* $\mathsf{At}(x) \triangleq \{a \in \mathsf{At} : a \leq x\}$ *is the set of* atoms of x.
(d) A lattice with 0 *is* atomic *if for every element* x *we have* $x = \sqcup \mathsf{At}(x)$.
(e) A KA is called Boolean *if its underlying lattice is a Boolean algebra (and hence a complete Boolean algebra). It is called* atomic *if its underlying lattice is atomic.*

Example 4 More generally than the concrete relation algebra REL, every abstract relation algebra is a KA. Such an *abstract relation algebra* (see e.g. [14]) is a tuple RA $= (R, \leq, ^-, 0, \top, ;, 1, ^\smile)$, where

(a) $(R, \leq, \overline{}, 0, \top)$ is a complete Boolean algebra with complement operation $\overline{}$, least element 0 and greatest element $\top$;
(b) $(R, ;, 1)$ is a monoid;
(c) Tarski's rule $x \neq 0 \Rightarrow \top ; x ; \top = \top$ holds;
(d) $\breve{\ } : R \to R$ is a unary operation such that Dedekind's rule

$$x ; y \sqcap z \ \leq \ (x \sqcap z ; y^{\smile}) ; (y \sqcap x^{\smile} ; z)$$

is satisfied. Equivalently, one may postulate the Schröder rule

$$x ; y \ \leq \ z \Leftrightarrow x^{\smile} ; \overline{z} \ \leq \ \overline{y} \Leftrightarrow \overline{z} ; y^{\smile} \ \leq \ \overline{x} \ .$$

The elements of R are called *abstract relations*; the operation $\breve{\ }$ forms the *converse* of a relation, whereas ; is called *relational composition*. It is customary to use the convention that ; binds tighter than $+$ and $\sqcap$. The reduct $(R, \leq, 0, \top, ;, 1)$ forms a Boolean KA.

For the remainder of the paper, we assume KAs to be Boolean.

3 Types

Definition 5 *A* type *of a Boolean KA is an element t with $t \leq 1$. The* negation *of a type t is $\neg t \triangleq \overline{t} \sqcap 1$.*

Note that, by monotonicity, the set of types is closed under composition. Moreover, it forms again a complete Boolean algebra. The definition of types is best illustrated in the KA REL. There, a type corresponds to a subset $T \subseteq M$ represented as the partial identity relation $I_T \triangleq \{(x, x) : x \in T\} \subseteq I$. Now, restriction of a relation $R \subseteq M \times M$ to arguments of type T, i.e. the relation $R \ \cap \ T \times M$, can also be described by composing R with I_T from the left: $R \ \cap \ T \times M \ = \ I_T ; R$. Similarly, co-restriction is modeled by composing a partial identity from the right. Finally, consider types $S, T \subseteq M$ and binary relation $R \subseteq M \times M$. Then $R \subseteq S \times T \ \Leftrightarrow \ I_S ; R ; I_T = R$. In other words, the "default typing" $M \times M$ of R can be narrowed down to $S \times T$ iff restriction to S and co-restriction to T do not change R. These observations are the basis for our view of types as subidentities and our algebraic treatment of restriction and co-restriction. For a different, but related, approach see [8].

Lemma 6 *Assume a Boolean KA. Then the following hold:*

(a) All types are idempotent, i.e. $t \leq 1 \Rightarrow t \cdot t = t$.
(b) The infimum of two types is their product: $s, t \leq 1 \Rightarrow s \cdot t = s \sqcap t$. In particular, all types commute under the $\cdot$ operation.
(c) $s, t \leq 1 \Rightarrow (s \sqcap t) \cdot a = s \cdot a \sqcap t \cdot a$.

(d) $t \leq 1 \Rightarrow \overline{t \cdot \top} = \neg t \cdot \top$.
(e) *Restriction by a type can be expressed as a meet:* $t \leq 1 \Rightarrow t \cdot a = a \sqcap t \cdot \top$.
(f) *Restriction by a type distributes through meet and can be shifted between meet partners:* $t \leq 1 \Rightarrow t \cdot (a \sqcap b) = t \cdot a \sqcap t \cdot b = t \cdot a \sqcap b = a \sqcap t \cdot b$.
(g) *For all families* L *of types,* $(\sqcap L) \cdot \top = \sqcap (L \cdot \top)$.

PROOF. We first note that by monotonicity and neutrality, for $s, t \leq 1$ we have that $s \cdot t \leq s \cdot 1 = s$ and, symmetrically, $s \cdot t \leq t$, so that $s \cdot t \leq s \sqcap t$.

(a) Using the above remark together with Boolean laws, neutrality, disjunctivity and monotonicity, we get $t = (t + \neg t) \cdot t = t \cdot t + \neg t \cdot t \leq t \cdot t \leq t$.
(b) We have already shown that $s \cdot t$ is a lower bound for s and t. Using (a) and monotonicity, we also have $s \sqcap t = (s \sqcap t) \cdot (s \sqcap t) \leq s \cdot t$. Hence $s \cdot t = s \sqcap t$.
(c)

$$
\begin{array}{ll}
 & (s \sqcap t) \cdot a \\
\leq & \quad \{\!\![\text{ monotonicity }]\!\!\} \\
 & s \cdot a \sqcap t \cdot a \\
= & \quad \{\!\![\text{ neutrality and Boolean algebra }]\!\!\} \\
 & (s + \neg s) \cdot (s \cdot a \sqcap t \cdot a) \\
= & \quad \{\!\![\text{ disjunctivity }]\!\!\} \\
 & s \cdot (s \cdot a \sqcap t \cdot a) + \neg s \cdot (s \cdot a \sqcap t \cdot a) \\
\leq & \quad \{\!\![\text{ monotonicity and associativity }]\!\!\} \\
 & s \cdot t \cdot a + \neg s \cdot s \cdot a \\
= & \quad \{\!\![\text{ by strictness, since } \neg s \cdot s = \neg s \sqcap s = 0 \text{, and by (b) }]\!\!\} \\
 & (s \sqcap t) \cdot a \ .
\end{array}
$$

(d) First, by (c), $t \cdot \top \sqcap \neg t \cdot \top = (t \sqcap \neg t) \cdot \top = 0 \cdot \top = 0$. Second, $t \cdot \top + \neg t \cdot \top = (t + \neg t) \cdot \top = 1 \cdot \top = \top$.
(e)

$$
\begin{array}{ll}
 & t \cdot \top \sqcap a \\
= & \quad \{\!\![\text{ Boolean algebra } (\top = a + \overline{a}) \text{, and disjunctivity }]\!\!\} \\
 & (t \cdot a + t \cdot \overline{a}) \sqcap a \\
= & \quad \{\!\![\text{ distributivity }]\!\!\} \\
 & (t \cdot a \sqcap a) + (t \cdot \overline{a} \sqcap a) \\
= & \quad \{\!\![\ t \cdot \overline{a} \leq \overline{a} \text{ by monotonicity and neutrality }]\!\!\} \\
 & (t \cdot a \sqcap a) + 0 \\
= & \quad \{\!\![\ t \cdot a \leq a \text{ by monotonicity and neutrality }]\!\!\} \\
 & t \cdot a \ .
\end{array}
$$

