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A note on Reeb dynamics on the tight 3-sphere

F. Bourgeois, K. Cieliebak, and T. Ekholm

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Abstract

We show that a nondegenerate tight contact form on the 3-sphere has exactly two simple closed Reeb orbits if and only if the differential in linearized contact homology vanishes. Moreover, in this case the Floquet multipliers and Conley-Zehnder indices of the two Reeb orbits agree with those of a suitable irrational ellipsoid in 4-space.

1 Introduction

A *contact form* on a closed 3-manifold Y is a 1-form λ such that $\lambda \wedge d\lambda$ is a volume form on Y. The *contact structure* determined by a contact form λ is the tangent hyperplane field $\ker(\lambda) \subset TY$. The condition on $\lambda \wedge d\lambda$ guarantees that the contact structure is a completely non-integrable plane field. The *Reeb vector field* determined by the contact form λ is the vector field R_λ on Y uniquely determined by the conditions $\lambda(R_\lambda) = 1$ and $\iota_{R_\lambda} d\lambda = 0$. A periodic solution γ of the differential equation determined by R_λ is called a *(closed) Reeb orbit*.

An *overtwisted disk* in a contact 3-manifold Y is an embedded 2-disk $D \subset Y$ such that the foliation of D induced by the contact structure has exactly one singular point and such that the boundary ∂D of D is a closed leaf in this foliation. A contact 3-manifold which does not contain any overtwisted disk is called *tight*.

For a star-shaped (with respect to the origin) hypersurface $Y \subset \mathbb{R}^4 \cong \mathbb{C}^2$ the canonical 1-form $\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j)$, where $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$ are linear coordinates on $\mathbb{C}^2$, restricts to a contact form on Y. This contact form is tight, and by Eliashberg's uniqueness theorem [8] each tight contact form on the three-sphere S^3 arises in this way. Particularly simple hypersurfaces of this form are the *irrational*

ellipsoids

$$E(a_1, a_2) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} = 1 \right\}$$

for $a_1, a_2 > 0$ and $a_1/a_2 \notin \mathbb{Q}$.

In [9] Eliashberg, Givental, and Hofer introduced Symplectic Field Theory (SFT). It is a framework for extracting invariants of contact and symplectic manifolds via holomorphic curve counts.

For the tight 3-sphere there is a particularly simple SFT type invariant called "linearized contact homology". For the sake of completeness, we will recall its definition and a proof of its invariance in Section 2. Roughly speaking, linearized contact homology is defined as follows. Let λ be a contact form on S^3 such that all closed Reeb orbits are *nondegenerate*, i.e. no *Floquet multiplier* (eigenvalue of the linearized return map on a transverse section) equals 1. Then each closed Reeb orbit has a well-defined *Conley-Zehnder index* $\mathrm{CZ}(\gamma) \in \mathbb{Z}$, see [13]. Following [13], we call a contact form λ *dynamically convex* if all closed Reeb orbits are nondegenerate and have Conley-Zehnder index at least 3. E.g. this is the case for the induced contact forms on irrational ellipsoids, and more generally for the contact forms on hypersurfaces in $\mathbb{R}^4$ bounding strictly convex domains [13]. For a dynamically convex contact form λ let $CC_*(S^3, \lambda)$ be the $\mathbb{Q}$-vector space generated by the "good" (see Section 2) closed Reeb orbits, graded by their *degree* $|\gamma| = \mathrm{CZ}(\gamma) - 1$. Fixing an $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R} \times S^3$ compatible with λ, we define a differential $\partial : CC_*(S^3, \lambda) \to CC_{*-1}(S^3, \lambda)$ by counting rigid J-holomorphic cylinders connecting Reeb orbits. We show in Section 2 that $\partial^2 = 0$, that the *linearized contact homology* $HC^{\mathrm{lin}}(S^3) = \ker \partial / \mathrm{im} \partial$ is independent of the choice of J and of λ, and furthermore that

$$HC_k^{\mathrm{lin}}(S^3) = \begin{cases} \mathbb{Q} & \text{for } k \geq 2 \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

In this note we prove the following result.

Theorem 1. *Let λ be a dynamically convex tight contact form on the 3-sphere. Then the following are equivalent:*

(i) *The differential in linearized contact homology vanishes.*

(ii) *There are precisely two simple Reeb orbits γ_1, γ_2.*

Moreover, in this case γ_1 and γ_2 are unknotted, elliptic, have linking number 1, and their Conley-Zehnder indices and Floquet multipliers agree with those of a suitable irrational ellipsoid.

Remark 2. The abstract perturbation theory developed in [14] will eventually lead to a definition and an invariance proof of linearized contact homology for arbitrary (not necessarily dynamically convex) tight contact forms on S^3 with nondegenerate closed Reeb orbits, cf. Remark 7 below. Using this in combination with formulas for Conley-Zehnder indices, see Section 3.1, it is immediate that condition (i) implies dynamical convexity and an additional argument, see Remark 14, shows that (ii) does as well. Thus, the restriction in Theorem 1 to the technically simpler dynamically convex case turns out not to be any restriction at all.

Remark 3. The implication that if there are precisely two simple closed Reeb orbits γ_1, γ_2 then their Floquet multipliers lie on the unit circle with irrational angles has been proved independently in [11].

Remark 4. It follows from [13] that in the situation of Theorem 1 the closed Reeb orbit of degree 2 is the binding of an open book decomposition with pages whose interiors are transverse to the Reeb vector field. The return map of a page is an area preserving diffeomorphism of the open disk with one fixed point and no other periodic points. However, the return map need not be conjugate to an irrational rotation (see [10]).

Remark 5. In view of the result in [13], Theorem 1 implies the following dichotomy for the Reeb dynamics on a strictly convex hypersurface in $\mathbb{R}^4$. Either the differential in linearized contact homology vanishes and there are precisely two simple closed orbits; or the differential does not vanish and there are infinitely many simple closed orbits. Moreover, generically the second case occurs (even in the class of star-shaped hypersurfaces).

This discussion motivates the following conjecture. *If, for a star-shaped hypersurface in $\mathbb{R}^4$, the differential in linearized contact homology does not vanish, then it carries infinitely many closed characteristics.* More optimistically, one could even conjecture that a star-shaped hypersurface in $\mathbb{R}^{2n}$ carries either precisely n (if the differential in linearized contact homology vanishes) or infinitely many (if it does not vanish) simple closed Reeb orbits. See [12] for an exposition of known multiplicity results.

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2 Linearized contact homology

In this section we define the linearized contact homology for a dynamically convex contact form on S^3 and prove its invariance. See [9] for details of the setup.

