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MORSE HOMOLOGY ON NONCOMPACT MANIFOLDS

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Abstract. Given a Morse function on a manifold whose moduli spaces of gradient flow lines for each action window are compact up to breaking one gets a bidirect system of chain complexes. There are different possibilities to take limits of such a bidirect system. We discuss in this note the relation between these different limits.

Contents

1. Introduction

In this note we assume that we have a Morse function f on a finite dimensional (possibly noncompact) Riemannian manifold (M, g) with the property that the moduli spaces of gradient flow lines in fixed action windows are compact up to breaking. Hence for an action window $[a, b] \subset \mathbb{R}$ we can define Morse homology groups

$$HM_*^{[a,b]} = HM_*^{[a,b]}(f, g).$$

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For our purposes the following notation turns out to be useful

$$HM_a^b := HM_*^{[a,b]}.$$

So the reader should be aware that the subscript for our homology groups does not refer to the grading but to the lower end of the action window. We actually suppress the reference to the grading since it plays a minor role in our discussion.

There are now different limits one can take from these homology groups. One possibility was carried out by H. Hofer and D. Salamon in [7]. They take a Novikov completion of the chain complex on which they get a well-defined boundary homomorphism. We denote by HM the homology of this complex. Other possibilities are to take direct and inverse limits of the homology groups HM_a^b. Hence we abbreviate

$$\overline{HM} = \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} HM_a^b$$

and

$$\underline{HM} = \varprojlim_{a\to-\infty} \varinjlim_{b\to\infty} HM_a^b.$$

The aim of this note is to study the relation between these three homology groups. We remark that there are canonical maps $\kappa\colon \overline{HM} \to \underline{HM}$, $\overline{\rho}\colon HM \to \overline{HM}$, and $\underline{\rho}\colon HM \to \underline{HM}$ whose definition we recall later. We summarize them into the diagram:

$$\begin{array}{ccc} HM & \xrightarrow{\ \overline{\rho}\ } & \overline{HM} \\ & {\scriptstyle\underline{\rho}}\searrow & \downarrow{\scriptstyle\kappa} \\ & & \underline{HM} \end{array} \tag{1}$$

Our main result is the following.

Theorem A. *Assume that the Morse homology groups are taken with field coefficients. Then the diagram* (1) *is commutative,* $\overline{\rho}$ *is an isomorphism, and* κ *and therefore also* $\underline{\rho}$ *are surjective.*

Remark 1. Theorem A might fail if one uses integer coefficients instead of field coefficients. We provide an example in the appendix.

Remark 2. Although we state Theorem A only for finite dimensional manifolds, it can be carried over to the semi-infinite dimensional case of Floer homology. The difficulty with giving a precise statement lies in the fact that up to now there is no precise definition what a Floer homology in general actually is. We hope to modify this unsatisfactory situation in the near future by using the newly established theory of H. Hofer, K. Wysocki, and E. Zehnder about scale structures [9]: From a usual Floer theory setup, their theory should provide a scale manifold equipped with a function so that the Floer homology can be interpreted as the Morse homology of the function on the scale manifold.

Alternatively, one can also give an axiomatized description of Morse homology for which Theorem A continues to hold. We explain that in Section 2.2.

Remark 3. We became interested in the relation of the different Morse homologies via Rabinowitz Floer homology. We defined in [1] Rabinowitz Floer homology as the Morse homology of the Rabinowitz action functional by taking Novikov sums as in [7]. On the other hand, in a joint work with A. Oancea [2] we are proving that Rabinowitz Floer homology is isomorphic to a variant of symplectic homology. To establish this isomorphism we need to work with $\overline{HM}$. Therefore it became important for us to know if $\overline{\rho}$ is an isomorphism or not.

Remark 4. In Floer homology the homology groups $\overline{HM}$ were successfully applied by K. Ono in his proof of the Arnold conjecture for weakly monotone symplectic manifolds [12]. In this paper K. Ono raises the question if $\overline{\rho}$ is an isomorphism. In the case of Floer homology for weakly monotone symplectic manifolds it was later shown by S. Piunikhin, D. Salamon, and M. Schwarz that the homology groups HM and $\overline{HM}$ coincide by direct computation. Theorem A gives an algebraic explanation for this fact. See also the remark at the end of Section 6.3 in [5] for an alternative algebraic explanation.

The following example shows that κ and therefore $\underline{\rho}$ do not need to be injective.

Example. Let $M = \bigsqcup_{n=1}^{\infty} R_n$ where each $R_n \cong \mathbb{R}$ and for each $n \in \mathbb{N}$ the Morse function $f|_{R_n}$ has one single maximum $\overline{c}_n$ and one single minimum $\underline{c}_n$ with

$$f(\overline{c}_n) = n, \quad f(\underline{c}_n) = -n.$$

It follows that there is precisely one gradient flow line from $\overline{c}_n$ to $\underline{c}_n$. Taking Morse homology with coefficients in the abelian group Γ we obtain for $(a,b) \in \mathbb{R}^2$

$$HM_a^b = \Bigg(\bigoplus_{b<n\leq -a} \Gamma \cdot \underline{c}_n\Bigg) \oplus \Bigg(\bigoplus_{-a<n\leq b} \Gamma \cdot \overline{c}_n\Bigg).$$

We conclude that

$$HM^b = \varprojlim HM_a^b = \prod_{n>b} \Gamma \cdot \underline{c}_n$$

and

$$HM_a = \varinjlim HM_a^b = \bigoplus_{n>-a} \Gamma \cdot \overline{c}_n.$$

We get

$$\underline{HM} = \varprojlim HM_a = 0, \quad \overline{HM} = \varinjlim HM^b \neq 0$$

which shows that κ does not need to be injective.

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2. Morse homology

2.1. Morse tuples

In this section we introduce the notion of a Morse tuple on a (not necessarily compact) finite dimensional manifold M. A Morse tuple (f, g) consists of a Morse function f on M and a Riemannian metric g meeting a transversality and a compactness condition which ensure that Morse homology for each action window can be defined as usual, see [15]. We then proceed by explaining the maps which connect the Morse homology groups for different action windows.

If (M, g) is a Riemannian manifold and $f \in C^\infty(M)$ is a Morse function on M we denote by ∇f the gradient of f with respect to the metric g. A gradient flow line $x \in C^\infty(\mathbb{R}, M)$ is a solution of the ordinary differential equation

$$\partial_s x(s) = \nabla f(x(s)), \quad s \in \mathbb{R}. \tag{2}$$

We denote by $||\cdot||$ the norm on TM induced from the metric g. The energy of any $x \in C^\infty(\mathbb{R}, M)$ not necessarily satisfying (2) is given by

$$E(x) = \int_{-\infty}^{\infty} ||\partial_s x||^2 ds.$$

If x is a gradient flow line, then its energy equals

$$E(x) = \limsup_{s\in\mathbb{R}} f(x(s)) - \liminf_{s\in\mathbb{R}} f(x(s)) = \lim_{s\to\infty} f(x(s)) - \lim_{s\to-\infty} f(x(s)).$$

In particular, if $x(s)$ converges to critical points $x^\pm$ of f as s goes to $\pm\infty$ we obtain

$$E(x) = f(x^+) - f(x^-).$$

We abbreviate

$$\mathcal{G} = \big\{x \in C^\infty(\mathbb{R}, M) : \; x \text{ solves (2)}, \; E(x) < \infty\big\}$$

the moduli space of all finite energy flow lines of ∇f. For a two dimensional vector $(a, b) \in \mathbb{R}^2$ we denote

$$\mathcal{G}_a^b = \big\{x \in \mathcal{G} : a \le f(x(s)) \le b \text{ for all } s \in \mathbb{R}\big\}.$$

Note that $\mathbb{R}$ acts on $\mathcal{G}$ by time shift

$$r_* x(s) = x(s + r), \quad x \in \mathcal{G}, \; s, r \in \mathbb{R}.$$

This action is semifree in the sense that in the complement of its fixed points it acts freely. We abbreviate

$$\mathcal{C} = \mathrm{Fix}(\mathbb{R}) \subset \mathcal{G}.$$

The fixed point set $\mathcal{C}$ can naturally be identified with the set of critical points $\mathrm{crit}(f)$ via the evaluation map

$$\mathrm{ev} \colon \mathcal{G} \to M, \quad x \mapsto x(0).$$

Moreover, we endow the set $\mathcal{C}$ with the structure of a graded set where the grading is given by the Morse index. For $(a,b) \in \mathbb{R}^2$ we further denote

$$\mathcal{C}_a^b = \mathcal{C} \cap \mathcal{G}_a^b.$$

Note that $\mathcal{C}_a^b$ corresponds to the critical points of f in the action window $[a,b]$. Again this set is graded by the Morse index. We further remark that $\mathcal{C}_a^b$ only depends on the Morse function f and not on the metric g. Our first hypothesis is the following compactness assumption. To state it we endow the space $C^\infty(\mathbb{R}, M)$ with the C^∞_{loc}-topology.