(f) By (e) we have $t \cdot (a \sqcap b) = a \sqcap b \sqcap t \cdot \top$, from which the claim is immediate by commutativity, associativity and idempotence of $\sqcap$ as well as (e) again.
(g) We show $\overline{(\sqcap L) \cdot \top} = \overline{\sqcap (L \cdot \top)}$ by using de Morgan, (d), disjunctivity, and de Morgan and (d) again, in this order:

$$
\begin{aligned}
\overline{\sqcap (L \cdot \top)} &= \sqcup \{\overline{t \cdot \top} : t \in L\} = \sqcup \{\neg t \cdot \top : t \in L\} \\
&= (\sqcup \{\neg t : t \in L\}) \cdot \top = (\neg \sqcap \{t : t \in L\}) \cdot \top = \overline{(\sqcap L) \cdot \top} \ .
\end{aligned}
$$

4 Domain and Codomain

Definition 7 *In a KA* $(K, \leq, \top, \cdot, 0, 1)$*, we can define, for* $a \in K$*, the* domain $\ulcorner a$ *via the Galois connection* $\forall\, t : t \leq 1 \Rightarrow (\ulcorner a \leq t \overset{\text{def}}{\Leftrightarrow} a \leq t \cdot \top)$.

This is well-defined because of Lemma 6(g) [1,2]. Hence the operation $\ulcorner$ is universally disjunctive and therefore monotonic and strict. Moreover the definition implies $a \leq \ulcorner a \cdot \top$. The *co-domain* $a\urcorner$ is defined symmetrically by the Galois connection $a\urcorner \leq t \overset{\text{def}}{\Leftrightarrow} a \leq \top \cdot t$. We list a number of useful properties of the domain operation (see also [1] and consider [12] for the proofs); analogous ones hold for the co-domain operation.

Lemma 8 *Consider a KA* $(K, \leq, \top, \cdot, 0, 1)$ *and* $a, b, c \in K$.

(a) $\ulcorner a = \sqcap\{t : t \leq 1 \wedge t \cdot a = a\}$.
(b) $\ulcorner a \cdot a = a$.
(c) $t \leq 1 \wedge t \cdot a = a \Rightarrow \ulcorner a \leq t$.
(d) $\ulcorner(a \cdot b) \leq \ulcorner a$.
(e) $t \leq 1 \Leftrightarrow \ulcorner t = t$.
(f) $\ulcorner \top = 1$.
(g) $t \leq 1 \Rightarrow \ulcorner(t \cdot a) = t \cdot \ulcorner a$.
(h) $\ulcorner(a \cdot \top) = \ulcorner a$.
(i) $a \cdot \top \leq \ulcorner a \cdot \top$.
(j) $\ulcorner(a \cdot b) \leq \ulcorner(a \cdot \ulcorner b)$.
(k) $a\urcorner \sqcap \ulcorner b = 0 \Rightarrow a \cdot b = 0$.
(l) $\ulcorner a = 0 \Leftrightarrow a = 0$.

5 Locality of Composition

Note that the converse inequality of Lemma 8(j) does not follow from our axiomatization. A counterexample will be given in Section 8.2. Its essence is that composition does not work "locally" in that only the "near end", i.e. the domain, of the right factor of a composition does decide "composability". This observation is the motivation for the term "local composition" defined below.

Definition 9 *A KA has* left-local composition *if it satisfies* $\ulcorner b = \ulcorner c \Rightarrow \ulcorner(a \cdot b) = \ulcorner(a \cdot c)$. Right-locality *of composition is defined symmetrically. A KA has* local composition *if its composition is both left-local and right-local.*

Lemma 10 *(a) A KA has left-local composition iff it satisfies*

$$\ulcorner(a \cdot b) = \ulcorner(a \cdot \ulcorner b)\ . \tag{1}$$

(b) If a KA has left-local composition then $\ulcorner(\ulcorner a \cdot b) = \ulcorner a \cdot \ulcorner b = \ulcorner a \sqcap \ulcorner b$.

PROOF.

(a) $(\Rightarrow)$ Immediate from the assumption, since by Lemma 8(e) $\ulcorner(\ulcorner b) = \ulcorner b$.

(⇐) Assume $\ulcorner b = \ulcorner c$. Using Equation (1) twice, we get

$$\ulcorner(a \cdot b) = \ulcorner(a \cdot \ulcorner b) = \ulcorner(a \cdot \ulcorner c) = \ulcorner(a \cdot c) \ .$$

(b) By Equation (1) and Lemmas 8(e) and 6(b),

$$\ulcorner(\ulcorner a \cdot b) = \ulcorner(\ulcorner a \cdot \ulcorner b) = \ulcorner a \cdot \ulcorner b = \ulcorner a \sqcap \ulcorner b \ .$$

Analogous properties hold for right-locality. In the sequel we only consider KAs with local composition. All examples given in Section 2 satisfy that property.

We conclude this section by establishing a Galois connection between domain and range. It will be useful in Section 6.3 about modal operators.

Lemma 11 *If $s, t \leq 1$, then $\ulcorner(a \cdot t) \leq \neg s \ \Leftrightarrow\ (s \cdot a)\urcorner \leq \neg t$.*

PROOF. By Boolean algebra and Lemma 6(b), the claim is equivalent to $s \cdot \ulcorner(a \cdot t) = 0 \ \Leftrightarrow\ (s \cdot a)\urcorner \cdot t = 0$. By Lemma 8(e) and local composition, $s \cdot \ulcorner(a \cdot t) = \ulcorner(s \cdot \ulcorner(a \cdot t)) = \ulcorner(s \cdot a \cdot t)$. Symmetrically, $(s \cdot a)\urcorner \cdot t = (s \cdot a \cdot t)\urcorner$. Now the claim is immediate from Lemma 8(l).

6 Candidate Characterizations of Determinacy

6.1 Candidates from Relation Algebra

In a relation algebra, an element R is called a *(partial) function* or a *map* or *deterministic* iff it satisfies $R^{\smile} ; R \subseteq I$. By the Schröder laws this is equivalent to the requirement $R \,;\, \overline{I} \subseteq \overline{R}$.