Fix a dynamically convex contact form λ on S^3. If γ is a simple closed orbit of the Reeb field R_λ, we denote by γ^k its k-th iterate. We call γ^k *good* if the Conley-Zehnder indices (see Section 3) of γ and γ^k have the same parity. Otherwise, we call γ^k *bad*. Let $CC_*(S^3,\lambda)$ be the $\mathbb{Q}$-vector space generated by the good closed Reeb orbits, graded by their degree $|\gamma| = \mathrm{CZ}(\gamma) - 1$.

An $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R}\times S^3$ is called *compatible with* λ if it preserves $\xi = \ker\lambda$, if $d\lambda(\cdot, J\cdot)$ defines a metric on ξ, and if $J\frac{\partial}{\partial t} = R_\lambda$, where t denotes the coordinate on $\mathbb{R}$. Fix such a J. For good closed orbits $\gamma, \gamma_1, \dots, \gamma_r$ of R_λ of periods $T, T_1, \dots, T_r$, let $\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_r)$ be the moduli space consisting of equivalence classes of tuples $(x, p; y_1, p_1 \dots, y_r, p_r; f)$, where $x, y_1, \dots, y_r$ are distinct points on S^2 with directions $p, p_1, \dots, p_r$ and $f = (a,u) : S^2\backslash\{x, y_1, \dots, y_r\} \to \mathbb{R}\times S^3$ is a map with the following properties:

- $df + J\circ df\circ j = 0$,
- $\lim_{z\to x} a(z) = +\infty$ and $\lim_{z\to y_i} a(z) = -\infty$,
- in polar coordinates $(\rho,\theta) \in]0,1]\times\mathbb{R}/\mathbb{Z}$ around x in which p corresponds to $\theta = 0$ we have $\lim_{\rho\to 0} u(\rho,\theta) = \gamma(-T\theta)$, and
- in polar coordinates $(\rho_i,\theta_i) \in]0,1]\times\mathbb{R}/\mathbb{Z}$ around y_i in which p_i corresponds to $\theta_i = 0$ we have $\lim_{\rho_i\to 0} u(\rho_i,\theta_i) = \gamma_i(T_i\theta_i)$.

Two such tuples $(x, p; y_1, p_1 \dots, y_r, p_r; f)$ and $(x', p'; y_1', p_1' \dots, y_r', p_r'; f')$ are equivalent if there exists a biholomorphism h of S^2 such that $h(x) = x'$, $h(y_i) = y_i'$, $d_x h\cdot p = p'$, $d_{y_i}h\cdot p_i = p_i'$ and $f = f'\circ h$. Since J is $\mathbb{R}$-invariant, $\mathbb{R}$ acts on these moduli spaces by translation and we denote the quotient by $\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_r)/\mathbb{R}$. Using an appropriate functional analytic setup, the moduli spaces can be described as the zero locus of a Fredholm section of a certain bundle and have expected dimension (determined by the Fredholm index) given by

$$\dim\big(\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_r)/\mathbb{R}\big) = |\gamma| - \sum_{j=1}^{r}|\gamma_j| \ - 1.$$

If this dimension is zero and if the moduli space is compact and cut out transversally, then it consists of finitely many points. One can

associate a sign to each of these points via coherent orientations on the moduli spaces [3] and we denote by $n(\gamma;\gamma_1,\dots,\gamma_r)$ the algebraic count of the elements in $\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_r)/\mathbb{R}$. Furthermore, we denote by κ_γ the multiplicity of the Reeb orbit γ. Define the linear map

$$\partial : CC_*(S^3,\lambda) \to CC_{*-1}(S^3,\lambda), \qquad \gamma \mapsto \sum_{|\gamma'|=|\gamma|-1} \frac{n(\gamma;\gamma')}{\kappa_{\gamma'}}\gamma'.$$

Thus ∂ counts rigid J-holomorphic cylinders interpolating between closed Reeb orbits γ and γ'.

Theorem 6. *Let λ be a dynamically convex contact form on S^3. Then for a generic S^1-dependent compatible almost complex structure J the map ∂ is well-defined and satisfies $\partial^2 = 0$. Moreover, the* linearized contact homology $HC^{\mathrm{lin}}(S^3) = \ker\partial/\mathrm{im}\partial$ *is independent of the choice of J and λ, and is given by*

$$HC_k^{\mathrm{lin}}(S^3) = \begin{cases} \mathbb{Q} & \text{for } k \geq 2 \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows the familiar scheme from Floer homology (see e.g. [15, 9]), provided we can prove that 1-dimensional moduli spaces of holomorphic cylinders are regular and compact up to breaking into pairs of cylinders. As usual, one needs to prove this in three cases:

- in a symplectization (to establish [9, Proposition 1.9.1]),
- in a cobordism (to establish [9, Proposition 1.9.3]), and
- for a homotopy of almost complex structures on a cobordism (to establish [9, Proposition 1.9.4]).

We will explain the argument in the case of a homotopy, the other two cases being analogous but easier. Once invariance of $HC_k^{\mathrm{lin}}(S^3)$ is established it can be easily computed for an irrational ellipsoid using the formulae for the Conley-Zehnder indices in Section 3 ($\partial = 0$ in this case).

Step 1: Let $Y = S^3$. Consider $X = \mathbb{R}\times Y$ with an exact symplectic form $d\lambda$ which coincides near $\{\pm\infty\}\times Y$ with $d(e^t\lambda_\pm)$ for dynamically convex contact forms $\lambda_\pm$ on Y. Let $(J_\tau)_{\tau\in[0,1]}$ be a homotopy of almost complex structures on $\mathbb{R}\times Y$ such that for every τ, J_τ is compatible with $d\lambda$ and coincides near $\pm\infty$ with fixed $\mathbb{R}$-invariant almost complex structures $J_\pm$ compatible with $\lambda_\pm$. Fix closed Reeb orbits $\gamma_\pm$ for $\lambda_\pm$ of equal degrees $|\gamma_+| = |\gamma_-|$ and consider the 1-dimensional moduli space

$$\mathcal{M}_{[0,1]} = \cup_{\tau\in[0,1]}\mathcal{M}(\gamma_+;\gamma_-;J_\tau).$$

By the SFT compactness theorem [2, 6], every sequence f_k, $k = 1, 2, \ldots$ in $\mathcal{M}_{[0,1]}$ has a subsequence converging as $k \to \infty$ to a *broken holomorphic sphere* $F = \{F_\alpha\}_{\alpha \in T}$. Here T is a directed tree (i.e. each edge is directed) with the following properties. For each vertex $\alpha \in T$, F_α is a punctured holomorphic sphere with exactly one positive puncture and any number of negative punctures in $(\mathbb{R} \times Y, J_+)$, in $(\mathbb{R} \times Y, J_-)$, or in (X, J_τ) for some $\tau \in [0, 1]$. Each edge e of T is labeled by a closed Reeb orbit γ_e of $\lambda_\pm$. If e is directed from a vertex α to a vertex β, then γ_e is the asymptotic Reeb orbit at the unique positive puncture of F_α and at one negative puncture of F_β. Conversely, each puncture on any F_α corresponds to a unique edge in this way except for two *free punctures*, a positive one asymptotic to γ_+ and a negative one asymptotic to γ_-.