(H1) For all $(a,b) \in \mathbb{R}$ the set $\mathcal{G}_a^b$ is a compact subset of $C^\infty(\mathbb{R}, M)$.

Before stating our second hypothesis we show in the following lemma that hypothesis (H1) implies that each finite energy gradient flow line converges asymptotically to critical points.

Lemma 2.1. *Assume hypothesis* (H1). *If $x \in \mathcal{G}$, then there exists $x^\pm \in \mathcal{C}$ such that*

$$\lim_{r \to \pm\infty} r_* x = x^\pm.$$

Proof. Choose $a, b \in \mathbb{R}$ such that $x \in \mathcal{G}_a^b$. For $\nu \in \mathbb{N}$ consider the sequence

$$x_\nu = \nu_* x$$

of gradient flow lines. Since $\mathcal{G}_a^b$ is $\mathbb{R}$-invariant it follows that

$$x_\nu \in \mathcal{G}_a^b, \quad \nu \in \mathbb{N}.$$

By hypothesis (H1) it follows that there exists a subsequence ν_j and $x^+ \in \mathcal{G}_a^b$ such that x_{ν_j} converges to x^+ in the C^∞_{loc}-topology as j goes to infinity. It remains to show that x^+ is a constant gradient flow line, hence a critical point. Fix $s > 0$. To see that x^+ is constant we have to show that $f(x^+(0)) = f(x^+(s))$. We argue by contradiction and assume that there exists $\epsilon > 0$ satisfying

$$f(x^+(s)) - f(x^+(0)) = \epsilon.$$

Since x_{ν_j} converges to x^+ in the C^∞_{loc}-topology there exists j_0 such that for every $j \geq j_0$ the inequality

$$f(x_{\nu_j}(s)) - f(x_{\nu_j}(0)) \geq \frac{\epsilon}{2}$$

holds. By definition of x_{ν_j} this means

$$f(x(\nu_j)) - f(x(\nu_j + s)) \geq \frac{\epsilon}{2}.$$

For $\ell \in \mathbb{N}$ we define recursively

$$j_\ell = \min\big\{ j : \nu_{j_{\ell-1}} + s \leq \nu_j \big\}.$$

Choose ℓ_0 satisfying

$$\ell_0 > \frac{2E(x)}{\epsilon}.$$

We estimate using the gradient flow equation

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} ||\partial_s x||^2 ds \\ &\geq \sum_{\ell=0}^{\ell_0-1} \int_{\nu_{j_\ell}}^{\nu_{j_\ell}+s} ||\partial_s x||^2 ds \\ &= \sum_{\ell=0}^{\ell_0-1} \int_{\nu_{j_\ell}}^{\nu_{j_\ell}+s} \frac{d}{ds} f(x(s)) ds \\ &= \sum_{\ell=0}^{\ell_0-1} \Big(f(x(\nu_{j_\ell}+s)) - f(x(\nu_{j_\ell})) \Big) \\ &\geq \frac{\ell_0 \epsilon}{2} \\ &> E(x). \end{aligned}$$

This contradiction shows that the assumption that x^+ was nonconstant had to be wrong. Hence x^+ is a critical point and since f is Morse, the gradient flow line converges at the positive asymptotic to x^+. A completely analogous reasoning shows that x converges at the negative asymptotic, too. This proves the lemma. □

Our second assumption is that (f, g) meet the Morse-Smale condition. We do not suppose that the flow of ∇f exists for all times. So instead of assuming that the stable and unstable manifolds for each pair of critical points of the Morse function intersect transversely the Morse-Smale condition has to be rephrased in the assumption that the operator coming from the linearization of the gradient flow is surjective as in [15]. In order to recall this operator we have to introduce some notation. It follows from Lemma 2.1 that asymptotically each gradient flow line converges to critical points. For critical points $x^\pm \in \mathcal{C}$ abbreviate by $\mathcal{H} = \mathcal{H}(x^-, x^+)$ the Hilbert manifold of $W^{1,2}$-paths from $\mathbb{R}$ to M which converge to $x^\pm$ for $s \to \pm\infty$. Let $\mathcal{E}$ be the bundle over $\mathcal{H}$ whose fiber at a point $x \in \mathcal{H}$ is given by

$$\mathcal{E}_x = L^2(\mathbb{R}, x^*TM).$$

Consider the section

$$\varsigma \colon \mathcal{H} \to \mathcal{E}, \quad x \mapsto \partial_s x - \nabla f(x).$$

The zero set of this section are gradient flow lines from x^- to x^+. If $x \in \varsigma^{-1}(0)$ there is a canonical splitting of the tangent space

$$T_x\mathcal{E} = \mathcal{E}_x \otimes T_x\mathcal{H}.$$

Denote by

$$\pi \colon T_x\mathcal{E} \to \mathcal{E}_x$$

the projection along $T_x\mathcal{H}$. The *vertical differential* at a zero of the section ς at a zero $x \in \varsigma^{-1}(0)$ is given by

$$D\varsigma(x) = \pi \circ d\varsigma(x)\colon T_x\mathcal{H} = W^{1,2}(\mathbb{R}, x^*TM) \to \mathcal{E}_x.$$

We can now formulate our second hypothesis

(H2) For each $x \in \mathcal{G}$ the operator $D\varsigma(x)$ is surjective.

Definition 2.2. A tuple (f, g) consisting of a Morse function f and a Riemannian metric g on the manifold M such that (H1) and (H2) hold is called a *Morse tuple for M*.

Remark. Hypothesis (H1) is actually much more important than hypothesis (H2) in order to define Morse homology. Even if transversality fails one can define Morse homology by using abstract perturbation theory provided compactness is guaranteed. However, we assume in this paper hypothesis (H2) so that we can avoid discussions about abstract perturbations.

In the following we assume that we have fixed a Morse tuple (f, g) on M. Fix further a field $\mathbb{F}$. The Morse complex

$$\big(CM_a^b, \partial_a^b\big) = \big(CM_a^b(f;\mathbb{F}), \partial_a^b(f,g;\mathbb{F})\big)$$

is defined in the following way. The chain group

$$CM_a^b = \mathcal{C}_a^b \otimes \mathbb{F}$$

is the $\mathbb{F}$-vector space generated by the critical points of f in the action window $[a, b]$. Note that CM_a^b is a finite dimensional vector space. Indeed, it follows from (H1) that the set $\mathcal{C}_a^b$ is compact. Since f is Morse, it is also discrete and hence finite. The boundary operator ∂_a^b is given by counting gradient flow lines. For $x^\pm \in \mathcal{C}$ abbreviate

$$\mathcal{G}(x^-, x^+) = \{x \in \mathcal{G} : \lim_{r\to\pm\infty} r_* x = x^\pm\}.$$

If $x^- \neq x^+$, then $\mathbb{R}$ acts freely on $\mathcal{G}(x^-, x^+)$. Moreover, if the Morse indices satisfy $\mu(x^-) = \mu(x^+) - 1$, then it is well known that it follows from hypotheses (H1) and (H2) that the quotient $\mathcal{G}(x^-, x^+)/\mathbb{R}$ is a finite set, see [15]. In this case we define the integer

$$m(x^-, x^+) = \#_\sigma\big(\mathcal{G}(x^-, x^+)/\mathbb{R}\big),$$

where $\#_\sigma$ refers to the signed count of the set. The sign is determined by the choice of a coherent orientation for the moduli spaces of gradient flow lines. For $c \in \mathcal{C}_a^b$, we put

$$\partial_a^b c = \sum_{\substack{c' \in \mathcal{C}_a^b \\ \mu(c') = \mu(c) - 1}} m(c', c) c'.$$