It is well known that an element is a function if and only if it is left-distributive through intersection [14]. However, the equivalence is not Kleene valid, as we shall show in Section 8.3. In Kleene algebras, left-distributivity is only equivalent to a stronger property that results by generalizing the constant I in the inclusion $R \,;\, \overline{I} \subseteq \overline{R}$ to a variable (after making it visible in the right hand side, see the definition of SC below). These observations lead to our first three candidates for characterizations of determinate objects (the formula involving converse is not usable in Kleene algebras). We attach names to the characterizing predicates for easier reference.

$$\begin{array}{rll} \mathrm{LD}(a) & \stackrel{\mathrm{def}}{\Leftrightarrow} \forall\, b, c : a \cdot (b \sqcap c) = a \cdot b \sqcap a \cdot c & \text{(left-distributivity)} \\ \mathrm{SC}(a) & \stackrel{\mathrm{def}}{\Leftrightarrow} \forall\, b : a \cdot \overline{b} \leq \overline{a \cdot b} & \text{(subsumption of complement)} \\ \mathrm{SC1}(a) & \stackrel{\mathrm{def}}{\Leftrightarrow} a \cdot \overline{1} \leq \overline{a} & \text{(subsumption of complement of 1)} \end{array}$$

Lemma 12 $\mathrm{LD}(a) \Leftrightarrow \mathrm{SC}(a)$.

PROOF. $(\Rightarrow)$ $0 = a \cdot 0 = a \cdot (c \sqcap \overline{c}) = a \cdot c \sqcap a \cdot \overline{c}$ by $\mathrm{LD}(a)$.

$$
\begin{array}{ll}
(\Leftarrow) & a \cdot b \ \sqcap \ a \cdot c \\
= & \quad \{\![\text{ Boolean algebra }]\!\} \\
 & a \cdot ((b \sqcap c) + (b \sqcap \overline{c})) \ \sqcap \ a \cdot c \\
= & \quad \{\![\text{ distributivities }]\!\} \\
 & (a \cdot (b \sqcap c) \sqcap \ a \cdot c) + (a \cdot (b \sqcap \overline{c}) \sqcap a \cdot c) \\
= & \quad \{\![\text{ lattice algebra }]\!\} \\
 & a \cdot (b \sqcap c) + (a \cdot (b \sqcap \overline{c}) \sqcap a \cdot c) \\
= & \quad \{\![\text{ since } a \cdot (b \sqcap \overline{c}) \sqcap a \cdot c \leq a \cdot \overline{c} \sqcap a \cdot c = 0 \text{ by } \mathrm{SC}(a) \ ]\!\} \\
 & a \cdot (b \sqcap c) \ .
\end{array}
$$

Moreover, we clearly have $\mathrm{SC}(a) \Rightarrow \mathrm{SC1}(a)$ (set $b = 1$).

In Section 8.3, we show that the reverse implication is not valid in all Kleene algebras. However, it holds in LAN, PAT and RA. To understand this, let us elaborate on the case of the Kleene algebra LAN of formal languages over an alphabet A. There we have $\overline{1} = A^+$, and so for $U \subseteq A^*$ we get $\mathrm{SC1}(U) \Leftrightarrow U \bullet A^+ \subseteq \overline{U}$. In other words, a proper extension of a word in U must not lie in U again. This is equivalent to U being a prefix-free language (i.e. none of the strings of U is a proper prefix of another; in coding theory, this is known as the Fano condition; in process algebra, sets with this property are called *prefix-free* or *prefix antichains*). The same applies to the algebra PAT of sets of paths in a graph, modeled as sets of strings of nodes.

Assume now $\mathrm{SC1}(U)$ and $x \in U \bullet V \cap U \bullet W$ for $V, W \subseteq A^*$, say $x = u_1 \bullet v = u_2 \bullet w$ for some $u_1, u_2 \in U$, $v \in V$ and $w \in W$. By local linearity of the prefix relation we obtain that u_1 must be a prefix of u_2 or the other way around. By prefix-freeness of U this means $u_1 = u_2$ and cancellativity of $\bullet$ shows $v = w \in V \cap W$. Therefore also $x \in U \bullet (V \cap W)$, i.e. $\mathrm{LD}(U)$ holds. But this is equivalent to $\mathrm{SC}(U)$ as stated in the above lemma.

6.2 Domain-Oriented Characterizations

In view of the previous section it appears that LD, SC and SC1 are not appropriate characterizations of (partial) functions, since in a prefix-free set of paths still different paths may emanate from the same starting node. Therefore, a function should rather be characterized in a domain-oriented way: every point in the domain should have a unique "extension".

Now, in PAT, a node starts a unique path in a path set a iff removal of this path removes that node from the domain of a. Using the strict-order $<$ given by $c < d \overset{\text{def}}{\Leftrightarrow} c \leq d \wedge c \neq d$, this can be captured in a purely order-theoretic way by the property

$$\mathrm{DD}(a) \overset{\text{def}}{\Leftrightarrow} \forall\, b : b < a \;\Rightarrow\; \ulcorner b < \ulcorner a \qquad \text{(decrease of domain).}$$

Note that all atoms satisfy DD. Property DD(a) is easily shown to be equivalent to ED(a) and SO(a), where

$$\begin{aligned}
&\mathrm{ED}(a) \overset{\text{def}}{\Leftrightarrow} \forall\, b : b \leq a \wedge \ulcorner b = \ulcorner a \;\Rightarrow\; b = a && \text{(equality of domain),}\\
&\mathrm{SO}(a) \overset{\text{def}}{\Leftrightarrow} \ulcorner : \{b : b \leq a\} \to \{t : t \leq \ulcorner a\} && \\
&\qquad\qquad \text{is an order-isomorphism} && \text{(subobject lattice).}
\end{aligned}$$

A calculationally more pleasant characterization is provided by

Lemma 13 DD(a) $\Leftrightarrow$ CD(a)*, where* CD(a) (characterization by domain) *is given by one of the following four equivalent formulations:*

(a) $\forall b : b \leq a \;\Rightarrow\; b = \ulcorner b \cdot a$.

(b) $\forall\, b : b \sqcap a = \ulcorner(b \sqcap a) \cdot a$.

(c) $\forall b : \ulcorner(b \sqcap a) \cdot a \leq b$.

(d) $\forall b : \ulcorner(b \sqcap a) \cdot \ulcorner(\overline{b} \sqcap a) = 0$.

