A geometric obstruction to the contact type property

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1 Introduction

Prompted by the existence results for periodic orbits on energy surfaces, in 1979 A. Weinstein introduced the following concept ([We2]): Let (M, ω) be a symplectic manifold, i.e. a manifold M of even dimension $2n$ with a nondegenerate closed 2-form ω. A hypersurface $S \subset M$ (throughout this paper all hypersurfaces are assumed to be smooth without boundary) is said to be of *contact type* if there exists a 1-form λ on S such that

(i) $d\lambda = \omega|_S$, and
(ii) $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on S.

This condition has proved extremely fruitful, mainly for the following two properties of a hypersurface S of contact type:

1. (Stability): There exists a diffeomorphism $\phi : [-\epsilon, \epsilon] \times S \to W$ onto a tubular neighborhood W of S in M, $\phi(\{0\} \times S) = S$, such that all hypersurfaces $S_\rho = \phi(\{\rho\} \times S)$ are conformally symplectomorphic to S, i.e. (S_ρ, ω) is symplectomorphic to $(S, r\omega)$ for some constant $r(\rho) > 0$. In particular, the characteristic foliations on all S_ρ are conjugate. Here the *characteristic foliation* of a hypersurface $S \subset (M, \omega)$ is the 1-dimensional foliation consisting of the integral curves of the line bundle $\ker(\omega|_S) \to S$. Its leaves are called *characteristics*.
2. (J-convexity): Suppose that the hypersurface S is cooriented by a normal vector field ν. Then in a neighborhood of S we can speak of the *interior* and *exterior*, defined by the condition that ν points to the exterior of S. We say that (S, ν) is *ω-convex* if there exists a 1-form λ on S satisfying (i) and (ii) such that $\lambda \wedge (d\lambda)^{n-1}$ is a positive multiple of $i_\nu(\omega^n)|_S$, where i_ν is the contraction with ν. In this case there exists a *compatible almost*

complex structure J on (M, ω), i.e. an almost complex structure J such that $\omega(\cdot, J\cdot)$ is a Riemannian metric, for which S is *J-convex* in the sense that no J-holomorphic curve can touch S from the interior.

Given a cooriented hypersurface $(S, \nu) \subset (M, \omega)$ it is in general difficult to decide whether it is of contact type. An obvious necessary condition arises from the closed characteristics. Suppose that $\omega|_S$ is exact. Then to a closed characteristic $x : S^1 := \mathbf{R}/\mathbf{Z} \to S$ which is homologically trivial, $[x] = 0 \in H_1(S)$, and oriented *positively*, i.e. such that $\omega(\dot{x}, \nu) > 0$, we associate its *action*

$$A(x) := -\int_{S^1} x^*\alpha \,,$$

where α is any 1-form on S with $d\alpha = \omega|_S$. This definition does not depend on the choice of α. If (S, ν) is ω-convex with contact form λ, then $A(x) = -\int_x \lambda > 0$ for all homologically trivial closed characteristics x (similarly < 0 if S is ω-concave).
More generally, let us fix a vector field X generating $\ker(\omega|_S)$ with $\omega(X, \nu) > 0$. Then every finite Borel measure μ on S acts on 1-forms β on S via

$$\langle \mu, \beta \rangle := \int_S \beta(X) d\mu.$$

We say that μ is exact as a current if $\langle \mu, \beta \rangle = 0$ for all closed 1-forms β. Then we have the following criterion due to D.Sullivan and D.McDuff ([Su], [McD], see also [EG]): A closed hypersurface $S \subset (M, \omega)$ with $\omega|_S = d\alpha$ is of contact type if and only if there exists a constant c such that $\langle \mu, \alpha \rangle \geq c \cdot \mu(S)$ for every finite X-invariant Borel measure μ on S which is exact as a current.

