ANNALES

DE

L'INSTITUT FOURIER

Kai CIELIEBAK, Urs FRAUENFELDER & Gabriel P. PATERNAIN

Stability is not open

Tome 60, nº 7 (2010), p. 2449-2459.

<http://aif.cedram.org/item?id=AIF_2010__60_7_2449_0>

Ann. Inst. Fourier, Grenoble
60, 7 (2010) 2449-2459

STABILITY IS NOT OPEN

by Kai CIELIEBAK,
Urs FRAUENFELDER & Gabriel P. PATERNAIN (*)

ABSTRACT. — We give an example of a symplectic manifold with a stable hypersurface such that nearby hypersurfaces are typically unstable.

RÉSUMÉ. — Nous donnons un exemple d'une variété symplectique contenant une hypersurface stable telle que les hypersurfaces voisines sont instables.

1. Introduction

A closed hypersurface Σ in a symplectic manifold (M, Ω) is called *stable* if a neighbourhood of Σ can be foliated by hypersurfaces whose characteristic foliations are conjugate. Here the characteristic foliation on a hypersurface Σ is defined by the 1-dimensional distribution $\ker(\Omega|_\Sigma)$. Stability was introduced in [12] as a condition on hypersurfaces for which the Weinstein conjecture can be proved. More recently, it has attained importance as the condition needed for the compactness results underlying Symplectic Field Theory [7, 2, 5] and Rabinowitz Floer homology [3, 4].

Let us consider, in a fixed symplectic manifold (M, Ω), the space $\mathcal{HS}$ of closed hypersurfaces equipped with the C^∞-topology and its subset $\mathcal{SHS}$ of stable hypersurfaces. It is easy to see that $\mathcal{SHS}$ is not closed: For example, the horocycle flow on a hyperbolic surface defines a hypersurface which is unstable but the smooth limit of stable ones; see [4] for many more examples. On the other hand, $\mathcal{SHS}$ contains open components, e.g. those corresponding to hypersurfaces of contact type. This prompted the question whether the set $\mathcal{SHS}$ is actually open in $\mathcal{HS}$. The result of this paper shows that this is not the case.

Keywords: Stability, Hamiltonian structure, characteristic foliation.
Math. classification: 53D40, 53D25.
(*) The first author was partially supported by DFG grant CI 45/1-3. The second author was partially supported by the Basic research fund 2010-0007669 funded by the Korean government.

Theorem 1.1. — *There exists a stable closed hypersurface Σ in a symplectic 6-manifold such that nearby hypersurfaces are typically unstable in the following sense: There exists a neighbourhood of Σ in $\mathcal{HS}$ which contains an open dense set consisting of unstable hypersurfaces.*

The theorem continues to hold if the C^∞ topology is replaced by the C^k topology for some $k \geqslant 2$ and hypersurfaces are only assumed to be of class C^k.

The theorem can be rephrased in terms of *stable Hamiltonian structures* [2, 5, 6]. A two-form ω on an odd-dimensional manifold Σ is called a *Hamiltonian structure* if it is closed and maximally nondegenerate in the sense that its kernel distribution is one-dimensional. It is called *stable* if there exists a one-form λ such that $\lambda|_{\ker\omega} \neq 0$ and $\ker\omega \subset \ker d\lambda$. Then a hypersurface Σ in a symplectic manifold (M, Ω) is stable iff $\Omega|_\Sigma$ defines a stable Hamiltonian structure, and every stable Hamiltonian structure arises as a stable hypersurface in some symplectic manifold [5]. Now Theorem 1.1 can be rephrased as follows: *There exists a stable Hamiltonian structure ω on a closed 5-manifold Σ such that nearby Hamiltonian structures with the same cohomology class as ω are typically unstable.*

Theorem 1.1 has implications on the foundations of holomorphic curve theories such as Symplectic Field Theory [7, 2, 5] and Rabinowitz Floer homology [3, 4]. For the construction of those theories one needs to perturb a given stable Hamiltonian structure to make all closed characteristics nondegenerate. Theorem 1.1 suggests that such a perturbation may not be possible within the class of stable Hamiltonian structures (see also [6] for a result pointing in the same direction). In Rabinowitz Floer homology this problem can be overcome in the following way [4]: One chooses an additional Hamiltonian perturbation of the Rabinowitz action functional. For a generic small perturbation the Rabinowitz action functional becomes Morse, but for the perturbed action functional one might lose compactness. However, one can still define a boundary operator by taking into account only gradient flow lines close to the original ones. We wonder if a similar strategy can be applied to SFT as well.