We define ∂_a^b on CM_a^b by $\mathbb{F}$-linear extension of the formula above. Again it is well known, see [15], that under hypothesis (H1) and (H2) the homomorphism ∂_a^b is a boundary operator, i.e.,

$$\left(\partial_a^b\right)^2 = 0.$$

Hence we get a graded vector space

$$HM_a^b = HM_a^b(f, g; \Gamma) = \frac{\ker\partial_a^b}{\mathrm{im}\partial_a^b}.$$

If $a_1 \leq a_2$ we denote by $\underline{\mathcal{C}}_{a_1}^{a_2}$ the set generated by critical points in the half open action interval $[a_1, a_2)$. In particular, if a_2 lies not in the spectrum of f the graded set $\underline{\mathcal{C}}_{a_1}^{a_2}$ equals $\mathcal{C}_{a_1}^{a_2}$. We abbreviate

$$\underline{CM}_{a_1}^{a_2} = \underline{\mathcal{C}}_{a_1}^{a_2} \otimes \Gamma. \tag{3}$$

If $a_1 \leq a_2 \leq b$, then the disjoint union

$$\mathcal{C}_{a_1}^b = \underline{\mathcal{C}}_{a_1}^{a_2} \sqcup \mathcal{C}_{a_2}^b$$

leads to the direct sum

$$CM_{a_1}^b = \underline{CM}_{a_1}^{a_2} \oplus CM_{a_2}^b.$$

We denote by

$$p_{a_2,a_1}^b \colon CM_{a_1}^b \to CM_{a_2}^b$$

the projection along $\underline{CM}_{a_1}^{a_2}$. Since the action is increasing along gradient flow lines the projections commute with the boundary operators in the sense that

$$p_{a_2,a_1}^b \circ \partial_{a_1}^b = \partial_{a_2}^b \circ p_{a_2,a_1}^b. \tag{4}$$

Moreover, for $a_1 \leq a_2 \leq a_3 \leq b$, their composition obviously meets

$$p_{a_3,a_2}^b \circ p_{a_2,a_1}^b = p_{a_3,a_1}^b \tag{5}$$

and for $a \leq b$

$$p_{a,a}^b = \mathrm{id}|_{CM_a^b}. \tag{6}$$

It follows from (4) that p_{a_2,a_1}^b induces homomorphisms

$$Hp_{a_2,a_1}^b \colon HM_{a_1}^b \to HM_{a_2}^b$$

which satisfy

$$Hp_{a_3,a_2}^b \circ Hp_{a_2,a_1}^b = Hp_{a_3,a_1}^b \tag{7}$$

by (5) and

$$Hp_{a,a}^b = \mathrm{id}|_{HM_a^b} \tag{8}$$

by (6). Similarly, for $a \leq b_1 \leq b_2$ the inclusions $\mathcal{C}_a^{b_1} \mapsto \mathcal{C}_a^{b_2}$ induce maps

$$i_a^{b_2,b_1} \colon CM_a^{b_1} \to CM_a^{b_2}.$$

Again the inclusions commute with the boundary operators

$$i_a^{b_2,b_1} \circ \partial_a^{b_1} = \partial_a^{b_2} \circ i_a^{b_2,b_1}, \tag{9}$$

their composition satisfies

$$i_a^{b_3,b_2} \circ i_a^{b_2,b_1} = i_a^{b_3,b_1} \tag{10}$$

for $a \le b_1 \le b_2 \le b_3$ and

$$i_a^{b,b} = \mathrm{id}|_{CM_a^b} \tag{11}$$

for $a \le b$. Moreover, inclusions and projections commute in the sense that if $a_1 \le a_2 \le b_1 \le b_2$, then

$$i_{a_2}^{b_2,b_1} \circ p_{a_2,a_1}^{b_1} = p_{a_2,a_1}^{b_2} \circ i_{a_1}^{b_2,b_1}. \tag{12}$$

It follows from (9) that $i_a^{b_2,b_1}$ induces homomorphisms

$$Hi_a^{b_2,b_1} \colon HM_a^{b_1} \to HM_a^{b_2}.$$

By (10) they satisfy

$$Hi_a^{b_3,b_2} \circ Hi_a^{b_2,b_1} = Hi_a^{b_3,b_1}, \tag{13}$$

by (11)

$$Hi_a^{b,b} = \mathrm{id}|_{HM_a^b}, \tag{14}$$

and by (15)

$$Hi_{a_2}^{b_2,b_1} \circ Hp_{a_2,a_1}^{b_1} = Hp_{a_2,a_1}^{b_2} \circ Hi_{a_1}^{b_2,b_1}. \tag{15}$$

We can summarize the results of this section in the following proposition.

Proposition 2.3. *For a Morse tuple (f, g) on the manifold M the quadruple (CM, p, i, ∂) is a bidirect system of chain complexes* (*see Section 3.3 for the definition*).

2.2. An axiomatic approach

Following a suggestion of D. Salamon we can axiomatize the results of the previous subsection in the following way. Via this axiomatized approach one can get Theorem A also in the infinite dimensional case of Floer homology provided one has the necessary compactness.

Definition 2.4. A *Floer triple*

$$\mathcal{F} = (\mathcal{C}, f, m)$$

consists of a set $\mathcal{C}$, a function $f \colon \mathcal{C} \to \mathbb{R}$ and a function $m \colon \mathcal{C} \times \mathcal{C} \to \mathbb{F}$ such that the following condition holds.

(i) For each $a \le b$ the set $\mathcal{C}_a^b = \{c \in \mathcal{C} : a \le f(c) \le b\}$ is finite.
(ii) If $c_1, c_2 \in \mathcal{C}$ and $m(c_1, c_2) \neq 0$, then it follows that $f(c_1) < f(c_2)$.
(iii) If $c_1, c_3 \in \mathcal{C}$, then $\sum_{c_2 \in \mathcal{C}} m(c_1, c_2) m(c_2, c_3) = 0$.

Assertions (i) and (ii) make sure that the sum in assertion (iii) is finite. Elements of $\mathcal{C}$ are referred to as critical points, the value of f as their action value and the number $m(c_1, c_2)$ as the number of gradient flow lines between c_1 and c_2. Assertion (i) can then be rephrased by saying that in each finite action window there are only finitely many critical points, assertion (ii) says that the action is increasing along gradient flow lines, and assertion (iii) guarantees that on each action window a boundary operator can be defined by counting gradient flow lines. As in the previous subsection one can associate to each Floer triple a bidirect system of chain complexes. The assumption to have a Morse tuple in order that Theorem A holds can be generalized to arbitrary Floer triples.

3. Algebraic preliminaries

3.1. Direct and inverse limits

We first recall that a quasi ordered set is a tuple $\mathcal{A} = (A, \leq)$ where A is a set and $\leq$ is a reflexive and transitive binary relation. More sophisticatedly, one might think of $\mathcal{A}$ as a category with precisely one morphism from a_1 to a_2 whenever $a_1 \leq a_2$. A quasi ordered set is called partially ordered if the binary relation is also antisymmetric.