Since the Fredholm index is additive under joining spheres at Reeb orbits and since the only two free (not paired across edges) punctures of F are asymptotic to $\gamma_\pm$, the expected dimensions (i.e. the Fredholm indices) $\operatorname{ind}(F_\alpha)$ of the moduli spaces of $J_\pm$-, or J_τ-holomorphic spheres (τ fixed) which contain the F_α satisfy

$$\sum_{\alpha \in T} \operatorname{ind}(F_\alpha) = |\gamma_+| - |\gamma_-| = 0.$$

The structure of the tree T can be described as follows. Let $\alpha_\pm \in T$ be the vertices such that $F_{\alpha_\pm}$ contains the free punctures asymptotic to $\gamma_\pm$. Define the *stem* S of the tree T to be the unique linear (i.e. at most two edges meet at each vertex) subtree $S \subset T$ connecting α_+ and α_-. Define the *branches* $B_1, \ldots, B_k$ to be the connected components of $T - S$. Since each branch B_i has precisely one free puncture which is positive and asymptotic to a Reeb orbit γ_i, its total index $\operatorname{ind}(B_i) = \sum_{\alpha \in B_i} \operatorname{ind}(F_\alpha)$ satisfies

$$\operatorname{ind}(B_i) = |\gamma_i| \geq 2$$

by dynamical convexity.

The *orbit cylinder* over a closed Reeb orbit γ for $\lambda_\pm$ is the $J_\pm$-holomorphic cylinder $\mathbb{R} \times \gamma \subset \mathbb{R} \times Y$. We call a punctured holomorphic sphere *good* if it is not a branched cover of an orbit cylinder. Now let us assume that the following regularity condition holds.

(R) All good components F_α with $\alpha \in S$ in the stem are *regular*, i.e. transversely cut out by the 1-parameter family of Cauchy-Riemann operators corresponding to the homotopy $(J_\tau)_{\tau \in [0,1]}$.

We conclude that each good F_α in $(\mathbb{R} \times Y, J_\pm)$ is transversely cut out and thus satisfies $\operatorname{ind}(F_\alpha) \geq 1$ because of translation invariance (see Step 2 below for a full explanation of how to reach this conclusion). Moreover, there is a finite collection of exceptional τ-values

$0 < \tau_1 < \cdots < \tau_m < 1$ such that the following hold. For every $\tau \neq \tau_j$, $j = 1, \ldots, m$, each good F_α in (X, J_τ) and in the stem belongs to a moduli space which is transversely cut out by the $\bar{\partial}_{J_\tau}$-equation and hence $\mathrm{ind}(F_\alpha) \geq 0$. For $\tau = \tau_j$ some $j \in \{1, \ldots, m\}$, there exists a unique exceptional good punctured holomorphic sphere $C_\tau \in \mathcal{M}(\beta_0; \beta_1, \ldots, \beta_r; J_\tau)$, which may belong to the stem, with the property that the linearized $\bar{\partial}_{J_\tau}$-operator at C_τ has 1-dimensional cokernel; at all other good spheres $F_\alpha \neq C_\tau$ in the stem, the linearized $\bar{\partial}_{J_\tau}$-operator is surjective. Since the cokernel has dimension 1, it follows that $\mathrm{ind}(C_\tau) \geq -1$.

By Lemma 8 below, a branched cover F_α over an orbit cylinder has index $\mathrm{ind}(F_\alpha) \geq 0$. So we see that $\mathrm{ind}(F_\alpha) \geq 0$ for each component F_α in the stem with $F_\alpha \neq C_\tau$.

Recall that each edge $e \in T$ was labeled with a Reeb orbit γ_e. We order the edges by the actions $\mathcal{A}(\gamma_e)$. For area reasons, this order strictly increases in the direction of the tree and therefore the special component C_τ can occur at most once among the F_α with $\alpha \in S$. In particular, in view of the preceding discussion the total index of the stem S satisfies

$$\mathrm{ind}(S) \geq -1.$$

Thus, since $\mathrm{ind}(T) = 0$ and since $\mathrm{ind}(B_i) \geq 2$ for any branch B_i, we conclude that there are no branches B_i, and hence $T = S$ is a linear tree. In particular, this excludes branched covers of orbit cylinders and hence all components F_α are good. Moreover, there are only the following two possibilities for the dimensions of the F_α, $\alpha \in T$.

Case 1: $\mathrm{ind}(F_\alpha) \geq 0$ for all $\alpha \in T$, in which case T has only one vertex α, no breaking occurs, and the sequence (f_k) converges in $\mathcal{M}_{[0,1]}$ to $F_\alpha \in \mathcal{M}_{[0,1]}$.

Case 2: $\mathrm{ind}(F_{\beta^\pm}) = \pm 1$ for unique vertices $\beta^\pm \in T$ and $\mathrm{ind}(F_\alpha) = 0$ for all vertices $\alpha \in T - \{\beta^+, \beta^-\}$. Since the linear tree T contains only one component in (X, J_τ) and components in $(\mathbb{R} \times Y, J_\pm)$ have index at least 1, we conclude that $\beta^\pm$ are the only vertices in T. Hence the limit curve F is a pair of holomorphic cylinders, which is precisely what is needed for the chain homotopy property. This concludes the proof modulo the assumption that regularity condition (R) above holds.

Step 2: We adapt the technique used in [7] to achieve the regularity condition (R).

To three distinct points z_0, z_1, z_2 on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and a tangent direction $p_0 \in T_{z_0} S^2 / \mathbb{R}_+$ at z_0, where $\mathbb{R}_+$ acts on tangent vectors by scalar multiplication, we associate an angle $w(z_0, p_0, z_1, z_2) \in S^1$ as follows. Let $\phi \in \mathrm{Aut}(S^2)$ be the unique Möbius

transformation with $\phi(z_0) = 0$, $\phi(z_1) = 1$, and $\phi(z_2) = \infty$, and define

$$w(z_0, p_0, z_1, z_2) = d_{z_0}\phi \cdot p_0 \in T_0\mathbb{C}/\mathbb{R}_+ \cong S^1.$$

The map w is clearly invariant under simultaneous action of Möbius transformations on (z_0, p_0, z_1, z_2) and thus it induces a diffeomorphism

$$w : \mathcal{M}_3^{\$} \to S^1,$$

where $\mathcal{M}_3^{\$}$ is the decorated Deligne-Mumford space of 3 ordered distinct points (z_0, z_1, z_2) on the Riemann sphere with a specified direction p_0 at z_0 (see [2]). One easily sees that this map extends to arbitrary trees of spheres for which z_2 lies between z_0 and z_1 (i.e. z_2 lies on the unique embedded path connecting z_0 and z_1).