2. Preliminaries on Anosov Hamiltonian structures

Anosov Hamiltonian structures. Recall that the flow ϕ_t of a vector field F on a closed manifold Σ is *Anosov* if there is a splitting $T\Sigma = \mathbb{R}F \oplus E^s \oplus E^u$ and positive constants λ and C such that for all $x \in \Sigma$

$$|d_x\phi_t(v)| \leqslant Ce^{-\lambda t}|v| \text{ for } v \in E^s \text{ and } t \geqslant 0,$$

$$|d_x\phi_{-t}(v)| \leqslant Ce^{-\lambda t}|v| \text{ for } v \in E^u \text{ and } t \geqslant 0.$$

If an Anosov vector field F is rescaled by a positive function its flow remains Anosov [1, 15]. It will be useful for us to know how the bundles E^s and E^u change when we rescale F by a smooth positive function $r\colon \Sigma \to \mathbb{R}_+$. Let $\widetilde{\phi}$ be the flow of rF and $\widetilde{E}^s$ its stable bundle. Then (*cf.* [15])

$$\widetilde{E}^s(x) = \big\{v + z(x,v)F(x)\colon v \in E^s(x)\big\}, \tag{2.1}$$

where $z(x,v)$ is a continuous 1-form (*i.e.* linear in v and continuous in x). Moreover, if we let $l = l(t,x)$ be (for fixed x) the inverse of the diffeomorphism

$$t \mapsto \int_0^t r(\phi_s(x))^{-1}\, ds$$

then

$$d\widetilde{\phi}_t(v + z(x,v)F(x)) = d\phi_l(v) + z(\phi_l(x), d\phi_l(v))F(\phi_l(x)). \tag{2.2}$$

This shows that for closed Σ the flow $\widetilde{\phi}_t$ is again Anosov. There is a similar expression for $\widetilde{E}^u$. It is clear from the discussion above that the weak bundles $\mathbb{R}F \oplus E^s$ and $\mathbb{R}F \oplus E^u$ do not change under rescaling of F (the strong bundles $E^{s,u}$ are indeed affected by rescaling as we have just seen).

Let (Σ,ω) be a Hamiltonian structure. We say that the structure is *Anosov* if the flow of any vector field F spanning $\ker\omega$ is Anosov.

We say that an Anosov Hamiltonian structure satisfies the $1/2$-*pinching condition* or that it is *1-bunched* [10, 9] if for any vector field F spanning $\ker\omega$ with flow ϕ_t there are functions $\mu_f, \mu_s\colon \Sigma\times\mathbb{R}_+ \to \mathbb{R}_+$ such that

- $\lim_{t\to\infty}\sup_{x\in\Sigma}\frac{\mu_s(x,t)^2}{\mu_f(x,t)} = 0$;
- $\mu_f(x,t)|v| \leqslant |d\phi_t(v)| \leqslant \mu_s(x,t)|v|$ for all $x \in \Sigma$, $t > 0$ and $v \in E^s(x)$, and $\mu_f(x,t)|v| \leqslant |d\phi_{-t}(v)| \leqslant \mu_s(x,t)|v|$ for all $x \in \Sigma$, $t > 0$ and $v \in E^u(\phi_t x)$.

We remark that the $1/2$-pinching condition is invariant under rescaling. Indeed, consider the flow $\widetilde{\phi}_t$ of rF. It is clear from (2.1) and (2.2) that there is a positive constant κ such that

$$\frac{1}{\kappa}\mu_f(x, l(t,x))|\tilde{v}| \leqslant |d\widetilde{\phi}_t(\tilde{v})| \leqslant \kappa\mu_s(x, l(t,x))|\tilde{v}|$$

for $t > 0$ and $\tilde{v} \in \widetilde{E}^s$ (with a similar expression for $\widetilde{E}^u$). We know that given $\varepsilon > 0$, there exists $T > 0$ such that for all $x \in \Sigma$ and all $t > T$ we have

$$\frac{\mu_s(x,t)^2}{\mu_f(x,t)} < \varepsilon.$$

On the other hand, there exists $a > 0$ such that $l(t,x) \geqslant at$ for all $x \in \Sigma$ and $t > 0$. Hence, if we choose $t > T/a$ we have

$$\frac{\mu_s(x, l(t,x))^2}{\mu_f(x, l(t,x))} < \varepsilon$$

for all $x \in \Sigma$. Therefore

$$\lim_{t\to\infty} \sup_{x\in\Sigma} \frac{\mu_s(x, l(t,x))^2}{\mu_f(x, l(t,x))} = 0$$

and thus $\widetilde{\phi}_t$ is also 1/2-pinched.