To define direct and inverse limits the notion of a *direct system* is needed. For the applications we have in mind we have to work in the category of graded vector spaces. For simplicity we skip the reference to the grading. Hence a direct system is a tuple

$$\mathcal{D} = (G, \pi),$$

where G is a family of vector spaces indexed by a quasi ordered set $\mathcal{A} = (A, \leq)$, i.e.,

$$G = \{G_a\}_{a \in A},$$

and

$$\pi = \{\pi_{a_2,a_1}\}_{a_1 \leq a_2;\, a_1,a_2 \in A}$$

is a family of homomorphisms

$$\pi_{a_2,a_1} \colon G_{a_1} \to G_{a_2}$$

satisfying

$$\pi_{a,a} = \mathrm{id}|_{G_a},\ a \in A, \quad \pi_{a_3,a_1} = \pi_{a_3,a_2} \circ \pi_{a_2,a_1},\ a_1 \leq a_2 \leq a_3.$$

If one thinks of a quasi ordered set as a category, then a direct system is a functor to the category of vector spaces. For a direct system the *inverse limit* or just *limit* is defined as the vector space

$$\varprojlim G := \varprojlim_{\mathcal{A}} G := \left\{ \{x_a\}_{a \in A} \in \prod_{a \in A} G_a : a_1 \leq a_2 \Rightarrow \pi_{a_2,a_1}(x_{a_1}) = x_{a_2} \right\}.$$

For $a \in A$ let

$$\pi_a \colon \varprojlim G \to G_a$$

be the (not necessarily surjective) projection to the a-th component. These maps satisfy for $a_1 \le a_2$ the relation

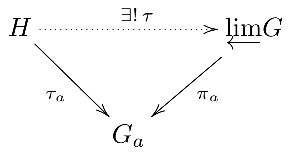
$$\pi_{a_2} = \pi_{a_2,a_1} \circ \pi_{a_1}.$$

The inverse limit is characterized by the following universal property. Given a vector space H and a family of homomorphisms $\tau_a \colon H \to G_a$ for $a \in A$ which satisfies

$$\tau_{a_2} = \pi_{a_2,a_1} \circ \tau_{a_1}, \quad a_1 \le a_2,$$

then there exists a unique homomorphism $\tau \colon H \to \varprojlim G$ such that for any $a \in A$ the following diagram commutes:

(16)



The *direct limit* or *colimit* is constructed dually to the inverse limit. To make the notation easier adaptable to our later purposes we denote in the definition of the direct limit the family of homomorphisms by ι, i.e., our direct system reads now

$$\mathcal{D} = (G, \iota).$$

Moreover, the index set is now denoted by $\mathcal{B} = (B, \le)$ and subscripts are replaced by superscripts. For $b \in B$ let

$$\lambda^b \colon G^b \to \bigoplus_{b' \in B} G^{b'}$$

be the b-th injection into the sum of the abelian groups G^b. Define the subgroup $S_{\mathcal{D}}$ of $\bigoplus G^b$ by

$$S_{\mathcal{D}} = \Big\{ \lambda^{b_2} \iota^{b_2,b_1}(x) - \lambda^{b_1}(x) : x \in G^{b_1},\ b_1 \le b_2 \Big\}.$$

The direct limit is now defined as the vector space

$$\varinjlim G := \varinjlim_{\mathcal{B}} G := \Big(\bigoplus_{b \in B} G^b \Big) \Big/ S_{\mathcal{D}}.$$

The direct limit is characterized by the following universal property dual to the characterization of the inverse limit. For $b \in B$ let

$$\iota^b \colon G^b \to \varinjlim G$$

be the (not necessarily injective) homomorphism induced from the inclusion of G^b into the sum $\bigoplus G^b$. Assume that H is an vector space and τ^b for $b \in B$ is a family of homomorphisms $\tau^b \colon G^b \to H$ satisfying

$$\tau^{b_1} = \tau^{b_2} \circ \iota^{b_2,b_1}, \quad b_1 \le b_2.$$

Then there exists a unique homomorphism $\tau\colon \varinjlim G \to H$ such that the following diagram commutes for any $b \in B$:

$$\begin{array}{ccc} \varinjlim G & \overset{\exists!\,\tau}{\dashrightarrow} & H \\ & {\scriptstyle \iota^b}\nwarrow \quad \nearrow{\scriptstyle \tau^b} & \\ & G^b & \end{array}$$

3.2. The canonical homomorphism

Direct and inverse limits do not necessarily commute. However, there is a canonical homomorphism

$$\kappa\colon \varinjlim\varprojlim G \to \varprojlim\varinjlim G$$

which we describe next. We consider two quasi ordered sets $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ and a double indexed family of abelian groups G_a^b with $a \in A$ and $b \in B$. We suppose that for every $b \in B$ and every $a_1 \leq a_2 \in A$ there exists a homomorphism

$$\pi^b_{a_2,a_1}\colon G^b_{a_1} \to G^b_{a_2}$$

and for every $a \in A$ and $b_1 \leq b_2 \in B$ there exists a homomorphism

$$\iota_a^{b_2,b_1}\colon G_a^{b_1} \to G_a^{b_2}$$

such that the following holds. For any fixed $b \in B$ and any fixed $a \in A$ the tuples $\big(G^b, \pi^b\big)$ and $\big(G_a, \iota_a\big)$ are direct systems. Moreover, π and ι are required to commute in the following sense

$$\iota_{a_2}^{b_2,b_1} \circ \pi^{b_1}_{a_2,a_1} = \pi^{b_2}_{a_2,a_1} \circ \iota_{a_1}^{b_2,b_1}\colon G_{a_1}^{b_1} \to G_{a_2}^{b_2}, \quad a_1 \leq a_2,\ b_1 \leq b_2.$$

This can be rephrased by saying that for every $a_1 \leq a_2$ and $b_1 \leq b_2$ the square

$$\begin{array}{ccc} G^{b_1}_{a_1} & \xrightarrow{\pi^{b_1}_{a_2,a_1}} & G^{b_1}_{a_2} \\ {\scriptstyle \iota^{b_1,b_2}_{a_1}}\big\downarrow & & \big\downarrow{\scriptstyle \iota^{b_2,b_1}_{a_2}} \\ G^{b_2}_{a_1} & \xrightarrow[\pi^{b_2}_{a_2,a_1}]{} & G^{b_2}_{a_2} \end{array} \tag{17}$$

is commutative. We refer to the triple $\big(G, \pi, \iota\big)$ as a *bidirect system*. Due to the commutation relation between π and ι for $a_1 \leq a_2 \in A$ the map

$$\pi_{a_2,a_1}\colon \varinjlim_{\mathcal{B}} G_{a_1} \to \varinjlim_{\mathcal{B}} G_{a_2}, \quad \big[\{x^b\}_{b\in B}\big] \mapsto \big[\{\pi^b_{a_2,a_1}(x^b)\}_{b\in B}\big]$$

is a well defined homomorphism. Analoguously, for $b_1 \leq b_2 \in B$, we have a well defined homomorphism

$$\iota^{b_2,b_1}\colon \varprojlim_{\mathcal{A}} G^{b_1} \to \varprojlim_{\mathcal{A}} G^{b_2}, \quad \big\{x_a\big\}_{a\in A} \mapsto \big\{\iota_a^{b_2,b_1}(x_a)\big\}_{a\in A}.$$

Moreover, both $(\varinjlim G, \pi)$ and $(\varprojlim G, \iota)$ are direct systems.

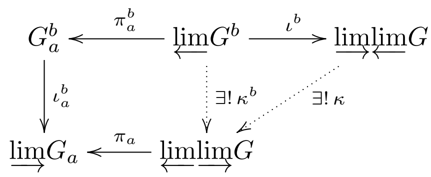
Proposition 3.1. *For a bidirect system* (G, π, ι) *there exists for every* $b \in B$ *a unique homomorphism*

$$\kappa^b \colon \varprojlim_{\mathcal{A}} G^b \to \varprojlim_{\mathcal{A}} \varinjlim_{\mathcal{B}} G$$

and a unique homomorphism

$$\kappa \colon \varinjlim_{\mathcal{B}} \varprojlim_{\mathcal{A}} G \to \varprojlim_{\mathcal{A}} \varinjlim_{\mathcal{B}} G$$

such that for each $a \in A$ *and* $b \in B$ *the following diagram commutes*:



Proof. A straightforward computation shows that for $a_1 \leq a_2 \in A$ and $b \in B$ the formula

$$\iota_{a_2}^b \pi_{a_2}^b = \pi_{a_2,a_1} \iota_{a_1}^b \pi_{a_1}^b$$

holds. Hence existence and uniqueness of the κ^b for $b \in B$ follows from the universal property of the inverse limit. For $b_1 \leq b_2 \in B$ and $a \in A$ one computes using the already establishes commutativity in the left square that

$$\pi_a \kappa^{b_2} \iota^{b_2,b_1} = \iota_a^{b_1} \pi_a^{b_1} = \pi_a \kappa^{b_1}.$$

Using uniqueness we conclude that

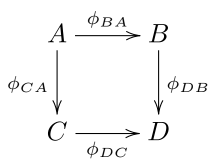
$$\kappa^{b_2} \iota^{b_2,b_1} = \kappa^{b_1}.$$

Now existence and uniqueness of κ follows from the universal property of the direct limit. □

Conditions under which the canonical homomorphism κ is an isomorphism were obtained by B. Eckmann and P. Hilton in [3] and by A. Frei and J. Macdonald in [4]. We first remark that in general κ is neither necessarily injective nor surjective. The example at the end of Section 1 shows that injectivity might fail. An example were surjectivity fails is described in [3, p. 117].