PROOF. We first show the equivalence of the four formulations of CD. The implication (a) $\Rightarrow$ (b) is clear by $b \sqcap a \leq a$. For the reverse, for $b \leq a$ we have $b = b \sqcap a = \ulcorner(b \sqcap a) \cdot a = \ulcorner b \cdot a$.
(b) $\Leftrightarrow$ (c) follows, since by Lemma 8(b), Boolean algebra and monotonicity $b \sqcap a = \ulcorner(b \sqcap a) \cdot (b \sqcap a) \leq \ulcorner(b \sqcap a) \cdot a$ is valid anyway. Moreover, the reverse inequation can be simplified to the form stated, since $\ulcorner(b \sqcap a) \cdot a \leq a$ by monotonicity.
(c) $\Leftrightarrow$ (d) is shown using Boolean algebra and Lemmas 6(f), 8(l) and 8(g) by

$$\ulcorner(b \sqcap a) \cdot a \leq b \;\Leftrightarrow\; \ulcorner(b \sqcap a) \cdot a \sqcap \overline{b} = 0 \;\Leftrightarrow\; \ulcorner(b \sqcap a) \cdot (a \sqcap \overline{b}) = 0 \;\Leftrightarrow$$

$$\ulcorner(\ulcorner(b \sqcap a) \cdot (a \sqcap \overline{b})) = 0 \;\Leftrightarrow\; \ulcorner(b \sqcap a) \cdot \ulcorner(a \sqcap \overline{b}) = 0\ .$$

Finally we prove ED(a) $\Leftrightarrow$ CD(a). For ($\Leftarrow$) suppose $b \leq a$ and $\ulcorner b = \ulcorner a$. Then by CD($a$) and Lemma 8(b), we get $b = \ulcorner b \cdot a = \ulcorner a \cdot a = a$. For ($\Rightarrow$) let $c \triangleq b + \neg\ulcorner b \cdot a$. Because $b \leq a$ and by monotonicity, $c \leq a$. Also, $\ulcorner c = \ulcorner(b + \neg\ulcorner b \cdot a) = \ulcorner b + (\neg\ulcorner b \sqcap \ulcorner a) = \ulcorner b + \ulcorner a = \ulcorner a$, where the disjunctivity of $\ulcorner$, Lemma 8(g), the monotonicity of $\ulcorner$ with the assumption $b \leq a$, and Boolean algebra have been used. Then $c = a$ follows from ED(a), whence $\ulcorner b \cdot a = \ulcorner b \cdot c = \ulcorner b \cdot (b + \neg\ulcorner b \cdot a) = \ulcorner b \cdot b + \ulcorner b \cdot \neg\ulcorner b \cdot a = b$.

However, CD and DD are not equivalent to LD. In LAN an element a satisfies DD(a) iff it contains at most one word, whereas LD(a) is equivalent to prefix-freeness of a. So in LAN the properties CD and DD imply LD, but not the other way around. We show in section 8.3 that the implication does not hold for arbitrary Kleene algebras. In REL the properties CD and DD are equivalent to the other characterizations of deterministic relations. However, in the case of an abstract relation algebra in RA, DD does not imply LD, as will be shown in Section 8.4.

Lemma 14 LD *implies* DD *in* RA.

PROOF. Assume LD(a), which is equivalent to $a^{\smile} ; a \leq 1$ in RA, and $b \leq a$. We show CD(a). Since $b = \ulcorner b ; b \leq \ulcorner b ; a$, we only need to prove $\ulcorner b ; a \leq b$. Using relational algebra, Dedekind, Boolean algebra (since $b \leq a \leq a ; \top^{\smile}$), monotonicity and the assumption $a^{\smile} ; a \leq 1$, we get

$$\ulcorner b ; a = b ; \top \sqcap a \leq (b \sqcap a ; \top^{\smile}) ; (\top \sqcap b^{\smile} ; a) = b ; b^{\smile} ; a \leq b ; a^{\smile} ; a \leq b.$$

6.3 A Modal Characterization

The modal operators diamond and box are quantifiers about the successor states of a state in a transition system. But they can also be viewed as assertion transformers dealing with sets of states. The (forward) diamond operator assigns to a set of states t the set of all states that have a successor in t. The (forward) box operator is the dual of the diamond operator; it assigns to a set of states t the set of all states for which all successors lie in t. The backward modal operators are defined symmetrically. In the setting of KAs, the role of assertions or sets of states is played by types. Hence we can define modal operators as type transformers. The operators of dynamic logic are obtained by setting

$$\langle a \rangle t \triangleq \ulcorner(a \cdot t) , \qquad [a]t \triangleq \neg\langle a \rangle \neg t .$$

We note that $[a]t = a \to t$, where $a \to b \triangleq \neg\ulcorner(a \cdot \neg\ulcorner b)$ is called *type implication*, an operation which is useful for dealing with assertions in demonic semantics and which enjoys many useful properties, see [5]. Moreover, in relational semantics of imperative programs, $[a]t = \mathsf{wlp}.a.t$ [2,7].

We now carry over the well-known modal characterization of deterministic relations (see e.g. [13]) to elements of Kleene algebras and call an element a *modally deterministic* iff MD(a) holds, where (using Boolean algebra)

$$\begin{aligned} \mathrm{MD}(a) &\overset{\text{def}}{\Leftrightarrow} \forall\, t : \langle a \rangle t \leq [a]t \\ &\Leftrightarrow \forall\, t : \ulcorner(a \cdot t) \cdot \ulcorner(a \cdot \neg t) = 0 . \end{aligned} \qquad (2)$$

The following properties are easily checked:

Corollary 15 *(a)* $\langle a\rangle 0 = 0$.
(b) $\langle a\rangle 1 = \ulcorner a$.
(c) $[a]0 = \neg\ulcorner a$.
(d) $[a]1 = 1$.
(e) In particular, $\langle a\rangle 0 \le [a]0$ *and* $\langle a\rangle 1 \le [a]1$.
(f) Suppose that the only types are 0 *and* 1 *(such as e.g. in the algebra* LAN *of formal languages). Then* MD(a) *holds for all* a.
(g) $\langle \_ \rangle$ *is monotonic and* $[\_]$ *is antitonic, i.e. for* $a \le b$ *and* $t \le 1$ *we have* $\langle a\rangle t \le \langle b\rangle t$ *and* $[b]t \le [a]t$.