Hence the Anosov property as well as the 1/2-pinching condition are invariant under rescaling and thus intrinsic properties of the Hamiltonian structure. One of the main consequences of the 1/2-pinching condition is that the weak bundles $\mathbb{R}F \oplus E^s$ and $\mathbb{R}F \oplus E^u$ are of class C^1 [9, Theorem 5] (see also [11]).

Stable Anosov Hamiltonian structures. Suppose now (Σ, ω) is a *stable* Anosov Hamiltonian structure satisfying the 1/2-pinching condition. Let λ be a stabilizing 1-form and R the Reeb vector field defined by $i_R\omega = \lambda_0$ and $\lambda(R) = 1$. Invariance under the flow implies that ω and λ both vanish on E^s and E^u. Since the flow ϕ_t of R is Anosov and $E^s \oplus E^u = \ker\lambda$ which is C^∞, it follows that $E^s = \ker\lambda \cap (\mathbb{R}F \oplus E^s)$ and E^u must be C^1. Under these conditions we can introduce the *Kanai connection* [13] which is defined as follows.

Let I be the $(1,1)$-tensor on Σ given by $I(v) = -v$ for $v \in E^s$, $I(v) = v$ for $v \in E^u$ and $I(R) = 0$. Consider the symmetric non-degenerate bilinear form given by

$$h(X,Y) := \omega(X, IY) + \lambda \otimes \lambda(X,Y).$$

The pseudo-Riemannian metric h is of class C^1 and thus there exists a unique C^0 affine connection ∇ such that:

(1) h is parallel with respect to ∇;
(2) ∇ has torsion $\omega \otimes R$.

This connection has the following desirable properties [8, 13]: it is invariant under ϕ_t and the Anosov splitting is invariant under ∇ (*i.e.* if X is any section of $E^{s,u}$ then $\nabla_v X \in E^{s,u}$ for any v).

The other good consequence of the 1/2-pinching condition, besides C^1 smoothness of the bundles, is the following lemma (*cf.* [13, Lemma 3.2]).

Lemma 2.1. — $\nabla(d\lambda) = 0$.

Proof. — Suppose τ is any invariant $(0,3)$-tensor annihilated by R. We claim that τ must vanish. To see this, consider for example a triple of vectors (v_1, v_2, v_3) where $v_1, v_2 \in E^s$ but $v_3 \in E^u$. Then there is a constant $C > 0$ such that for all $t \geqslant 0$

$$\begin{aligned}|\tau_x(v_1, v_2, v_3)| &= |\tau_{\phi_t x}(d\phi_t(v_1), d\phi_t(v_2), d\phi_t(v_3))| \\ &\leqslant C\mu_s(x,t)^2 \mu_f(x,t)^{-1}|v_1||v_2||v_3|.\end{aligned}$$

By the 1/2-pinching condition the last expression tends to zero as $t \to \infty$ and therefore $\tau_x(v_1, v_2, v_3) = 0$. The same will happen for other possible triples (v_1, v_2, v_3) when we let $t \to \pm\infty$.

Since $d\lambda$ and ∇ are ϕ_t-invariant, so is $\nabla(d\lambda)$. Since $i_R d\lambda = 0$, $\nabla(d\lambda)$ is also annihilated by R (to see that $\nabla_R(d\lambda) = 0$ use that $d\lambda$ is ϕ_t-invariant and that $\nabla_R = L_R$). Hence by the previous argument applied to $\tau = \nabla(d\lambda)$ we conclude that $\nabla(d\lambda) = 0$ as desired. □