To describe the result of A. Frei and J. Macdonald we need the following terminology. To a square

(18)



we can associate the sequence

$$A \overset{\{\phi_{BA}, \phi_{CA}\}}{\longrightarrow} B \oplus C \overset{\langle \phi_{DB}, -\phi_{DC} \rangle}{\longrightarrow} D. \tag{19}$$

The square (18) is commutative precisely if the sequence (19) is a complex. A commutative square is now called *exact*, *cartesian*, *cocartesian*, or *bicartesian* if and only if the corresponding sequence is exact, left exact, right exact, or a short exact sequence.

We further recall that a quasi ordered set $\mathcal{A} = (A, \leq)$ is *upward directed* if for any $a, a' \in A$ there exists $a'' \in A$ such that $a \leq a''$ and $a' \leq a''$. Dually it is called *downward directed* if for any $a, a' \in A$ there exists $a'' \in A$ such that $a'' \leq a$ and $a'' \leq a'$.

The following theorem follows from [4, Theorem 5.6].

Theorem 3.2. *Assume that $\mathcal{A}$ is upward directed, $\mathcal{B}$ is downward directed and the commutative square* (17) *is cartesian for any $a_1 \leq a_2$ and $b_1 \leq b_2$. Then the canonical homomorphism $\kappa\colon \varinjlim\varprojlim G \to \varprojlim\varinjlim G$ is an isomorphism.*

3.3. Bidirect systems of chain complexes

A bidirect system of chain complexes is a quadruple

$$\mathcal{Q} = (C, p, i, \partial),$$

where (C, p, i) is a bidirect system which in addition is endowed for each $a \in A$ and $b \in B$ with a boundary operator

$$\partial_a^b\colon C_a^b \to C_a^b$$

which commutes with i and p in the sense of (4) and (9). If

$$HC_a^b = \frac{\ker \partial_a^b}{\operatorname{im} \partial_a^b}$$

are the homology groups, and Hp_{a_2,a_1}^b and $Hi_a^{b_2,b_1}$ are the induced maps on homology the triple (HC, Hp, Hi) is a bidirect system. As in the previous subsection we let

$$\kappa\colon \varinjlim\varprojlim HC \to \varprojlim\varinjlim HC$$

be the canonical homomorphism on homology level. We refer to

$$k\colon \varinjlim\varprojlim C \to \varprojlim\varinjlim C$$

as the canonical homomorphism on chain level. Since ∂ commutes with i and p we obtain an induced map

$$Hk\colon H\big(\varinjlim\varprojlim C\big) \to H\big(\varprojlim\varinjlim C\big).$$

Moreover, for $a \in A$ and $b \in B$ the maps

$$Hi_a^b\colon HC_a^b \to H(\varinjlim C_a)$$

satisfy for $b_1 \leq b_2$

$$Hi_a^{b_1} = Hi_a^{b_2} \circ Hi_a^{b_2,b_1}$$

and hence by the universal property of the direct limit there exists a unique map

$$\mu_a\colon \varinjlim HC_a \to H(\varinjlim C_a)$$

such that for any $b \in B$ the diagram:

$$\begin{array}{ccc} \varinjlim HC_a & \xrightarrow{\mu_a} & H(\varinjlim C_a) \\ & {\scriptstyle \iota_a^b}\nwarrow \quad \nearrow{\scriptstyle Hi_a^b} & \\ & HC_a^b & \end{array}$$

commutes. Taking inverse limits of this diagram and using functoriality of the inverse limit gives a commutative diagram:

$$\begin{array}{ccc} \varprojlim \varinjlim HC & \xrightarrow{\varprojlim \mu} & \varprojlim H(\varinjlim C) \\ & {\scriptstyle \varprojlim \iota^b}\nwarrow \quad \nearrow{\scriptstyle \varprojlim Hi^b} & \\ & \varprojlim HC^b & \end{array}$$

Taking first inverse limits of the chain complexes and applying the procedure above gives a map

$$\mu\colon \varinjlim H(\varprojlim C) \to H(\varinjlim \varprojlim C)$$

which is uniquely characterized by the commutativity of the following diagram:

$$\begin{array}{ccc} \varinjlim H(\varprojlim C) & \xrightarrow{\mu} & H(\varinjlim \varprojlim C) \\ & {\scriptstyle \iota^b}\nwarrow \quad \nearrow{\scriptstyle Hi^b} & \\ & H(\varprojlim C^b) & \end{array}$$

Similarly by using the universal property of the inverse limit we obtain for each $b \in B$ a map

$$\nu^b\colon H(\varprojlim C^b) \to \varprojlim HC^b$$

such that for each $a \in A$ the diagram

$$\begin{array}{ccc} H(\varprojlim C^b) & \xrightarrow{\nu^b} & \varprojlim HC^b \\ & {\scriptstyle Hp_a^b}\searrow \quad \swarrow{\scriptstyle \pi_a^b} & \\ & HC_a^b & \end{array} \tag{20}$$

commutes. Taking the direct limit of this diagram we get the commutative diagram:

$$\begin{array}{ccc} \varinjlim H(\varprojlim C) & \xrightarrow{\varinjlim \nu} & \varinjlim \varprojlim HC \\ & {\scriptstyle \varinjlim Hp^b}\searrow \quad \swarrow{\scriptstyle \varinjlim \pi^b} & \\ & \varinjlim HC^b & \end{array}$$

Applying the direct limit already on chain level we obtain a map

$$\nu\colon H(\varprojlim\varinjlim C) \to \varprojlim H(\varinjlim C)$$

such that the following diagram commutes:

$$\begin{array}{ccc} H(\varprojlim\varinjlim C) & \xrightarrow{\quad\nu\quad} & \varprojlim H(\varinjlim C) \\ & {\scriptstyle Hp_a}\searrow \quad \swarrow{\scriptstyle \pi_a} & \\ & H(\varinjlim C_a) & \end{array}$$

We summarize the plethora of maps we passed by in the diagram:

$$\begin{array}{ccccc} H(\varinjlim\varprojlim C) & \xleftarrow{\ \mu\ } & \varinjlim H(\varprojlim C) & \xrightarrow{\varinjlim\nu} & \varinjlim\varprojlim HC \\ \downarrow{\scriptstyle Hk} & & & & \downarrow{\scriptstyle \kappa} \\ H(\varprojlim\varinjlim C) & \xrightarrow{\ \nu\ } & \varprojlim H(\varinjlim C) & \xleftarrow{\varprojlim\mu} & \varprojlim\varinjlim HC \end{array}$$

We do not know if the diagram above always commutes. But we make now an assumption on the bidirect system of chain complexes which guarantees commutativity of the diagram above.

Definition 3.3. A bidirect system of chain complexes is called *tame* if for any $a \in A$ and any $b \in B$ the maps μ_a and ν^b as well as the map μ are isomorphisms.