The modal characterization links to the domain-oriented characterizations as follows (for the relationship with our other characterizations see Section 8.7):

Lemma 16 *We have* CD(a) $\Rightarrow$ MD(a). *The reverse implication is not valid.*

PROOF. We use form (2) of MD(a) and calculate for type t:

$$
\begin{array}{ll}
& \ulcorner(a\cdot t)\cdot\ulcorner(a\cdot\neg t) \\
= & \quad \{\!\![\text{ by Lemma 6(f) }]\!\!\} \\
& \ulcorner(a\sqcap\top\cdot t)\cdot\ulcorner(a\sqcap\top\cdot\neg t) \\
= & \quad \{\!\![\text{ by Lemma 6(d) }]\!\!\} \\
& \ulcorner(a\sqcap\top\cdot t)\cdot\ulcorner(a\sqcap\overline{\top\cdot t}) \\
= & \quad \{\!\![\text{ by CD}(a)\text{ (Lemma 13(d), setting } b=\top\cdot t\text{) }]\!\!\} \\
& 0\ .
\end{array}
$$

To see that the reverse implication fails, consider a Kleene algebra (such as LAN) in which the only types are 0 and 1. Then we have DD(a) and hence, by Lemma 13, CD(a) iff a is an atom. However, by Corollary 15(f), MD(a) is always true.

7 Closure Properties

7.1 Downward Closure

A natural requirement is that a subobject of a determinate object should be determinate again:

Lemma 17 *The properties* SC*,* SC1*,* CD *and* MD *are closed under subobjects.*

PROOF. For SC and SC1 this is immediate from monotonicity, since we can restate these properties as $\forall\ b\ :\ a\cdot\overline{b}\sqcap a\cdot b\ =\ 0$ and $a\cdot\overline{1}\sqcap a\ =\ 0$, respectively. For CD, suppose $b \le a$ and $c \le b$. Then also $c \le a$, hence $c = \ulcorner c \cdot a = (\ulcorner c \sqcap \ulcorner b) \cdot a = \ulcorner c \cdot \ulcorner b \cdot a = \ulcorner c \cdot b$. Finally, for MD, the assertion is immediate from Corollary 15(g).

7.2 All Types are Determinate

Based on our original relational motivation, we would like to have that all types are determinate. By the previous section, to ensure this we only need to check that the largest type 1 satisfies all our characterizations. Fortunately, this indeed holds, as the following lemma shows.

Lemma 18 $\mathrm{LD}(1) \wedge \mathrm{SC1}(1) \wedge \mathrm{CD}(1) \wedge \mathrm{MD}(1)$.

PROOF. LD(1) and SC1(1) are trivial. For the third assertion, assume $t \le 1$. Then by Lemma 8(e) we have $t = t \cdot 1 = \ulcorner t \cdot 1$. Finally, for types s, t we have, again by Lemma 8(e), that$[1]t = t = \langle 1 \rangle t$.

7.3 Closure Under Compatible Join

In this section we show that all our characterizations are closed under join of pairwise compatible elements. Here, we call a, b *compatible* if they agree on the intersection of their domains, ie. if $\ulcorner b \cdot a = \ulcorner a \cdot b$. This is trivially satisfied if $\ulcorner a \cdot \ulcorner b = 0$. Moreover, every element is compatible with itself. A set $L \subseteq M$ is *compatible* if the elements of L are pairwise compatible.

As an auxiliary result we need the following lemma.

Lemma 19 *Let a, b be compatible.*

(a) $\ulcorner a \cdot b = a \sqcap b = \ulcorner b \cdot a$.
(b) If $\mathrm{LD}(a)$ *then* $a \cdot c \sqcap b \cdot d = \ulcorner b \cdot a \cdot (c \sqcap d)$.
(c) If $\mathrm{SC1}(a)$ *then* $a \cdot \overline{1} \sqcap b = 0$.
(d) If $\mathrm{CD}(a)$ *and* $c \le a$ *then* c *and* b *are compatible as well.*
(e) If $\mathrm{MD}(a)$ *and* $t \le 1$ *then* $\ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) = 0$.

PROOF.

(a) By Lemmas 8(b) and 6(f) we get $a \sqcap b = \ulcorner a \cdot a \sqcap b = a \sqcap \ulcorner a \cdot b = a \sqcap \ulcorner b \cdot a = \ulcorner b \cdot a$.
(b) $a \cdot c \sqcap b \cdot d$
$=$ {[by $a = \ulcorner a \cdot a$ and Lemma 6(f)]}
$a \cdot c \sqcap \ulcorner a \cdot b \cdot d$
$=$ {[compatibility]}

$$
\begin{array}{ll}
 & a \cdot c \sqcap {\urcorner}b \cdot a \cdot d \\
= & \quad \{\!\![\text{ by Lemma 6(f) }]\!\!\} \\
 & {\urcorner}b \cdot a \cdot c \sqcap {\urcorner}b \cdot a \cdot d \\
= & \quad \{\!\![\text{ by LD}(a) \text{ and downward closure of LD }]\!\!\} \\
 & {\urcorner}b \cdot a \cdot (c \sqcap d) \; .
\end{array}
$$

(c) By Lemma 6(f), compatibility, monotonicity and SC1(a),

$$a \cdot \overline{1} \sqcap b = a \cdot \overline{1} \sqcap {\ulcorner}a \cdot b = a \cdot \overline{1} \sqcap {\urcorner}b \cdot a \leq a \cdot \overline{1} \sqcap a = 0 \; .$$

(d) By CD(a), Lemma 8(g), compatibility, type commutativity, and CD(a),

$${\ulcorner}c \cdot b = {\ulcorner}({\ulcorner}c \cdot a) \cdot b = {\ulcorner}c \cdot {\ulcorner}a \cdot b = {\ulcorner}c \cdot {\urcorner}b \cdot a = {\urcorner}b \cdot {\ulcorner}c \cdot a = {\urcorner}b \cdot c \; .$$

(e)

$$
\begin{array}{ll}
 & {\ulcorner}(a \cdot t) \sqcap {\ulcorner}(b \cdot \neg t) \\
= & \quad \{\!\![\text{ by } a \cdot t \leq a \text{ and monotonicity }]\!\!\} \\
 & {\ulcorner}a \cdot {\ulcorner}(a \cdot t) \sqcap {\ulcorner}(b \cdot \neg t) \\
= & \quad \{\!\![\text{ by Lemma 6(f) }]\!\!\} \\
 & {\ulcorner}(a \cdot t) \sqcap {\ulcorner}a \cdot {\ulcorner}(b \cdot \neg t) \\
= & \quad \{\!\![\text{ by Lemma 10(b) }]\!\!\} \\
 & {\ulcorner}(a \cdot t) \sqcap {\ulcorner}({\ulcorner}a \cdot b \cdot \neg t) \\
= & \quad \{\!\![\text{ compatibility }]\!\!\} \\
 & {\ulcorner}(a \cdot t) \sqcap {\ulcorner}({\urcorner}b \cdot a \cdot \neg t) \\
\leq & \quad \{\!\![\text{ monotonicity }]\!\!\} \\
 & {\ulcorner}(a \cdot t) \sqcap {\ulcorner}(a \cdot \neg t) \\
= & \quad \{\!\![\text{ by MD}(a) \text{ }]\!\!\} \\
 & 0 \; .
\end{array}
$$

Lemma 20 *Let* P *range over* LD, SC, SC1, DD, ED, CD, SO, MD. *Let moreover* $L \subseteq K$ *be a compatible set. Then* $(\forall\, a \in L : \mathrm{P}(a)) \Rightarrow \mathrm{P}(\sqcup L)$.