Quasi-conformal Anosov Hamiltonian structures. Let ϕ_t be an Anosov flow on Σ endowed with a C^0-Riemannian metric. Consider the following functions on $\Sigma \times \mathbb{R}$:

$$K^s(x,t) = \frac{\max\{|d\phi_t(v)| \colon v \in E^s(x),\ |v| = 1\}}{\min\{|d\phi_t(v)| \colon v \in E^s(x),\ |v| = 1\}},$$

$$K^u(x,t) = \frac{\max\{|d\phi_t(v)| \colon v \in E^u(x),\ |v| = 1\}}{\min\{|d\phi_t(v)| \colon v \in E^u(x),\ |v| = 1\}}.$$

The flow ϕ_t is said to be *quasi-conformal* if K^u and K^s are both bounded on $\Sigma \times \mathbb{R}$. This property is clearly independent of the choice of Riemannian metric used to define K^s and K^u. Moreover it is shown in [18, Proposition 3.5] that quasi-conformality is independent of times changes, thus it makes sense to talk about quasi-conformal Anosov Hamiltonian structures. The next theorem will be useful for us.

THEOREM 2.2 ([18], Theorems 1.3 and 1.4). — *Let ϕ_t be a topologically mixing Anosov flow with $\dim E^s \geqslant 2$ and $\dim E^u \geqslant 2$. If ϕ_t is quasi-conformal, then the weak bundles are C^∞.*

Recall that ϕ_t is topologically mixing if for any two nonempty open sets U and V in Σ, there is a compact set $K \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus K$ we have $\phi_t(U) \cap V \neq \emptyset$. Recall also that ϕ_t is said to be transitive if there is a dense orbit. Our Anosov flows will always be transitive since they preserve a smooth volume form [14, Chapter 18].

3. A theorem

Theorem 3.1. — *Let (Σ, ω) be a 1/2-pinched Anosov Hamiltonian structure with $[\omega] \neq 0$, but $[\omega^2] = 0$. Suppose in addition that Σ fibres over a closed 3-manifold with fibres diffeomorphic to S^2 and transversal to the weak subbundles. Then, if (Σ, ω) is stable, the weak subbundles must be C^∞.*

Proof. — The proof of this theorem is very much inspired by the proof of Theorem 2 in [13]. We first make the following observation:

- E^s (E^u) cannot contain a nontrivial proper continuous subbundle.

Indeed since $\mathbb{R}R \oplus E^u$ is transversal to the fibres of the fibration $\Sigma \to M$ by 2-spheres, we can write $T\Sigma = V \oplus \mathbb{R}R \oplus E^u$ where V is the vertical subbundle of the fibration. Using this splitting we may define an isomorphism $E^s \mapsto V$ and since the tangent bundle of S^2 does not admit a nontrivial proper continuous subbundle, the same holds for E^s (and E^u).

Next we observe that the stabilizing 1-form λ cannot be closed. Indeed, write $\omega^2 = d\tau$ and note that if λ was closed, then the volume form $\lambda \wedge d\tau$ would be exact, which is absurd.

Since ω is non-degenerate, there exists a smooth bundle map $L\colon E^s \oplus E^u \to E^s \oplus E^u$ such that for sections X, Y of $E^s \oplus E^u$

$$d\lambda(X, Y) = \omega(LX, Y) = \omega(X, LY).$$

The map L is invariant under ϕ_t and preserves the decomposition $E^s \oplus E^u$, *i.e.* $L = L^s + L^u$, where $L^s\colon E^s \to E^s$ and $L^u\colon E^u \to E^u$. In particular, L commutes with I. By Lemma 2.1, the 1/2-pinching condition implies that $\nabla(d\lambda) = 0$ and thus L is parallel with respect to ∇. Note that by transitivity of ϕ_t, the characteristic polynomial of L^s_x is independent of $x \in \Sigma$. Let $\rho \in \mathbb{C}$ be an eigenvalue of L^s. Consider $A := L^s - \Re(\rho)\,\mathrm{Id}$. Note that A cannot be zero: Otherwise $d\lambda = c\,\omega$ for a constant $c \in \mathbb{R}$; since λ is not closed, $c \neq 0$, which in turns implies $[\omega] = 0$, contradicting the hypotheses of the theorem.