Let $\mathcal{J}_{S^1}$ be the space of S^1-dependent almost complex structures on $\mathbb{R} \times Y$ of the type considered above, i.e., which are compatible with $d\lambda$ and which coincides near $\pm\infty$ with fixed $\mathbb{R}$-invariant almost complex structures $J_\pm$. Each $J \in \mathcal{J}_{S^1}$ induces a Cauchy-Riemann operator acting on tuples (z_0, p_0, z_1, f) consisting of distinct points $z_0, z_1 \in S^2$, a direction p_0 at z_0, and a map $f : S^2 - \{z_0, z_1\} \to \mathbb{R} \times Y$ by

$$\bar\partial_J(z_0, p_0, z_1, f)(z) = \frac{1}{2}\Big(df(z) + J\big(w(z_0, p_0, z_1, z)\big) \circ df(z) \circ i\Big),$$

where $z \in S^2 - \{z_0, z_1\}$. This operator is clearly invariant under the simultaneous action of $\mathrm{Aut}(S^2)$ on (z_0, p_0, z_1, f). For fixed (z_0, p_0, z_1), after applying a Möbius transformation that sends z_0 to 0, z_1 to ∞ and p_0 to $\mathbb{R}_+$, we obtain a Cauchy-Riemann operator on maps $f\colon \mathbb{R} \times (\mathbb{R}/2\pi) \to \mathbb{R} \times Y$ where we think of the source as an infinite cylinder (using $\mathbb{C} - \{0\} \cong \mathbb{R} \times (\mathbb{R}/2\pi)$). In polar coordinates $(s, t) \in \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ this operator is given by

$$\bar\partial_J(f)(s, t) = \frac{1}{2}\Big(df(s, t) + J(e^{-it}) \circ df(s, t) \circ i\Big),$$

since $w(0, \mathbb{R}_+, \infty, e^{s+it}) = e^{-it}$. A standard argument (see e.g. [15]) shows that regularity for 1-parameter families of such Cauchy-Riemann operators can be achieved by choosing a generic path of S^1-dependent J in $\mathcal{J}_{S^1}$. More precisely, for a Baire set of paths $(J_\tau)_{\tau\in[0,1]} \in \mathcal{J}_{S^1}$ the following holds: at any pair (f, τ) such that $\bar\partial_{J_\tau} f = 0$ and such that f is not a branched cover of an orbit cylinder, the linearization of $\bar\partial_{J_\tau}$ at (f, τ) is surjective.

We use the regularity result for S^1-dependent almost complex structures to establish condition (R) as follows. Fix a point $p_{\bar\gamma}$ on each simple closed Reeb orbit $\bar\gamma$. For closed Reeb orbits $\gamma_0, \gamma_1, \ldots, \gamma_k$ with underlying simple orbits $\bar\gamma_i$ and $J \in \mathcal{J}_{S^1}$ denote by $\mathcal{M}(\gamma_0; \gamma_1, \ldots, \gamma_k; J)$ the moduli space of equivalence classes of tuples $(z_0, p_0, z_1, \ldots, z_k, f)$ of the following form:

- $z_0, \dots, z_k$ are distinct points in S^2;
- p_0 is a tangent direction at z_0;
- $f : S^2 - \{z_0, \dots, z_k\} \to \mathbb{R} \times Y$ is a map with $\bar{\partial}_J(z_0, p_0, z_1, f) = 0$, which has its positive puncture at z_0 where f is asymptotic to γ_0 and takes the tangent direction p_0 to the point $p_{\tilde{\gamma}_0}$, and which has negative punctures at $z_1, \dots, z_k$ where f is asymptotic to $\gamma_1, \dots, \gamma_k$, respectively;
- f is not a branched cover of an orbit cylinder.

Two tuples $(z_0, p_0, z_1, \dots, z_k, f)$ and $(z_0', p_0', z_1', \dots, z_k', f')$ are equivalent if they are related under the natural action of $\mathrm{Aut}(S^2)$.

The regularity result for the Cauchy-Riemann operator above implies that, for generic paths $(J_\tau)_{\tau \in [0,1]}$ in $\mathcal{J}_{S^1}$, any moduli space

$$\mathcal{M}_{[0,1]} = \cup_{\tau \in [0,1]} \mathcal{M}(\gamma_0; \gamma_1, \dots, \gamma_k; J_\tau)$$

is cut out transversally and is consequently a manifold of dimension

$$\dim(\mathcal{M}_{[0,1]}) = |\gamma_0| - \sum_{j=1}^{k} |\gamma_j| + 1.$$

Here a manifold of negative dimension is understood to be empty.

In order to establish (R) it remains to study the boundary of a 1-dimensional moduli space $\mathcal{M}_{[0,1]}$ as above. Let $f_k \in \mathcal{M}_{[0,1]}$ be a sequence as in Step 1 which converges to a broken holomorphic sphere F modelled on a tree T with stem S. Then each $\alpha \in S$ lies between the special vertices $\alpha_\pm$ corresponding to the free asymptotic orbits $\gamma_\pm$ (the asymptotic orbits of f_k), so any good holomorphic sphere F_α, $\alpha \in S$, in (X, J_τ) belong to some moduli space $\mathcal{M}(\gamma_0; \gamma_1, \dots, \gamma_k; J_\tau)$ of the type above. Thus the preceding discussion implies the following regularity properties for F_α. If F_α lies in $(\mathbb{R} \times Y, J_\pm)$ it belongs to a moduli space $\mathcal{M}(\gamma_0; \gamma_1, \dots, \gamma_k; J_\pm)$ which is transversely cut out, so $\mathrm{ind}(F_\alpha) \geq 1$ by $\mathbb{R}$-invariance. If F_α lies in (X, J_τ) for some τ it belongs to a moduli space $\mathcal{M}_{[0,1]}$ which is transversely cut out, so $\mathrm{ind}(F_\alpha) \geq -1$. Moreover, $\mathrm{ind}(F_\alpha) = -1$ occurs only for finitely many components C_{τ_j} at parameter values $\tau_1, \dots, \tau_m$ and all other components have $\mathrm{ind}(F_\alpha) \geq 0$. This proves that the regularity condition (R) holds and hence demonstrates Theorem 6. □

Remark 7. Theorem 6 defines linearized contact homology for dynamically convex contact forms on S^3, which turn out to be sufficient for the purposes of this paper, see Remark 2. Linearized contact homology is expected to exist more generally for any contact manifold (Y, λ) with an exact symplectic filling $(X, d\lambda)$. Here the boundary map between

closed Reeb orbits $\gamma_\pm$ will count punctured holomorphic spheres in $\mathbb{R}\times Y$ with one positive puncture asymptotic to γ_+ and any number of negative punctures asymptotic to Reeb orbits $\gamma_-, \gamma_1, \ldots, \gamma_k$, together with rigid holomorphic planes in X asymptotic to the γ_i, $i = 1, \ldots, k$. See [5, 1] for details of this construction. However, it seems that in this more general situation transversality cannot be achieved by the method in Theorem 6, but requires the use of abstract perturbations. Hofer, Wysocki and Zehnder are developing a theory of such perturbations called "polyfold Fredholm theory" [14]. However, at the time of this writing their theory is not yet completed.