PROOF. Due to the equivalences between some of the properties, it suffices to prove the lemma for LD, SC1, CD and MD.

(LD)

$$
\begin{array}{ll}
 & (\bigsqcup_{a \in L} a) \cdot c \sqcap (\bigsqcup_{a \in L} a) \cdot d \\
= & \quad \{\!\![\text{ distributivity of } \cdot \text{ over } \sqcup \text{ }]\!\!\} \\
 & (\bigsqcup_{a \in L} a \cdot c) \sqcap (\bigsqcup_{a \in L} a \cdot d) \\
= & \quad \{\!\![\text{ distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming }]\!\!\} \\
 & \bigsqcup_{a \in L} \bigsqcup_{b \in L} a \cdot c \sqcap b \cdot d \\
= & \quad \{\!\![\text{ by } \forall\, a \in L : \text{LD}(a) \text{ and Lemma 19(b) }]\!\!\} \\
 & \bigsqcup_{a \in L} \bigsqcup_{b \in L} {\urcorner}b \cdot a \cdot (c \sqcap d) \\
= & \quad \{\!\![\text{ distributivity of } \cdot \text{ over } \sqcup \text{ }]\!\!\} \\
 & (\bigsqcup_{a \in L} \bigsqcup_{b \in L} {\urcorner}b \cdot a) \cdot (c \sqcap d)
\end{array}
$$

$$
\begin{aligned}
= \quad & \{\!\![\text{ by } \forall\, a,b \in L : \ulcorner\!\!\ulcorner b \cdot a \le a \,\wedge\, \ulcorner a \cdot a \,=\, a \text{ }]\!\!\} \\
& (\bigsqcup_{a\in L} a) \cdot (c \sqcap d) \ .
\end{aligned}
$$

$$
\begin{aligned}
\text{(SC1)} \quad & (\bigsqcup_{a\in L} a) \cdot \bar{1} \sqcap (\bigsqcup_{a\in L} a) \\
= \quad & \{\!\![\text{ distributivity of } \cdot \text{ over } \sqcup \text{ }]\!\!\} \\
& (\bigsqcup_{a\in L} a \cdot \bar{1}) \sqcap (\bigsqcup_{a\in L} a) \\
= \quad & \{\!\![\text{ distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming }]\!\!\} \\
& \bigsqcup_{a\in L} \bigsqcup_{b\in L} a \cdot \bar{1} \sqcap b \\
= \quad & \{\!\![\text{ by } \forall\, a \in L : \mathrm{SC1}(a) \text{ and Lemma 19(c) }]\!\!\} \\
& 0 \ .
\end{aligned}
$$

(CD) We use Lemma 13(c) and calculate

$$
\begin{aligned}
& \ulcorner(c \sqcap \sqcup L) \cdot \sqcup L \\
= \quad & \{\!\![\text{ distributivity and disjunctivity }]\!\!\} \\
& \bigsqcup_{a\in L} \bigsqcup_{b\in L} \ulcorner(c \sqcap a) \cdot b \\
= \quad & \{\!\![\text{ by } c \sqcap a \le a \text{ and Lemma 19(d) }]\!\!\} \\
& \bigsqcup_{a\in L} \bigsqcup_{b\in L} \ulcorner\!\!\ulcorner b \cdot (c \sqcap a) \\
= \quad & \{\!\![\text{ disjunctivity and distributivity }]\!\!\} \\
& \ulcorner(\sqcup L) \cdot (c \sqcap \sqcup L) \\
\le \quad & \{\!\![\ \ulcorner(\sqcup L) \le 1 \text{ and Boolean algebra }]\!\!\} \\
& c \ .
\end{aligned}
$$

(MD) We use variant (2) of MD.

$$
\begin{aligned}
& \ulcorner((\bigsqcup_{a\in L} a) \cdot t) \sqcap \ulcorner((\bigsqcup_{a\in L} a) \cdot \neg t) \\
= \quad & \{\!\![\text{ distributivity of } \cdot \text{ and } \ulcorner \text{ over } \sqcup \text{ }]\!\!\} \\
& (\bigsqcup_{a\in L} \ulcorner(a \cdot t)) \sqcap (\bigsqcup_{a\in L} \ulcorner(a \cdot \neg t)) \\
= \quad & \{\!\![\text{ distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming }]\!\!\} \\
& \bigsqcup_{a\in L} \bigsqcup_{b\in L} \ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) \\
= \quad & \{\!\![\text{ by } \forall\, a \in L : \mathrm{MD}(a) \text{ and Lemma 19(e) }]\!\!\} \\
& 0 \ .
\end{aligned}
$$

For the case of CD we also have the reverse implication:

Lemma 21 *If* $\mathrm{CD}(\sqcup L)$ *then* $\forall\, a \in L : \mathrm{CD}(a)$, *and* L *is compatible.*

PROOF. The first part of the claim is immediate by downward closure of CD. For the second part, let $a, b \in L$. Note that by Lemma 10(b), $\ulcorner(\ulcorner a \cdot b) =$

$\ulcorner a \cdot \urcorner b \;=\; \urcorner b \cdot \ulcorner a \;=\; \ulcorner(\urcorner b \cdot a)$. Moreover, $\ulcorner a \cdot b \leq a + b$, $\urcorner b \cdot a \leq a + b$ and $\mathrm{CD}(a+b)$ by $a + b \leq \sqcup L$ and downward closure of CD. Hence, using Lemma 10(b), distributivity and Lemma 8(b),

$$\ulcorner a \cdot b = \ulcorner(\ulcorner a \cdot b) \cdot (a+b) = \ulcorner a \cdot b + \urcorner b \cdot a = \ulcorner(\urcorner b \cdot a) \cdot (a+b) = \urcorner b \cdot a \ .$$

Note that the corresponding property for $\mathrm{SC1}, \mathrm{LD}$ and MD does not hold. Indeed, $L \triangleq \{a, b\}$ in LAN $\triangleq (\mathcal{P}(\{a,b\}^*), \subseteq, \emptyset, \{a,b\}^*, \bullet, \varepsilon)$ can be used as a counterexample for all three cases.