Clearly A^2 has $\mu := -\Im(\rho)^2$ as an eigenvalue. Let $H \subset E^s$ denote the eigenspace of the eigenvalue μ. Since L^s is parallel it has the same dimension at every point $x \in \Sigma$ and since E^s cannot contain a nontrivial proper continuous subbundle, we deduce that $H = E^s$. Hence $A^2 = \mu\,\mathrm{Id}$. Moreover $\mu \neq 0$, otherwise $\ker A$ would be a nontrivial proper continuous subbundle of E^s. Therefore we have proved that

$$\mathbb{J}^s := \frac{1}{\Im(\rho)}(L^s - \Re(\rho)\,\mathrm{Id})$$

defines a parallel almost complex structure on E^s of class C^1 invariant under ϕ_t. Similarly we obtain an almost complex structure $\mathbb{J}^u$ on E^u.

Now choose a Riemannian metric on E^s (resp. E^u) which is invariant under $\mathbb{J}^s$ (resp. $\mathbb{J}^u$). By declaring E^s, E^u and $\mathbb{R}R$ orthogonal and R with norm 1, we obtain a metric (of class C^1) on Σ such that with respect to this metric

$$\frac{\max\{|d\phi_t(v)|\colon v \in E^s(x),\ |v|=1\}}{\min\{|d\phi_t(v)|\colon v \in E^s(x),\ |v|=1\}} = 1,$$

for all $t \in \mathbb{R}$ and $x \in \Sigma$. This is because ϕ_t preserves $\mathbb{J}^s$ and E^s has rank two. Similarly for E^u. This shows that (Σ, ω) is a quasi-conformal Anosov Hamiltonian structure.

Finally we note that if a transitive Anosov flow is not topologically mixing, then by a theorem of J. Plante [17] it must be a suspension with constant return function. In particular, this implies that there is a closed 1-form β such that $\beta(R) > 0$. The same argument above that proved that λ cannot be closed shows that such a β cannot exist. Hence ϕ_t is topologically mixing and by Theorem 2.2 the weak bundles must be C^∞. $\square$

Remark 3.2. — Note that the proof above only requires λ to be of class C^2.

4. The example

Let Γ be a discrete group of isometries of $\mathbb{H}^3$ such that $M := \Gamma \setminus \mathbb{H}^3$ is a closed orientable hyperbolic 3-manifold. We consider the geodesic flow acting on the unit sphere bundle SM and let α be the canonical contact 1-form.

The space of invariant 2-forms of the geodesic flow of $M = \Gamma \setminus \mathbb{H}^3$ has dimension two [13, Claim 3.3]. It is spanned by the 2-form $d\alpha$ and the additional 2-form ψ which we now describe. Given a unit vector $v \in T_x\mathbb{H}^3$, let $i(v)\colon T_x\mathbb{H}^3 \to T_x\mathbb{H}^3$ be the linear map defined by $i(v)(v) = 0$ and $i(v)$ rotates vectors in $\{v\}^\perp$ by $\pi/2$ according to the orientation of $\mathbb{H}^3$. Any vector $\xi \in T_vS\mathbb{H}^3$ can be written as $\xi = (\xi_H, \xi_V)$ with the usual identification of horizontal and vertical components (*cf.* [16]). Define $J_v\colon T_vS\mathbb{H}^3 \to T_vS\mathbb{H}^3$ as

$$J_v(\xi_H, \xi_V) = (i(v)\xi_V, i(v)\xi_H). \tag{4.1}$$

Then

$$\psi_v(\xi, \eta) := d\alpha_v(J_v\xi, \eta) = \langle i(v)\xi_V, \eta_V\rangle - \langle i(v)\xi_H, \eta_H\rangle. \tag{4.2}$$

Clearly this construction descends to SM where we use the same notation (ψ, α, etc.) In a moment we will check that ψ is invariant under ϕ_t, but before we do so, let us describe the stable and unstable bundles of ϕ_t and the action of $d\phi_t$ on them. Recall that $d\phi_t(\xi_H,\xi_V)=(Y(t),\dot{Y}(t))$ where Y is the unique Jacobi field (along the geodesic $\pi\phi_t(v)$, where $\pi\colon SM\to M$ is foot-point projection) with initial conditions (ξ_H,ξ_V). Solving the Jacobi equation $\ddot{Y}-Y=0$ we find:

$$\begin{aligned}E^s(v)&=\{(w,-w)\colon w\perp v\},\\ E^u(v)&=\{(w,w)\colon w\perp v\}.\end{aligned}$$

Note that J leaves E^s and E^u invariant. Moreover

$$\begin{aligned}d\phi_t(w,-w)&=e^{-t}(e_w(t),-e_w(t)),\\ d\phi_t(w,w)&=e^{t}(e_w(t),e_w(t)),\end{aligned}$$

where $e_w(t)$ is the parallel transport of w along the geodesic $\pi\phi_t(v)$. Since $e_{i(v)w}(t)=i(\pi\phi_t v)e_w(t)$ we see that $d\phi_t$ preserves J. Since $d\alpha$ is also ϕ_t invariant, it follows that ψ is invariant. Note that $i_R\psi=0$ for the Reeb vector field R of α.