3 Proof of Theorem 1

In this section we prove Theorem 1. To that end we study properties of Reeb orbits in dimension 3 and dynamical properties of translations on a flat torus.

3.1 Conley-Zehnder indices

The grading in the contact homology algebra is induced by Conley-Zehnder indices, see [13], [12]. Here we recall some properties of the Conley-Zehnder index in dimension 3 in the case of non-degenerate Reeb orbits. Let γ be a simple closed Reeb orbit and denote by γ^k its k-th iterate. Then the Floquet multipliers (eigenvalues of the linearized return map on a transverse section) occur in a pair $\mu, 1/\mu \in \mathbb{R}\setminus\{0, \pm 1\}$ if they are real, or $\mu, \bar{\mu} \in S^1 \setminus \{\pm 1\}$ if not. If γ is a Reeb orbit we write $\mathrm{CZ}(\gamma)$ for its Conley-Zehnder index. According to Section 8.1 in [12], we need to distinguish three cases (the first case is covered by Theorem 7 and the other two by Theorem 6).

Elliptic case: γ has a non-real Floquet multiplier $\mu = e^{2\pi i\alpha}$ with $\alpha \in (0, 1)$ irrational. Then

$$\mathrm{CZ}(\gamma^k) = 2kr + 2[k\alpha] + 1 = 2[k(r+\alpha)] + 1,$$

for some integer $r \in \mathbb{Z}$, where $[x]$ denotes the largest integer smaller than or equal to x.

Even hyperbolic case: γ has a positive real Floquet multiplier $\mu \in (0, 1)$. Then

$$\mathrm{CZ}(\gamma^k) = 2kr,$$

for some integer $r \in \mathbb{Z}$.

Odd hyperbolic case: γ has a negative real Floquet multiplier $\mu \in (-1, 0)$. Then

$$\mathrm{CZ}(\gamma^k) = (2r+1)k,$$

for some integer $r \in \mathbb{Z}$. Note that the even multiples of γ are "bad" in the sense of [9] and do not contribute to contact homology.

As mentioned in Section 1, the *degree* $|\gamma|$ of a closed Reeb orbit γ as a generator of the contact homology algebra as well as a generator of the linearized contact homology chain complex is given by

$$|\gamma| = \mathrm{CZ}(\gamma) - 1.$$

The following lemma was used in the proof of Theorem 6 above.

Lemma 8. *Let γ be a simple closed Reeb orbit all of whose iterates are nondegenerate. Then for any positive integers $k_1, \dots, k_s$*

$$|\gamma^{k_1+\cdots+k_s}| \geq |\gamma^{k_1}| + \cdots + |\gamma^{k_s}|.$$

Proof. Set $k = k_1 + \cdots + k_s$. We treat each type of γ separately. If γ is elliptic, then

$$|\gamma^k| - \sum_{i=1}^{s} |\gamma^{k_i}| = 2r(k - \sum_{i=1}^{s} k_i) + 2([k\alpha] - \sum_{i=1}^{s} [k_i\alpha]) \geq 0$$

because $[a+b] \geq [a] + [b]$ for any real numbers a, b. If γ is even hyperbolic, then

$$|\gamma^k| - \sum_{i=1}^{s} |\gamma^{k_i}| = (2kr - 1) - \sum_{i=1}^{s} (2k_i r - 1) = s - 1 \geq 0.$$

Finally, if γ is odd hyperbolic, then

$$|\gamma^k| - \sum_{i=1}^{s} |\gamma^{k_i}| = \big((2r+1)k - 1\big) - \sum_{i=1}^{s} \big((2r+1)k_i - 1\big) = s - 1 \geq 0.$$

□

3.2 Torus dynamics

Consider the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $(\xi_1, \dots, \xi_n) \in [0,1]^n$. Let $\tau \colon \mathbb{T}^n \to \mathbb{T}^n$ denote translation by $(\xi_1, \dots, \xi_n)$.

Lemma 9. *The orbit $\{\tau^m(0)\}_{m\in\mathbb{Z}}$ of $0 \in \mathbb{T}^n$ under τ is dense in a finite collection of translates of an l-torus $\mathbb{T}^l$ which is a subgroup of $\mathbb{T}^n$, where*

$$l + 1 = \dim_{\mathbb{Q}} \Big(\mathrm{Span}_{\mathbb{Q}}(\xi_1, \dots, \xi_n, 1) \Big).$$

Here $\mathrm{Span}_{\mathbb{Q}}(\xi_1, \dots, \xi_n, 1)$ denotes the vector subspace of $\mathbb{R}$ spanned by $\xi_1, \dots, \xi_n, 1$, where $\mathbb{R}$ is viewed as a vector space over $\mathbb{Q}$.

Remark 10. This lemma is a discrete version of the well-known fact that the geodesic through $0 \in T^m$ in direction $(\xi_1, \ldots, \xi_m)$ is dense in a subtorus $T^k \subset T^m$, where

$$k = \dim_{\mathbb{Q}}\Big(\mathrm{Span}_{\mathbb{Q}}(\xi_1, \ldots, \xi_m)\Big).$$

Proof. Assume first that $l = n$. Then by the preceding remark the geodesic γ through $0 \in T^{n+1}$ in direction $(\xi_1, \ldots, \xi_n, 1)$ is dense in the torus T^{n+1}. Since γ is transverse to the subtorus T^n where the last coordinate equals zero, the intersection $\gamma \cap T^n$ is dense in T^n. But this intersection is just the orbit $\{\tau^m(0)\}_{m\in\mathbb{Z}}$ of 0 in $\mathbb{T}^n$.

Consider next the case when $l < n$. In this case there is an equation of the form

$$m_1\xi_1 + m_2\xi_2 + \cdots + m_n\xi_n + m_{n+1} = 0,$$

where all m_j are integers. If d is the greatest common divisor of $m_1, \ldots, m_n$ we may rewrite this as

$$m_1'(d\xi_1) + \cdots + m_n'(d\xi_n) + m_{n+1} = 0.$$

Since $m_1', \ldots, m_n'$ do not have any common divisor, it follows that the hyperplane H in $\mathbb{R}^n$ given by the equation

$$m_1'(dx_1) + \cdots + m_n'(dx_n) + m_{n+1} = 0$$

contains a point with integer coordinates. It follows that any iterate $\tau^{kd}(0)$ which is a multiple of d lies in the torus $\mathbb{T}^{n-1} \subset \mathbb{T}^n$ which is the subgroup with preimage in $\mathbb{R}^n$ given by the integer translates of the hyperplane H. Any other iterate lies in a translate of this subgroup by $\tau^j(0)$, $j = 1, \ldots, d-1$.