We point out that for a tame bidirect system of chain complexes the map ν does not need to be an isomorphism. However, by functoriality of the inverse and direct limits the maps $\varprojlim\mu$ and $\varinjlim\nu$ are isomorphisms, too. Hence we can define maps

$$\rho\colon H(\varprojlim\varinjlim C) \to \varprojlim\varinjlim HC, \quad \rho = (\varinjlim\nu)\circ\mu^{-1}$$

and

$$\sigma\colon H(\varprojlim\varinjlim C) \to \varprojlim\varinjlim HC, \quad \sigma = (\varprojlim\mu)^{-1}\circ\nu.$$

In particular, the previous diagram simplifies to

$$\begin{array}{ccc} H(\varinjlim\varprojlim C) & \xrightarrow{\ \rho\ } & \varinjlim\varprojlim HC \\ \downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\ H(\varprojlim\varinjlim C) & \xrightarrow{\ \sigma\ } & \varprojlim\varinjlim HC \end{array} \tag{21}$$

Proposition 3.4. *Assume that the bidirect system is tame. Then the diagram* (21) *commutes and* ρ *is an isomorphism.*

Proof. That ρ is an isomorphism is clear since as we observed above $\varinjlim\nu$ is an isomorphism. We show commutativity in two steps.

Step 1: *For every $b \in B$ the following diagram commutes*:

$$\begin{array}{ccc} H(\varprojlim C^b) & \xrightarrow{\nu^b} & \varprojlim HC^b \\ \Big\downarrow{\scriptstyle Hk^b} & & \Big\downarrow{\scriptstyle \kappa^b} \\ H(\varprojlim\varinjlim C) & \xrightarrow{\sigma} & \varprojlim\varinjlim HC \end{array}$$

To prove Step 1 we enlarge the diagram to the following one:

$$\begin{array}{ccccccc} H(\varprojlim C^b) & \xrightarrow{\nu^b} & \varprojlim HC^b & \xrightarrow{\pi_a^b} & HC_a^b & & \\ \Big\downarrow{\scriptstyle Hk^b} & & \Big\downarrow{\scriptstyle \kappa^b} & & \Big\downarrow{\scriptstyle \iota_a^b} & \searrow{\scriptstyle Hi_a^b} & \\ H(\varprojlim\varinjlim C) & \xrightarrow{\sigma} & \varprojlim\varinjlim HC & \xrightarrow{\pi_a} & \varinjlim HC_a & \xrightarrow{\mu_a} & H(\varinjlim C_a) \end{array}$$

The triangle on the right and the middle square commute. We claim that the exterior square also commutes. Indeed, this square is obtained by applying the homology functor to the commutative square:

$$\begin{array}{ccc} \varprojlim C^b & \xrightarrow{p_a^b} & C_a^b \\ \Big\downarrow{\scriptstyle k^b} & & \Big\downarrow{\scriptstyle i_a^b} \\ \varprojlim\varinjlim C & \xrightarrow{p_a} & \varinjlim C_a \end{array}$$

Using the fact that ν^b and μ_a are isomorphisms we conclude that the diagram

$$\begin{array}{ccc} \varprojlim HC^b & \xrightarrow{\pi_a^b} & C_a^b \\ {\scriptstyle \sigma\circ Hk^b\circ(\nu^b)^{-1}}\Big\downarrow\Big\downarrow{\scriptstyle \kappa^b} & & \Big\downarrow{\scriptstyle \iota_a^b} \\ \varprojlim\varinjlim HC & \xrightarrow{\pi_a} & \varinjlim HC_a \end{array}$$

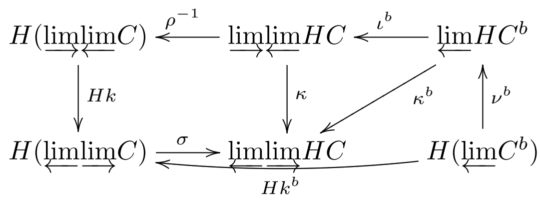
is commutative for both arrows. But by Proposition 3.1 the map κ^b is unique with this property. Hence

$$\kappa^b = \sigma \circ HK^b \circ (\nu^b)^{-1}$$

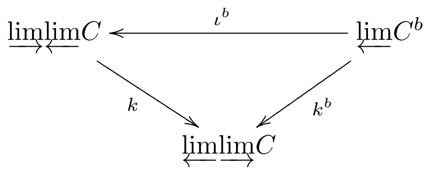
and Step 1 follows.

Step 2: *The diagram* (21) *commutes.*

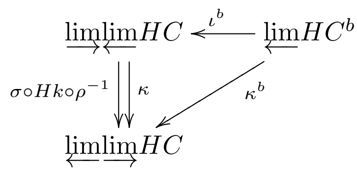
For $b \in B$ we enlarge diagram (21) to the diagram:



The exterior square is obtained by applying the homology functor to the commutative triangle



and is therefore commutative. Hence using Step 1 and the assumption that ν^b is an isomorphism we deduce that the diagram



is commutative for both arrows. Again by Proposition 3.1 we conclude that

$$\kappa = \sigma \circ Hk \circ \rho^{-1}.$$

This finishes the proof of Step 2 and hence of the proposition. □

3.4. The Mittag-Leffler condition

Given a direct system of chain complexes (C, p, ∂) there is a canonical map $\nu\colon H(\varprojlim C) \to \varprojlim HC$ defined as in (20). An important tool to study surjectivity and bijectivity properties of the map ν is the Mittag-Leffler condition. Following A. Grothendieck, see [6, (13.1.2)], this condition reads as follows.

Definition 3.5. A direct system (G, π) of vector spaces indexed on the quasi-ordered set $(\mathbb{R}, \leq)$ is said to satisfy the *Mittag-Leffler condition* if for any $a \in A$ there exists $a' = a'(a) \leq a$ such that for any $a'' \leq a'$ the following holds

$$\mathrm{im}\pi_{a,a''} = \mathrm{im}\pi_{a,a'} \subset G_a.$$

The following lemma gives two criteria under which the Mittag-Leffler condition holds true.

Lemma 3.6. *The Mittag-Leffler condition holds in the following two cases.*

(i) *For every $a_1 \leq a_2$ the homomorphism $\pi_{a_2,a_1}\colon G_{a_1} \to G_{a_2}$ is surjective.*

(ii) *For any $a \in \mathbb{R}$ the vector space G_a is finite dimensional.*

Proof. That the Mittag-Leffler condition holds in case (i) is obvious. To show that it holds in case (ii) we first observe that the relation $\pi_{a,a''} = \pi_{a,a'} \circ \pi_{a',a''}$ for $a'' \leq a' \leq a$ implies that

$$\mathrm{im}\pi_{a,a''} \subset \mathrm{im}\pi_{a,a'} \subset G_a. \tag{22}$$

Using that G_a is finite dimensional the function

$$\varrho_a \colon (-\infty, a] \to \mathbb{N} \cup \{0\}, \quad a' \mapsto \dim\big(\mathrm{im}\pi_{a,a'}\big)$$

is well-defined and it is monotone increasing by (22). Since it is bounded from below and takes only discrete values there exists

$$m_a = \min \varrho_a \in \mathbb{N} \cup \{0\}.$$

We choose $a' = a'(a)$ in such a way that

$$\varrho_a(a') = m_a.$$

With this choice it follows that for every $a'' \leq a'$ it holds that

$$\dim\big(\mathrm{im}\pi_{a,a''}\big) = \dim\big(\mathrm{im}\pi_{a,a'}\big).$$

Hence by (22) we get

$$\mathrm{im}\pi_{a'',a} = \mathrm{im}\pi_{a',a}$$

which finishes the proof of the Mittag-Leffler condition. □

For the following theorem, see [6, Proposition 13.2.3] or [17, Proposition 3.5. 7, Theorem 3.5.8].