Lemma 4.1. — *The invariant 2-form ψ is closed but not exact. The 4-form ψ^2 is exact and (SM,ψ) is a stable Hamiltonian structure with stabilizing 1-form α and Reeb vector field R.*

Proof. — The 3-form $d\psi$ is invariant under ϕ_t and is annihilated by R. Then the proof of Lemma 2.1 shows that $d\psi=0$ (obviously ϕ_t is 1/2-pinched). In order to show that $[\psi]\neq 0$, consider S_x the 2-sphere of unit vectors in $T_x\mathbb{H}^3$. A tangent vector $\xi\in T_vS_x$ has the form $\xi=(0,w)$ where $w\perp v$. If we take two tangent vectors $\xi=(0,w)$, $\eta=(0,u)\in T_vS_x$, from (4.1) and (4.2) we see that

$$\psi_v(\xi,\eta)=\langle i(v)w,u\rangle.$$

This implies that

$$\int_{S_x}\psi\neq 0$$

and thus $[\psi]\neq 0$. Consider now the invariant 4-form ψ^2 and the invariant 5-form $\alpha\wedge\psi^2$. By transitivity, there is a constant k such that $\alpha\wedge\psi^2=k\,\alpha\wedge(d\alpha)^2$. Contracting with R we see that ψ^2 must be $k\,(d\alpha)^2$ and therefore exact. Finally, it is immediate from the definition (4.2) of ψ that its restriction to $E^s\oplus E^u=\ker\alpha$ is non-degenerate. Hence (SM,ψ) is a Hamiltonian structure with stabilizing 1-form α and Reeb vector field R. $\square$

Now let $X:=SM\times(-\varepsilon,\varepsilon)$ and $\tau\colon X\to SM$ the obvious projection. Define $\omega_X:=d(r\tau^*\alpha)+\tau^*\psi$, where $r\in(-\varepsilon,\varepsilon)$. For ε small enough (X,ω_X) is a symplectic manifold and $r=0$ is the stable hypersurface (SM,ψ).

We have now come to our main result which implies Theorem 1.1 in the introduction.

THEOREM 4.2. — *A typical hypersurface $\Sigma \subset X$ near SM is not stable.*

Proof. — Consider a hypersurface Σ near $r = 0$ and let ω be ω_X restricted to Σ. By Lemma 4.1, $[\omega] \neq 0$, but $[\omega^2] = 0$. Since SM fibres over M with fibres given by 2-spheres transveral to the weak bundles the same holds true for Σ (recall that under perturbations the stable and unstable bundles vary continuously). Finally we note that (Σ, ω) is 1/2-pinched. Indeed, recall that for the geodesic flow of M, we have

$$|d\phi_t(\xi)| = e^{-t}|\xi| \text{ for } \xi \in E^s,$$
$$|d\phi_t(\xi)| = e^{t}|\xi| \text{ for } \xi \in E^u.$$

Thus for a flow φ_t which is C^1 close to ϕ_t we get

$$\frac{1}{C}|\xi|e^{-At} \leqslant |d\varphi_t(\xi)| \leqslant C|\xi|e^{-at} \ \text{ for } \xi \in E^s \ \text{ and } t \geqslant 0,$$
$$\frac{1}{C}|\xi|e^{-At} \leqslant |d\varphi_{-t}(\xi)| \leqslant C|\xi|e^{-at} \ \text{ for } \xi \in E^u \ \text{ and } t \geqslant 0,$$

where all the constants C, A, a are close to 1. Thus (Σ, ω) is 1/2-pinched.