To finish the proof we use induction. The intersection of the lattice $\mathbb{Z}^n$ and the hyperplane H is again a lattice generated by vectors with integer coordinates. Let $v_1, \ldots, v_{n-1}$ be a basis. Writing

$$(d\xi_1, \ldots, d\xi_n) = \eta_1 v_1 + \cdots + \eta_{n-1}v_{n-1},$$

we find that the vector spaces over $\mathbb{Q}$ spanned by $\eta_1, \ldots, \eta_{n-1}$ and the vector space spanned by $\xi_1, \ldots, \xi_n$ are equal. Hence

$$\dim_{\mathbb{Q}}\Big(\mathrm{Span}_{\mathbb{Q}}(\eta_1, \ldots, \eta_{n-1}, 1)\Big) = l.$$

Using the argument above we can either confine the orbits to translates of lower dimensional tori (if $l < n-1$) or the orbit is dense (if $l = n-1$). The lemma follows after $n - l$ steps. □

3.3 Jump sequences

To each real number $\xi \in (0,1)$ we associate its *jump sequence* $j(\xi) = \{j_n(\xi)\}_{n\in\mathbb{N}}$ via

$$j_n(\xi) = [n/\xi], \qquad n \in \mathbb{N}.$$

Thus $j_n = j_n(\xi)$ is the unique integer satisfying

$$j_n\xi \le n < (j_n+1)\xi,$$

so the n-th jump in the sequence $\{[k\xi]\}_{k\in\mathbb{N}}$ occurs at $k = j_n(\xi)$. The jump sequence $j(\xi)$ determines ξ via

$$\lim_{n\to\infty} \frac{n}{j_n(\xi)} = \xi.$$

Lemma 11. *For $i = 1,2,3$ let $\xi_i \in (0,1)$ be irrational with jump sequences $j(\xi_i)$.*

(a) *If $j(\xi_2)$ is a subsequence of $j(\xi_1)$, then there exist a linear relation*

$$\xi_2 = p\xi_1 + q, \qquad p,q \in \mathbb{Q}, \quad p > 0.$$

(b) *If $j(\xi_2)$ and $j(\xi_3)$ are both subsequences of $j(\xi_1)$, then ξ_2 and ξ_3 have a common jump, i.e. there exist $m_2, m_3 \in \mathbb{N}$ such that $j_{m_2}(\xi_2) = j_{m_3}(\xi_3)$.*

Example 12. The situation in (a) occurs e.g. for $\xi_2 = \xi_1/k$, $k \in \mathbb{N}$, in which case $j_n(\xi_2) = j_{nk}(\xi_1)$. An example with $q \neq 0$ is given by $\xi_2 = \xi_1 - \frac{1}{2}$: If $k = j_n(\xi_2)$ is in the jump sequence of ξ_2 then

$$k\xi_1 < n + \frac{k}{2} < (k+1)\xi_1 - \frac{1}{2}.$$

It follows that for k even,

$$k\xi_1 < n + \frac{k}{2} < (k+1)\xi_1,$$

and for k odd,

$$k\xi_1 < n + \frac{k}{2} + \frac{1}{2} < (k+1)\xi_1.$$

Hence in either case k is in the jump sequence of ξ_1.

Remark 13. D. Kotschick has pointed out that if $j(\xi_2)$ is a subsequence of $j(\xi_1)$ and $\xi_1 \le 1/2$, then $\xi_1 = k\xi_2$ for some $k \in \mathbb{N}$. This can be seen as follows. Suppose $j_n(\xi_2) = j_{m(n)}(\xi_1)$ for a sequence $m : \mathbb{N} \to \mathbb{N}$. It easily follows that m is a quasi-morphism with defect (deviation from being a semi-group homomorphism) at most $[3\xi_1] \le 1$. A more careful

estimate shows that for $\xi_1 \leq 1/2$ the defect equals zero, so m is a group homomorphism, i.e. $m(n) = kn$ for some $k \in \mathbb{N}$.

As the preceding example shows, this result fails as soon as $\xi_1 > 1/2$. It might be interesting to characterize all pairs (ξ_1, ξ_2) for which $j(\xi_2)$ is a subsequence of $j(\xi_1)$.

Proof of Lemma 11. Consider (a). Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let $\tau\colon \mathbb{T}^2 \to \mathbb{T}^2$ be translation by the vector (ξ_1, ξ_2). Lemma 9 implies that if ξ_1, ξ_2 and 1 are linearly independent over $\mathbb{Q}$, then the orbit of 0 under τ is dense in $\mathbb{T}^2$. Thus there exists an iterate $\tau^k(0) = (x_1, x_2) \in (0,1)^2$ such that $x_1 + \xi_1 < 1$ and $x_2 + \xi_2 > 1$. So k is a jump of ξ_2 but not of ξ_1, contradicting the hypothesis. We conclude that ξ_1, ξ_2 and 1 satisfy a linear relation

$$\xi_2 = p\xi_1 + q, \qquad p, q \in \mathbb{Q}.$$

The proof of Lemma 9 shows that the orbit of 0 is dense in the straight line through 0 with slope p. If $p < 0$, this implies that there is an iterate $\tau^k(0)$ with representative $(x_1, x_2) \in (-\frac{1}{2}, \frac{1}{2})^2$ arbitrarily close to 0 and such that $x_1 > 0$, $x_2 < 0$. So k is a jump of ξ_2 but not of ξ_1, contradicting the hypothesis. Hence the slope p is positive.

Consider (b). Part (a) implies that there are linear relations

$$\xi_j = p_j\xi_1 + q_j, \qquad p_j, q_j \in \mathbb{Q}, \quad p_j > 0, \quad j = 2, 3.$$

It follows that there is a linear relation

$$\xi_3 = p\xi_2 + q, \qquad p, q \in \mathbb{Q}, \quad p > 0.$$

Again by the proof of Lemma 9, the orbit of 0 under the translation $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ by (ξ_2, ξ_3) is dense in the straight line through 0 with slope p. This implies that there is an iterate $\tau^k(0)$ with representative $(x_1, x_2) \in (-\frac{1}{2}, \frac{1}{2})^2$ arbitrarily close to 0 and such that $x_1 < 0$, $x_2 < 0$. So k is a jump of both ξ_2 and ξ_3. □

3.4 Proof of Theorem 1

Assume that the differential in linearized contact homology vanishes. Then in view of the discussion preceding Theorem 1, there is a unique (not necessarily simple) closed Reeb orbit of every positive even degree and no other Reeb orbits. In particular, there are no even hyperbolic orbits. We distinguish three cases.

Case 1: There are at least two simple Reeb orbits which are odd hyperbolic.