7.4 Closure under Composition

Another natural requirement for determinate objects is that they should be closed under composition. First, it is straightforward that LD (and hence SC) is closed under composition. Moreover, we have

Lemma 22 MD *is closed under composition.*

PROOF. By definition of $\langle \_ \rangle$ and local composition, $\langle a \cdot b \rangle t = \ulcorner(a \cdot b \cdot t) = \ulcorner(a \cdot \ulcorner(b \cdot t)) = \langle a \rangle(\langle b \rangle t)$, from which by duality we also get $[a \cdot b]t = [a]([b]t)$. Now the claim is immediate by monotonicity of $\langle a \rangle$, which follows directly from its definition.

Properties SC1 and DD are *not* closed under composition as will be shown in Sections 8.5 and 8.6. However, we can show (see [6] for the proof):

Lemma 23 *Suppose that the Boolean algebra underlying our KA is atomic with atom set* At*. Then* DD *is closed under composition iff the set* $\mathsf{SAt} \triangleq \mathsf{At} \cup \{0\}$ *of subatoms is closed under composition.*

Let us see how this applies to the semantics of while loops. Classically, a loop of the form $\mathsf{while}\ G\ \mathsf{do}\ B\ \mathsf{od}$ with guard G and body B is modeled in Kleene algebra as follows (see e.g. [8]). The guard is represented by a type g characterizing all states that satisfy G. The semantics of the body B is given by an element b of the underlying KA. Then the semantics of the loop itself is $(g \cdot b)^* \cdot \neg g$, representing the informal view that the loop repeats the body B as long as the guard G stays true and terminates as soon as a state is reached in which G becomes false. Now we have (for the proof see again [6]):

Lemma 24 *If* CD *is closed under composition, then*

$$\mathrm{CD}(b) \;\Rightarrow\; \mathrm{CD}((g \cdot b)^* \cdot \neg g) \ .$$

8 Counterexamples

8.1 A Technique for Constructing Kleene Algebras

In this section, various finite Kleene algebras are constructed in the following way. We head for algebras in which the underlying lattice is an *atomic* Boolean algebra. In each case we list the set At of atoms; the other elements are then given by all possible joins of atoms (including the empty join). If there are n atoms, the algebra thus has 2^n elements. The Boolean operations are defined via the atom sets of the resulting elements:

$$\mathsf{At}(p+q) \triangleq \mathsf{At}(p) \cup \mathsf{At}(q)\ , \qquad \mathsf{At}(p \sqcap q) \triangleq \mathsf{At}(p) \cap \mathsf{At}(q)\ ,$$
$$\mathsf{At}(\overline{p}) \triangleq \mathsf{At}\backslash\mathsf{At}(p)\ , \qquad \mathsf{At}(0) \triangleq \emptyset\ , \qquad \mathsf{At}(\top) \triangleq \mathsf{At}\ .$$

The meet of two atoms is of course 0. Obviously, this defines an atomic Boolean algebra.

Composition $\cdot$ is given by a table for the atoms only. Composition of the other elements is obtained through disjunctivity, thus satisfying this axiom by construction. E.g., for atoms a, b, c, d we set

$$(a+b)\cdot(c+d) \;\triangleq\; a\cdot c + a\cdot d + b\cdot c + b\cdot d\ .$$

If the composition of atoms is associative, by disjunctivity this propagates to sums of atoms, i.e. to the other elements. In the same way, neutrality of 1 propagates from atoms to sums of atoms.

8.2 Concerning Local Composition

Here we present a KA that does not have local composition. It has two atoms 1 and a (and thus four elements). Its composition table is shown in Fig. 1(a). There are only two types, viz. 0 and 1. Hence by Lemma 8(l) we have $\ulcorner a = 1$. Now, $\ulcorner(a\cdot a) = \ulcorner 0 = 0$, but $\ulcorner(a\cdot \ulcorner a) = \ulcorner(a\cdot 1) = \ulcorner a = 1$.

This algebra is a special case of a whole class of algebras similar to LAN, but with words of bounded length. Specifically, let A be any set and, for $i \in \mathbb{N}$, define $S_n \triangleq \{w \in A^* : |w| \le n\}$, where $|w|$ is the length of word w. For $U, V \subseteq S_n$, define *bounded concatenation* by $U \odot V \triangleq \{u \bullet v : u \in U \wedge v \in V \wedge |u\bullet v| \le n\}$. Then $\mathrm{LAN}_n \triangleq (\mathcal{P}(S_n), \subseteq, S_n, \odot, \emptyset, \varepsilon)$ is a Kleene algebra

(a)

$\cdot$	1	a
1	1	a
a	a	0

(b)

$\cdot$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

(c)

;	1	a	b	c
1	1	a	b	c
a	a	a	$a+b$	$\top$
b	b	$a+b$	$\bar{b}$	$b+c$
c	c	$\top$	$b+c$	c

Fig. 1. (a) An algebra without local composition; (b) SC1(a) does not imply SC(a); (c) DD does not imply LD

in which locality of composition does not hold. The example given above is obtained by starting from a set A with a single element and setting $n = 1$.

8.3 Concerning SC1*,* SC *and* DD

In this section we show that SC1(a) $\not\Rightarrow$ SC(a) and that DD does not imply either of SC1 and SC. The counter-example consists of a finite Kleene algebra with three atoms $1, a, b$ and the composition table shown in Fig. 1(b) (which obviously is associative and satisfies locality of composition). This algebra is isomorphic to the algebra generated by the following concrete relations under relational composition: $a = \{(0,1),(1,2),(2,2)\}$, $b = \{(0,2),(1,2),(2,2)\}$ and $1 = \{(0,0),(1,1),(2,2)\}$. In this algebra we have SC1(a) and DD(a), but not SC(a) (and hence not LD(a)). Moreover, DD(b) holds (since b is an atom), but SC1(b) does not.

8.4 Concerning DD *and* LD

We show that DD does *not* imply LD, even for RAs. The counterexample is McKenzie's non-representable 16-element RA [9] (also Appendix A in [14] [2]). The algebra has four atoms $1, a, b, c$ (and thus sixteen elements). The composition table of the atoms is shown in Fig. 1(c). Since a is an atom, it satisfies DD. Now, $a \,;(b \sqcap c) = 0$, but $a \,; b \sqcap a \,; c = (a+b) \sqcap \top = a + b$.

8.5 Non-Closure of SC1 *under Composition*

Consider again the algebra of Section 8.3. There, SC1 is not closed under composition, since a satisfies SC1, but the composition $b = a \cdot a$ does not.