Theorem 3.7. *Assume that (C, p, ∂) is a direct system of chain complexes indexed on the set $(\mathbb{R}, \leq)$. If (C, p) satisfies the Mittag-Leffler condition, then the homomorphism $\nu \colon H(\varprojlim C) \to \varprojlim HC$ is surjective. If in addition (HC, Hp) satisfies the Mittag-Leffler condition, too, then ν is an isomorphism.*

Under the assumptions of Theorem 3.7 if (C, p) satisfies the Mittag-Leffler condition, then the kernel of ν can be described with the help of the first derived functor $\varprojlim^1$ of the inverse limit. If (G, π) is a direct system of abelian groups indexed on the real line, $\varprojlim^1 G$ can be described in the following way. Choose a sequence $a_j \in \mathbb{R}$ such that $a_{j+1} \leq a_j$ for every $j \in \mathbb{N}$ and a_j converges to $-\infty$, i.e., $\{a_j\}_{j \in \mathbb{N}}$ is a cofinal sequence in $\mathbb{R}$. Consider the map

$$\Delta \colon \prod_{j=1}^{\infty} G_{a_j} \to \prod_{j=1}^{\infty} G_{a_j}, \quad \big\{x_{a_j}\big\}_{j \in \mathbb{N}} \mapsto \big\{x_{a_j} - \pi_{a_j, a_{j+1}}(x_{a_{j+1}})\big\}_{j \in \mathbb{N}}$$

and set

$$\varprojlim^1 G = \mathrm{coker}\Delta.$$

It is straightforward to check that $\varprojlim^1 G$ only depends on the choice of the cofinal sequence up to canonical isomorphism. For a graded abelian group G

and $n \in \mathbb{Z}$ let $G[n]$ be the graded group obtained from G by shifting the grading by n. Theorem 3.7 follows from the following exact sequence

$$0 \to \varprojlim{}^1 HC[1] \to H(\varprojlim C) \xrightarrow{\nu} \varprojlim HC \to 0 \tag{23}$$

and the fact that the Mittag-Leffler condition implies the vanishing of $\varprojlim^1$. The sequence (23) is also known as *Milnor sequence* since it appeared in a slightly different context in the work of Milnor, see [10].

Remark. One can also define higher derived functors $\varprojlim^n$ of the inverse limit. This was carried out by J. Roos in [14] and G. Nöbeling in [11]. However, if the direct system is indexed on the reals the functors $\varprojlim^n$ vanish for $n \geq 2$.

4. Proof of Theorem A

Let (CM, p, i, ∂) be the bidirect system of chain complexes associated to a Morse tuple (f, g) on a manifold M or more generally to a Floer triple $\mathcal{F} = (\mathcal{C}, f, m)$. Recall from Definition 3.3 the notion of a tame bidirect system of chain complexes. We need the following lemma.

Lemma 4.1. *The bidirect system (CM, p, i, ∂) is tame.*

Proof. Since $\mathbb{R}$ is upward directed the direct limit functor commutes with the homology functor [16, Theorem IV.7]. Consequently the homomorphism μ and the homomorphisms μ_a for any $a \in \mathbb{R}$ are isomorphisms. Because the projections $p^b_{a_2,a_1}$ are surjective it follows from assertion (i) in Lemma 3.6 that for any $b \in \mathbb{R}$ the direct system of abelian groups (CM^b, p^b) satisfies the Mittag-Leffler condition. Since all the vector spaces HM^b_a are finite dimensional assertion (ii) of Lemma 3.6 implies that the direct system (HM^b, Hp^b) satisfies the Mittag-Leffler condition, too. Hence it follows from Theorem 3.7 that the homomorphisms ν^b for any $b \in \mathbb{R}$ are also isomorphisms. This proves that (CM, p, i, ∂) is tame. □

In view of Proposition 3.4 and Lemma 4.1 the following diagram commutes and ρ is an isomorphism:

$$\begin{array}{ccc} H(\varinjlim\varprojlim CM) & \xrightarrow{\rho} & \varinjlim\varprojlim HM = \overline{HM} \\ \downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\ H(\varprojlim\varinjlim CM) & \xrightarrow{\sigma} & \varprojlim\varinjlim HM = \underline{HM} \end{array} \tag{24}$$

For the following lemma recall that HM is the Morse homology obtained by taking the Novikov completion of the chain groups CM^b_a.

Lemma 4.2. *The homomorphism k and Hk are isomorphisms and*

$$H(\varprojlim\varinjlim CM) = H(\varinjlim\varprojlim CM) = HM. \tag{25}$$

Proof. If k is an isomorphism, then Hk obviously is an isomorphism, too. To see that k is an isomorphism observe that the elements of both $\varprojlim\varinjlim CM$ and $\varinjlim\varprojlim CM$ are given by Novikov sums

$$\xi = \sum_{c\in\mathcal{C}} \gamma_c c, \quad \gamma_c \in \mathbb{F}, \quad \#\{c \in \mathcal{C} : \gamma_c \neq 0,\ f(c) > b\} < \infty,\ \forall\, b \in \mathbb{R}.$$

This additionally implies the second equality in (25). □

Before continuing with the proof of Theorem A we remark that the fact that k is an isomorphism can also be deduced from Theorem 3.2 in view of the following lemma.

Lemma 4.3. *For the bidirect system* (CM, p, i, ∂) *each diagram* (17) *is bicartesian and hence in particular cartesian.*

Proof. We have to show that for each $a_1 \le a_2$ and $b_1 \le b_2$ the sequence

$$CM_{a_1}^{b_1} \overset{\left\{i_{a_1}^{b_2,b_1},\, p_{a_2,a_1}^{b_1}\right\}}{\longrightarrow} CM_{a_1}^{b_2} \oplus CM_{a_2}^{b_1} \overset{\left\langle p_{a_2,a_1}^{b_2},\, -i_{a_2}^{b_2,b_1}\right\rangle}{\longrightarrow} CM_{a_2}^{b_2} \tag{26}$$

is short exact. Since $i_{a_1}^{b_2,b_1}$ is injective the first map is an injection and since $p_{a_2,a_1}^{b_2}$ is surjective the second map is a surjection. It remains to show exactness. Let

$$\Delta_{a_2}^{b_1} \subset CM_{a_2}^{b_1} \oplus CM_{a_2}^{b_1}$$

be the diagonal. Via the embedding $CM_{a_2}^{b_1} \oplus CM_{a_2}^{b_1} \hookrightarrow CM_{a_1}^{b_2} \oplus CM_{a_2}^{b_1}$ we think of $\Delta_{a_2}^{b_1}$ as a subvectorspace of $CM_{a_1}^{b_2} \oplus CM_{a_2}^{b_1}$. Recall the notation $\underline{CM}_{a_1}^{a_2}$ from (3). We then have

$$\mathrm{im}\left\{i_{a_1}^{b_2,b_1}, p_{a_2,a_1}^{b_1}\right\} = \Delta_{a_2}^{b_1} \cup \left(\underline{CM}_{a_1}^{a_2} \oplus \{0\}\right) = \ker\left\langle p_{a_2,a_1}^{b_2}, -i_{a_2}^{b_2,b_1}\right\rangle.$$

This shows exactness and hence the lemma is proved. □

End of proof of Theorem A. Setting $\overline{\rho} = \rho$ and $\underline{\rho} = \sigma \circ Hk$ we conclude from the diagram (24) using Lemma 4.2 that the diagram (1) is commutative with $\overline{\rho}$ an isomorphism. It remains to show that $\underline{\rho}$ is surjective. Using the formula

$$\underline{\rho} = \sigma \circ Hk = (\varprojlim \mu)^{-1} \circ \nu \circ Hk$$

and the fact that $\varprojlim\mu$ and Hk are isomorphisms we are reduced to show that ν is surjective. Since for any $a_1 \le a_2$ the homomorphism $\varinjlim p_{a_2,a_1}$ is surjective we conclude that the bidirect system $(\varinjlim CM, \varinjlim p)$ satisfies the Mittag-Leffler condition. Hence it follows again from Theorem 3.7 that ν is surjective. We are done with the proof of Theorem A. □

Remark. Using Milnor's exact sequence (23) one observes that the kernel of the canonical homomorphism κ is given by

$$\ker\kappa = \varprojlim{}^1 H(\varinjlim CM) = \varprojlim{}^1 \varinjlim HM.$$

Appendix A. Integer coefficients

The homomorphism $\overline{\rho}\colon HM \to \overline{HM}$ need not be an isomorphism any more if one uses integer coefficients. We show this in an example. We consider the following Floer triple $\mathcal{F} = (\mathcal{C}, f, m)$. The critical set $\mathcal{C}$ is given by

$$\mathcal{C} = \big\{\overline{c}_n : n \in \mathbb{N} \cup \{0\}\big\} \cup \big\{\underline{c}_n : n \in \mathbb{N}\big\}.$$

The function f satisfies

$$f(\overline{c}_n) = -n, \quad f(\underline{c}_n) = -n-1$$

and the nonvanishing entries of m are

$$m(\underline{c}_n, \overline{c}_{n-1}) = 1, \quad m(\underline{c}_n, \overline{c}_n) = -2, \qquad n \in \mathbb{N}.$$

We point out again that in the following theorem we use integer coefficients.