In this case, the formula for the Conley-Zehnder index shows that suitable odd multiplies of the simple orbits have the same degree. This contradicts vanishing differential. Hence, Case 1 is ruled out.

Case 2: There is exactly one simple Reeb orbit which is odd hyperbolic.

Denote this simple orbit by γ_1. Write $\mathrm{CZ}(\gamma_1) = 2r_1 + 1$. Dynamical convexity implies $r_1 \geq 1$. Odd multiples $\gamma_1^{2\ell-1}$ of γ_1 have degrees $|\gamma_1^{2\ell-1}| = (2r_1+1)(2\ell-1)-1$. Since $\{(2r_1+1)(2\ell-1)-1\}_{\ell=1}^{\infty}$ does not contain all positive even integers, there exists some other orbit of even degree. Since γ_1 is assumed to be the only odd hyperbolic orbit we conclude that there exists an elliptic orbit. Let γ_2 be a simple elliptic orbit. Write $\mathrm{CZ}(\gamma_2) = 2[r_2+\alpha_2]+1 = 2r_2+1$, where $r_2 \in \mathbb{Z}$ and $\alpha_2 \in (0,1)$ is irrational. Dynamical convexity implies $r_2 \geq 1$. Multiples γ_2^k of γ_2 have degrees $2[k(r_2+\alpha_2)]$. We claim that there exists multiples k and ℓ such that

$$(2r_1+1)(2\ell-1)-1 = 2[k(r_2+\alpha_2)]. \tag{1}$$

This claim implies that $|\gamma_1^{2\ell-1}| = |\gamma_2^k|$ which contradicts vanishing differential and allows us to rule out Case 2.

To verify the claim we argue as follows. Let $v = 2r_1+1$ and note that the numbers in the left hand side of (1) can be written as $(\ell-1)(2v)+(v-1)$, $\ell = 1,2,\ldots$. Consider the circle $\mathbb{R}/(v\cdot\mathbb{Z})$ and the irrational rotation $\tau\colon \mathbb{R}/(v\cdot\mathbb{Z}) \to \mathbb{R}/(v\cdot\mathbb{Z})$, $\tau(x) = x+(r_2+\alpha_2)$. Lemma 9 implies that there exists k such that

$$\mathrm{dist}\left(\tau^k(0), \left(\frac{v-1}{2}+\frac{1}{4}\right)\right) < 10^{-80}.$$

This means that there exists $\ell \geq 1$ such that

$$\left|(\ell-1)v + \frac{v-1}{2} + \frac{1}{4} - k(r_2+\alpha_2)\right| < 10^{-80}.$$

It follows that $\left|((2r_1+1)(2\ell-1)-1+\frac{1}{2}) - 2k(r_2+\alpha_2)\right| < 10^{-79}$. This implies the claim.

Case 3: All simple Reeb orbits are elliptic.

Let γ_j, $j = 1,2,\ldots$ denote the simple orbits. Write $\mathrm{CZ}(\gamma_j) = 2[r_j+\alpha_j]+1 = 2r_j+1$ where $r_j \in \mathbb{Z}$ and $\alpha_j \in (0,1)$ is irrational and note that dynamical convexity implies $r_j \geq 1$. Multiples γ_j^k of γ_j have degrees $2[k(r_j+\alpha_j)]$. For fixed j the set $\left\{2[k(r_j+\alpha_j)]\right\}_{k=1}^{\infty}$ does not contain all even multiples of r_j: For every multiplicity k such that there exists an integer m with

$$k\alpha_j < m < (k+1)\alpha_j,$$

the equation

$$|\gamma_j^{k+1}| - |\gamma_j^k| = 2[(k+1)(r_j+\alpha_j)] - 2[k(r_j+\alpha_j)] = 2(r_j+1) \geq 4$$

holds. We conclude that there are at least two simple orbits.

Now we invoke a result proved in [4]: For all $k \in \mathbb{N}$ there exist isomorphisms $HC^{\mathrm{lin}}_{2k+2}(S^3) \cong HC^{\mathrm{lin}}_{2k}(S^3)$ between linearized contact homology induced by chain maps counting holomorphic curves in the symplectization. Since the $d\lambda$-area of a holomorphic curve is positive, this implies the following, in the situation under consideration:

(O) The two orderings of the set $\{\gamma_j^k\}$, $j,k = 1,2,\dots$ of all closed Reeb orbits by increasing degree and by increasing action, respectively, coincide.

If $\mathcal{A}_j$ denotes the action of the simple Reeb orbit γ_j, then the action of γ_j^k equals $k\mathcal{A}_j$. It follows from (O) that each action ratio $\frac{\mathcal{A}_i}{\mathcal{A}_j}$, $i \neq j$, must be irrational: if it were rational some multiples of γ_i and γ_j would have the same action and hence the same degree, which is impossible since the differential vanishes.

Let γ_1 denote the simple orbit of smallest action. It follows from (O) and the fact that the minimal degree is 2 that $|\gamma_1| = 2[r_1 + \alpha_1] = 2r_1 = 2$. Hence $r_1 = 1$ and the degrees of adjacent multiples of γ_1 differ either by 2 or by 4. Consider a simple orbit γ_j, $j \neq 1$. Since $\frac{\mathcal{A}_1}{\mathcal{A}_j} \in (0,1)$ is irrational, for any integer m there is a unique integer k such that $k\mathcal{A}_1 < m\mathcal{A}_j < (k+1)\mathcal{A}_1$, or equivalently

$$k\frac{\mathcal{A}_1}{\mathcal{A}_j} < m < (k+1)\frac{\mathcal{A}_1}{\mathcal{A}_j}. \tag{2}$$

In particular, (O) implies that $|\gamma_1^{k+1}| - |\gamma_1^k| = 4$, or equivalently

$$k\alpha_1 < m' < (k+1)\alpha_1, \tag{3}$$

for some integer m'. In the language of Section 3.3, this means:

(J) For each $j > 1$ the jump sequence of $\xi_j = \frac{\mathcal{A}_1}{\mathcal{A}_j}$ is a subsequence of the jump sequence of $\xi_1 = \alpha_1$.

Now we prove that there are at most two simple orbits. To see this, assume that there are three distinct orbits γ_j, $j = 1,2,3$. Condition (J) and Lemma 11 (b) imply that $\xi_j = \frac{\mathcal{A}_1}{\mathcal{A}_j}$, $j = 2,3$, have a common jump, i.e. that there exist integers k, m_2, and m_3 such that

$$k\frac{\mathcal{A}_1}{\mathcal{A}_2} < m_2 < (k+1)\frac{\mathcal{A}_1}{\mathcal{A}_2}, \qquad k\frac{\mathcal{A}_1}{\mathcal{A}_3} < m_3 < (k+1)\frac{\mathcal{A}_1}{\mathcal{A}_3}.$$

This implies that both $m_2\mathcal{A}_2$ and $m_3\mathcal{A}_3$ lie between $k\mathcal{A}_1$ and $(k+1)\mathcal{A}_1$. Since $|\gamma_1^{k+1}| - |\gamma_1^k| \leq 4$, it follows from (O) that $|\gamma_2^{m_2}| = |\gamma_3^{m_3}|$, which contradicts vanishing differential. Hence there are at most two simple

Reeb orbits. Since we already know that there are at least two orbits, this proves that (i) implies (ii).