Let us mention that the algebras from Sections 8.2 and 8.6 below cannot be used as counterexamples. There, all atoms satisfy SC1. Moreover, in the latter

[2] The entry for $c \cdot b$ in Fig. A.2.3 of [14] should be changed to z.

$\cdot$	a	b	c	d	e	f	i	j	k
a	0	$d+e$	$d+f$	0	0	0	0	a	0
b	0	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	0	d
e	0	0	0	0	0	0	0	0	e
f	0	0	0	0	0	0	0	0	f
i	a	0	0	d	e	f	i	0	0
j	0	b	c	0	0	0	0	j	0
k	0	0	0	0	0	0	0	0	k

Fig. 2. Non-closure of DD under composition

we have for all $x \in \{b, c, d, e, f\}$ that $x \cdot \overline{1} = 0$ which makes SC1 closed under composition in that algebra.

8.6 Non-Closure of DD *under Composition*

To show that DD (and hence CD) is not closed under composition we use a KA with nine atoms, a, b, c, d, e, f, i, j, k. Composition of the atoms is given by the table in Fig. 2. The identity of composition is given by $1 \triangleq i+j+k$. It is slightly tedious to verify that composition is associative, although this is facilitated by the fact that most entries of the composition table are 0. Another manner to verify associativity is through the following relational model on the set $\{0, 1, 2, 3, 4\}$, for which standard relational composition gives the above table (in fact, we started from this concrete model rather than from the abstract one). Associativity follows from the fact that relational composition is associative.

$$\begin{array}{lll} a = \{(0,1)\} & d = \{(0,2)\} & i = \{(0,0)\} \\ b = \{(1,2),(1,3)\} & e = \{(0,3)\} & j = \{(1,1)\} \\ c = \{(1,2),(1,4)\} & f = \{(0,4)\} & k = \{(2,2),(3,3),(4,4)\} \end{array}$$

From the composition table and the definition of domain, one obtains the following euqations, from which it is easy to check that locality of composition is satisfied:

$$\ulcorner a = \ulcorner d = \ulcorner e = \ulcorner f = \ulcorner i = i \ , \qquad \ulcorner b = \ulcorner c = \ulcorner j = j \ , \qquad \ulcorner k = k \ ,$$
$$i\urcorner = i \ , \qquad a\urcorner = j\urcorner = j \ , \qquad b\urcorner = c\urcorner = d\urcorner = e\urcorner = f\urcorner = k\urcorner = k \ .$$

Now, because a and b are atoms, they satisfy DD. Also, $a \cdot b = d + e$, so that $d < a \cdot b$. But $\ulcorner d = \ulcorner(a \cdot b) = i$, so that DD$(a \cdot b)$ is violated.

The same algebra can be used to show that DD (or CD) does not imply LD. The element a is an atom and thus satisfies DD. However, $a \cdot (b \sqcap c) = a \cdot 0 = 0 \neq d = (d+e) \sqcap (d+f) = a \cdot b \sqcap a \cdot c$.

Moreover, the algebra is another counterexample to the implication $\mathrm{SC1}(a) \Rightarrow \mathrm{SC}(a)$, since $a \cdot \overline{1} = a \cdot (a+b+c+d+e+f) = d+e+f \leq b+c+d+e+f+1 = \overline{a}$, while $a \cdot \overline{b} = a \cdot (a+c+d+e+f+1) = a+d+f$ and $\overline{a \cdot b} = \overline{d+e} = a+b+c+f+1$. By Lemma 12, the implication $\mathrm{SC1}(a) \Rightarrow \mathrm{SC}(a)$ is also violated; this is illustrated by $a \cdot (b \sqcap c) \neq a \cdot b \sqcap a \cdot c$.

Another counter-example to $\mathrm{SC1}(a) \Rightarrow \mathrm{SC}(a)$ is obtained from the above one by replacing the three atomic subidentities i, j, k by a single atomic identity 1. However, the resulting algebra does not satisfy locality of composition.

8.7 Concerning SC, SC1 *and* MD

First we note that MD does not imply SC, since otherwise, by Lemmas 13 and 16, and transitivity of implication, we would obtain that DD implies SC, in contradiction to Section 8.4 and Lemma 12. Second, MD does not imply SC1 either. The algebra in Section 8.3 has 0 and 1 as its only types. Hence, by Corollary 15(f), we have $\mathrm{MD}(b)$, but $\mathrm{SC1}(b)$ does not hold.

Concerning the reverse implications, let us first see the informal meaning of MD in the KA PAT. Consider a graph node y, viewed as an atomic type, and a set of paths a. Then $\langle a \rangle y$ is the set of all nodes from which *some* path in a leads to y, whereas a node x is in $[a]y$ iff *all* paths in a that start in x end in y. So $\mathrm{MD}(a)$ holds iff all paths in a that start in the same node also end in the same node. However, a may contain several different paths between two nodes.

Now, as we have seen in Section 6.1, in PAT the properties $\mathrm{SC}(a)$ and $\mathrm{SC1}(a)$ are equivalent to prefix-freeness of a. Hence for different nodes x, y, z the set $a = \{xy, xz\}$ of paths satisfies $\mathrm{SC}(a)$ and $\mathrm{SC1}(a)$ but not $\mathrm{MD}(a)$. Therefore neither SC nor SC1 implies MD. A consequence of this is that SC1 does not imply CD; indeed, if this were the case, we would have that SC1 implies MD, because of Lemma 16. By Lemma 13, SC1 does not imply DD either.

9 Summary of the Results

To give the reader a survey of what has been achieved in this paper, we summarize our results in a table of mutual (non-)implications and (non-)closure properties (see Figure 3). Equivalent properties are in the same line/column.

	SC1	SC LD	ED SO CD DD	MD	closed under subobjects	closed under composition	closed under compatible join
SC1	$\Leftrightarrow$	$\Leftarrow$			yes	no	yes
SC, LD	$\Rightarrow$	$\Leftrightarrow$			yes	yes	yes
ED, SO, CD, DD			$\Leftrightarrow$	$\Rightarrow$	yes	cond.	yes
MD			$\Leftarrow$	$\Leftrightarrow$	yes	yes	yes

Fig. 3. Relationship between properties

When there is a $\Rightarrow$ or a $\Leftarrow$, the reverse implication does not hold. When an entry is blank, it means that no implication holds. The entry "cond." means "conditionally valid"; see Lemma 23 for the precise result.

10 Conclusion

The theory of Kleene algebras offers new and surprising views on the notion of a function. Characterizations that are equivalent in relation algebras [14] differ in this generalized setting.

However, the second author has shown that the characterization CD is sufficient to reprove (in a simpler fashion!) all properties of overwriting that were shown relationally in [11]. So it seems that the generalized setting indeed has its merits.

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