We can now apply Theorem 3.1 to conclude that if Σ near $r = 0$ is stable, then the weak bundles must be C^∞. However, a theorem of Hasselblatt [10, Corollary 1.10] asserts that an open and dense set of symplectic Anosov systems does not have weak bundles of class $C^{2-\varepsilon}$. Thus a typical hypersurface Σ near $r = 0$ cannot be stable. ☐

Remark 4.3. — It is possible to prove the last theorem without appealing to Theorem 2.2. An inspection of the proof of Theorem 3.1 shows that since $d\phi_t$ preserves $\mathbb{J}$, all the closed orbits are actually 2-bunched in the terminology of [10], and the local perturbation argument in [10, Section 4] implies that an open and dense set of symplectic Anosov systems does not have all closed orbits being 2-bunched (this fact is actually used in the proof of [10, Corollary 1.10] quoted above). Of course, the conclusion of Theorem 3.1 is stronger if we use Theorem 2.2.

BIBLIOGRAPHY

[1] D. V. ANOSOV & J. G. SINAĬ, "Certain smooth ergodic systems", *Uspehi Mat. Nauk* **22** (1967), no. 5 (137), p. 107-172.

[2] F. BOURGEOIS, Y. ELIASHBERG, H. HOFER, K. WYSOCKI & E. ZEHNDER, "Compactness results in symplectic field theory", *Geom. Topol.* **7** (2003), p. 799-888.

[3] K. CIELIEBAK & U. A. FRAUENFELDER, "A Floer homology for exact contact embeddings", *Pacific J. Math.* **239** (2009), no. 2, p. 251-316.

[4] K. Cieliebak, U. A. Frauenfelder & G. P. Paternain, "Symplectic topology of Mañé's critical values", *Geometry and Topology* **14** (2010), p. 1765-1870.

[5] K. Cieliebak & K. Mohnke, "Compactness for punctured holomorphic curves", *J. Symplectic Geom.* **3** (2005), no. 4, p. 589-654, Conference on Symplectic Topology.

[6] K. Cieliebak & E. Volkov, "First steps in stable Hamiltonian topology", arXiv:1003.5084, 2010.

[7] Y. Eliashberg, A. Givental & H. Hofer, "Introduction to symplectic field theory", *Geom. Funct. Anal.* (2000), no. Special Volume, Part II, p. 560-673, GAFA 2000 (Tel Aviv, 1999).

[8] R. Feres, "Geodesic flows on manifolds of negative curvature with smooth horospheric foliations", *Ergodic Theory Dynam. Systems* **11** (1991), no. 4, p. 653-686.

[9] B. Hasselblatt, "Horospheric foliations and relative pinching", *J. Differential Geom.* **39** (1994), no. 1, p. 57-63.

[10] ———, "Regularity of the Anosov splitting and of horospheric foliations", *Ergodic Theory Dynam. Systems* **14** (1994), no. 4, p. 645-666.

[11] M. W. Hirsch, C. C. Pugh & M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin, 1977, ii+149 pages.

[12] H. Hofer & E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1994, xiv+341 pages.

[13] M. Kanai, "Differential-geometric studies on dynamics of geodesic and frame flows", *Japan. J. Math. (N.S.)* **19** (1993), no. 1, p. 1-30.

[14] A. Katok & B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza, xviii+802 pages.

[15] W. Parry, "Synchronisation of canonical measures for hyperbolic attractors", *Comm. Math. Phys.* **106** (1986), no. 2, p. 267-275.

[16] G. P. Paternain, *Geodesic flows*, Progress in Mathematics, vol. 180, Birkhäuser Boston Inc., Boston, MA, 1999, xiv+149 pages.

[17] J. F. Plante, "Anosov flows", *Amer. J. Math.* **94** (1972), p. 729-754.

[18] V. Sadovskaya, "On uniformly quasiconformal Anosov systems", *Math. Res. Lett.* **12** (2005), no. 2-3, p. 425-441.

Manuscrit reçu le 20 août 2009,
accepté le 15 octobre 2010.

Kai CIELIEBAK
Ludwig-Maximilians-Universität
Mathematisches Institut
80333 München (Germany)
kai@math.lmu.de

Urs FRAUENFELDER
Seoul National University
Department of Mathematics
Research Institute of Mathematics
151-747 Seoul (South Korea)
frauenf@snu.ac.kr

Gabriel P. PATERNAIN
University of Cambridge
Department of Pure Mathematics
and Mathematical Statistics
Cambridge CB3 0WB (UK)
g.p.paternain@dpmms.cam.ac.uk