Theorem A.1. *For the Floer triple $\mathcal{F}$ as above, $\overline{HM} = 0$, but $HM \neq 0$.*

Proof. We prove the theorem in three steps. For $n \in \mathbb{N} \cup \{0\}$ we use the abbreviation

$$\gamma_n = \sum_{j=0}^{n} 2^{n-j} \overline{c}_j.$$

Step 1: *For $b \geq 0$ and $a \leq -1$ with $k = \lfloor -a \rfloor$ we have*

$$HM_a^b = \mathbb{Z}[\gamma_{k-1}] \oplus \mathbb{Z}[\gamma_k].$$

We first observe that the chain group is given by

$$CM_a^b = \bigoplus_{j=0}^{k} \mathbb{Z}\overline{c}_j \oplus \bigoplus_{j=1}^{k-1} \mathbb{Z}\underline{c}_j.$$

We claim that

$$\text{im}\partial_a^b = \bigoplus_{j=1}^{k-1} \mathbb{Z}\underline{c}_j. \tag{27}$$

It is clear that the left-hand side is contained in the right-hand side since there are no gradient flow lines starting from a critical point $\overline{c}_n$. To see the other inclusion, observe that

$$\underline{c}_n = \partial_a^b \gamma_{n-1}, \quad n \in \{1, \dots, k-1\} \tag{28}$$

which implies (27). We next show that

$$\ker\partial_a^b = \mathbb{Z}\gamma_{k-1} \oplus \mathbb{Z}\gamma_k \oplus \text{im}\partial_a^b. \tag{29}$$

It is straightforward to check that the righthand side is contained in the kernel of the boundary operator. To see the other inclusion we observe that $\{\gamma_0, \dots, \gamma_k\}$ is another $\mathbb{Z}$-basis of the free abelian group $\bigoplus_{j=0}^{k} \mathbb{Z}\overline{c}_j$. Indeed, the two bases are related by an upper triangular matrix with diagonal entries one. In particular, the determinant of this matrix is one. Assertion (29) therefore follows from (28). Step 1 is an immediate consequence of (29).

Step 2: $\overline{HM} = 0$.

We first prove that for $b \geq 0$ we have

$$\lim_{\substack{\longleftarrow \\ a \to -\infty}} HM_a^b = 0. \tag{30}$$

To see this assume that

$$x \in \lim_{\substack{\longleftarrow \\ a \to -\infty}} HM_a^b.$$

Then

$$x = \{x_a\}_{a \leq b},$$

where $x_a \in HM_a^b$ and for $a_1 \leq a_2 \leq b$ the equation

$$Hp^b_{a_2,a_1}(x_{a_1}) = x_{a_2}$$

holds. We have to show that

$$x_a = 0, \quad a \leq b.$$

This is clear if $a > 0$ since in this case $HM_a^b = 0$, because there are no critical points of positive action. If $a \in (-1, 0]$, then

$$HM_a^b = \mathbb{Z}[\overline{c}_0] = \mathbb{Z}[\gamma_0].$$

and hence there exists $n_a \in \mathbb{Z}$ such that

$$x_a = n_a[\gamma_0], \quad a \in (-1, 0]. \tag{31}$$

Since there are no critical points in the action window $(-1, 0)$ we conclude

$$n_a = n_0, \quad a \in (-1, 0].$$

If $a \leq -1$, then by Step 1 there exist $n_a^1, n_a^2 \in \mathbb{Z}$ such that

$$x_a = n_a^1[\gamma_{\lfloor -a \rfloor -1}] + n_a^2[\gamma_{\lfloor -a \rfloor}], \quad a \leq -1. \tag{32}$$

Again since for each $k \in \mathbb{N}$ there are no critical points in the action window $(-k-1, -k)$ we conclude that

$$n_a^1 = n^1_{\lceil a \rceil},\ n_a^2 = n^2_{\lceil a \rceil}, \quad a \leq -1.$$

Hence to prove (30) we are left with showing

$$n_0 = 0, \qquad n^1_{-k} = n^2_{-k} = 0, \quad k \in \mathbb{N}. \tag{33}$$

For $k \in \mathbb{N} \cup \{0\}$ and $\ell \in \mathbb{N}$ we compute

$$p^b_{-k,-k-\ell}\gamma_{k+\ell} = 2^\ell \gamma_k, \quad p^b_{-k,-k-\ell}\gamma_{k+\ell-1} = 2^{\ell-1}\gamma_k. \tag{34}$$

Applying (34) with $k = 0$ we obtain using (31) and (32) the equation

$$n_0 = 2^{\ell-1}n^1_{-\ell} + 2^\ell n^2_{-\ell} = 2^{\ell-1}\big(2n^1_{-\ell} + n^2_{-\ell}\big), \quad \ell \in \mathbb{N}. \tag{35}$$

Since (35) holds for any $\ell \in \mathbb{N}$ but a nonzero integer is not divisible by an arbitrary high power of 2 we conclude from (35) that

$$n_0 = 0. \tag{36}$$

Applying (34) for $k \in \mathbb{N}$ and again using (31) and (32) we get the equation

$$n^1_{-k} = 0, \quad n^2_{-k} = 2^{\ell-1}\big(2n^1_{-k-\ell} + n^2_{-k-\ell}\big), \quad k, \ell \in \mathbb{N}. \tag{37}$$

The same reasoning which was used in the derivation of (36) leads now to

$$n^2_{-k} = 0, \quad k \in \mathbb{N}. \tag{38}$$

Hence the above three formulas give (33) and therefore (30). We conclude that

$$\overline{HM} = \lim_{\substack{\longrightarrow \\ b\to\infty}} \lim_{\substack{\longleftarrow \\ a\to-\infty}} HM^b_a = 0.$$

This finishes the proof of Step 2.

Step 3: $HM \neq 0$.

Choose a sequence $\{a_j\}_{j\in\mathbb{N}}$ with $a_j \in \mathbb{Z}$ for all $j \in \mathbb{N}$ which satisfies the following two conditions.

- $\lim_{k\to\infty} \sum_{j=1}^k 2^{j-1} a_j = \infty$,
- $0 < \frac{1}{2^k} \sum_{j=1}^k 2^{j-1} a_j < 3/4$ for all k

(Such a sequence can be easily constructed consisting only of zeroes and ones). We show that the element

$$\xi = \sum_{j=1}^\infty a_j \underline{c}_j$$

gives rise to a nonvanishing class in HM. Obviously ξ is in the kernel of the boundary operator. To show that it is not in the image we argue by contradiction and assume that there exists

$$\eta = \sum_{j=0}^\infty b_j \overline{c}_j$$

which coefficients $b_j \in \mathbb{Z}$ such that

$$\partial\eta = \xi.$$

It follows that

$$a_j = -2b_j + b_{j-1}, \quad j \in \mathbb{N}.$$

By induction on this formula we obtain

$$b_0 = \sum_{j=1}^k 2^{j-1} a_j + 2^k b_k$$

for each $k \in \mathbb{N}$. By our first assumption on the sequence a_j we can find $n \in \mathbb{N}$ such that

$$\sum_{j=1}^k 2^{j-1} a_j > b$$

for all $k \geq n$. The second assumption on the a_j then implies that

$$b_k = 2^{-k} b_0 - 2^{-k} \sum_{j=1}^{k} 2^{j-1} a_j \in (-1, 0)$$

for $k \geq n$ sufficiently large. But b_k is an integer. This contradiction shows that ξ does not lie in the image of ∂. This implies Step 3 and hence the theorem. □

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