For the converse implication suppose that the differential in linearized contact homology does not vanish. This can only happen if there is at least one even hyperbolic orbit γ_1 (whose iterates have odd degree) and one elliptic or odd hyperbolic orbit γ_2 (whose good iterates have even degree). Dynamical convexity implies $r_2 > 0$ and the iteration formulae for Conley-Zehnder indices show that multiples of γ_2 cannot attain all even degrees, so there must be a third orbit. Thus (ii) implies (i).

Next suppose that (i) and (ii) hold, so there are precisely two simple closed orbits γ_1, γ_2 with actions $\mathcal{A}_i$, Floquet multipliers α_i and rotation numbers r_i. We first show that these data are realized by a suitable irrational ellipsoid.

Since there are only two simple closed orbits, condition (O) implies that the jump sequences of $\xi_1 = \alpha_1$ and $\xi_2 = \frac{\mathcal{A}_1}{\mathcal{A}_2}$ agree. Since the ξ_i are determined by their jump sequences, it follows that

$$\alpha_1 = \frac{\mathcal{A}_1}{\mathcal{A}_2}. \tag{4}$$

Next note that (O) implies that for each m, if k is such that

$$k\mathcal{A}_1 < m\mathcal{A}_2 < (k+1)\mathcal{A}_1,$$

then, since $r_1 = 1$,

$$[k(1+\alpha_1)] < [m(r_2+\alpha_2)] < [(k+1)(1+\alpha_1)].$$

These inequalities combine to

$$\frac{[k(1+\alpha_1)]}{k+1} < \frac{[m(r_2+\alpha_2)]\mathcal{A}_1}{m\mathcal{A}_2} < \frac{[(k+1)(1+\alpha_1)]}{k},$$

which in the limit $m \to \infty$ (hence $k \to \infty$) yields

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{r_2+\alpha_2}{1+\alpha_1}. \tag{5}$$

Next note that the first gap in the degrees of multiples of γ_1 occurs at the degree $2([1/\alpha_1]+1)$. So this must be the degree of γ_2 and we conclude

$$r_2 = [1/\alpha_1] + 1. \tag{6}$$

Equations (4), (5) and (6) together with $r_1 = 1$ uniquely determine α_1, α_2, r_1 and r_2 in terms of the action ratio $\mathcal{A}_2/\mathcal{A}_1$:

$$\alpha_1 = \frac{\mathcal{A}_1}{\mathcal{A}_2}, \qquad \alpha_2 = \frac{\mathcal{A}_2}{\mathcal{A}_1} - \left[\frac{\mathcal{A}_2}{\mathcal{A}_1}\right], \qquad r_1 = 1, \qquad r_2 = \left[\frac{\mathcal{A}_2}{\mathcal{A}_1}\right] + 1. \tag{7}$$

It remains to show that any combination of actions $\mathcal{A}_i$, Floquet multipliers α_i and rotation numbers r_i satisfying equations (7) is realized by an ellipsoid

$$E = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{|z_1|^2}{a_1} + \frac{|z_1|^2}{a_1} = 1 \right\}$$

with suitable $0 < a_1 < a_2$ such that a_1/a_2 is irrational.

The Reeb flow on E is given by $z_i(t) = e^{2it/a_i}$, which is periodic of period $T_i = \pi a_i$ in the i-th component. Thus for a_1/a_2 irrational there are precisely two simple closed orbits $\gamma_1 = \{z_2 = 0\}$ and $\gamma_2 = \{z_1 = 0\}$ of period (= action)

$$\mathcal{A}_i = T_i = \pi a_i.$$

Define the a_i by this equation. From

$$z_2(T_1) = e^{2\pi i a_1/a_2}, \qquad z_1(T_2) = e^{2\pi i a_2/a_1}$$

we read off the Floquet multipliers

$$\alpha_1 = a_1/a_2, \qquad \alpha_2 = a_2/a_1 - [a_2/a_1]$$

and the rotation numbers

$$r_1 = 1, \qquad r_2 = [a_2/a_1] + 1$$

(the additional +1 result from the choice of trivializations that extend over disks). For $a_i = \mathcal{A}_i/\pi$ these equations agree with equations (7).

It is proved in [13] that for dynamically convex contact forms there exists a simple closed orbit γ_1 with $\mathrm{CZ}(\gamma_1) = 3$ (unique in our case) which is the binding of an open book decomposition. The pages are discs whose interiors are transverse to the Reeb vector field. The return map of a page possesses at least one fixed point (see [13]), which in our case must correspond to the second simple closed orbit γ_2. Hence γ_1 and γ_2 are both unknotted and have linking number 1, which concludes the proof of Theorem 1. □

Remark 14. In this remark we assume abstract perturbations exist so that the linearized contact homology is well defined for all tight contact forms on S^3 with nondegenerate closed Reeb orbits. As mentioned in Remark 2, it is immediate from the formulas for Conley-Zehnder indices together with the computation of the linearized contact homology of a tight contact form on S^3 that condition (i) in Theorem 1 implies dynamical convexity. We show that (ii) implies (i) and hence (ii) implies dynamical convexity as well.

Suppose that the differential in linearized contact homology does not vanish. This can only happen if there is at least one even hyperbolic

orbit γ_1 (whose iterates have odd degree) and one elliptic or odd hyperbolic orbit γ_2 (whose good iterates have even degree). If $r_2 \neq 0$ then we conclude that (i) holds exactly as in the proof above: the iteration formulas for Conley-Zehnder indices show that multiples of γ_2 cannot attain all even degrees, so there must be a third orbit. If $r_2 = 0$ then $|\gamma_2| = 0$, so we must have $|\gamma_1| = \pm 1$. If $|\gamma_1| = -1$, then $|\gamma_1^k| = -1$ for all $k \geq 1$, so that infinitely many orbits are required to eliminate these generators in homology. If $|\gamma_1| = 1$, then $|\gamma_1^k| = 2k-1$ for all $k \geq 1$, so that in order to eliminate these generators in homology and span exactly $HC^{\text{lin}}(S^3)$ in degrees 2 to $2k-2$, we must have $|\gamma_2^{2k-2}| < 2k-1 < |\gamma_2^{2k}|$ for all $k > 1$. This implies that γ_2 is elliptic and that $4\alpha_2 = \lim_{k\to\infty} \frac{|\gamma_2^{2k}|}{k} = 2$, a contradiction. This proves that (ii) implies (i).

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