Notice that Sullivan's criterion, like the original definition, depends on the characteristic foliation and thus on the C^1-type of the hypersurface. Under C^0-small perturbations the characteristic foliation may change drastically (see e.g. [Ci]). We will now give a geometric obstruction to the contact type property which is stable under C^0-small perturbations.
Consider the space $\mathbf{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the standard symplectic form $\omega_{2n} := d\lambda_{2n}$, where

$$\lambda_{2n} := \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j) \,.$$

Define the action of a loop $x : S^1 \to \mathbf{R}^{2n}$ as

$$A(x) := -\int_{S^1} x^* \lambda_{2n} \,.$$

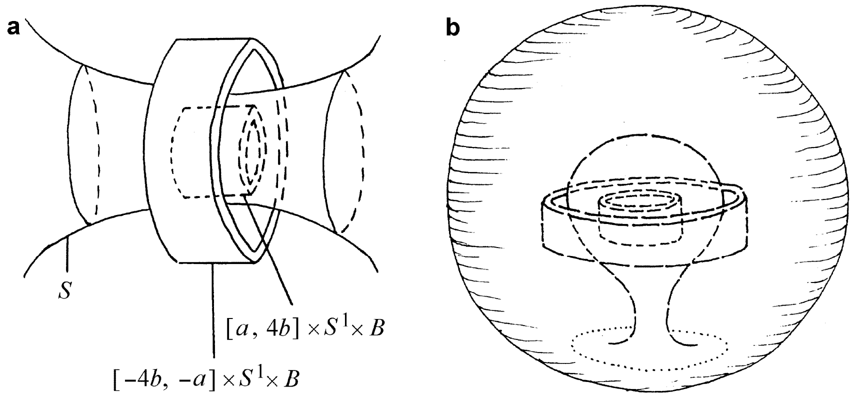

Fig. 1.

For a connected compact hypersurface $S \subset \mathbf{R}^{2n}$ (without boundary) we denote by $B(S)$ the bounded and by $U(S)$ the unbounded component of $\mathbf{R}^{2n} \setminus S$. By $B^k(r)$ we denote the closed ball around zero in $\mathbf{R}^k$ of radius r.

A hypersurface S in a symplectic manifold (M, ω) is said to be of *restricted contact type* if there exists a 1-form λ on S satisfying (i) and (ii) which extends to a 1-form $\bar{\lambda}$ on M with $d\bar{\lambda} = \omega$. Notice that if ω is exact and $H^1(S) = \{0\}$, then 'contact type' and 'restricted contact type' are equivalent conditions.

Theorem 1. *For $n \geq 2$ there exist numbers $0 < a \leq b$ and an embedding $f : [-4b, 4b] \times S^1 \times B^{2n-2}(4b) \hookrightarrow \mathbf{R}^{2n}$ with the following property: If $S \subset \mathbf{R}^{2n}$ is a connected compact hypersurface such that*

$$f\Big([-4b, -a] \times S^1 \times B^{2n-2}(4b)\Big) \subset B(S)\,,$$
$$f\Big([a, 4b] \times S^1 \times B^{2n-2}(4b)\Big) \subset U(S)\,,$$

then S is not *of restricted contact type.*

Figure 1 shows two examples of hypersurfaces satisfying the hypotheses of Theorem 1. In Fig. 1.a) the hypersurface S closes up to a hypersurface of type $S^1 \times S^{2n-2}$ surrounding the larger ring $[-4b, -a] \times S^1 \times B^{2n-2}(4b)$.

Remark. The embedding f is symplectic with respect to a twisted symplectic structure on $[-4b, 4b] \times S^1 \times B^{2n-2}(4b)$ in which the circles $\{\rho\} \times S^1 \times \{0\}^{2n-2}$ are nondegenerate closed characteristics on the hypersurfaces $\{\rho\} \times S^1 \times B^{2n-2}(4b)$ (see Sect. 3).

The *Hausdorff metric* on the space of all closed bounded subsets of a metric space (X, d) is defined as

$$d_H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}\,.$$

It is easy to see that every compact hypersurface $S \subset \mathbf{R}^{2n}$ can be approximated in the Hausdorff metric by compact hypersurfaces of restricted contact type: Approximate S in d_H by an embedded closed curve $L \subset \mathbf{R}^{2n} \setminus S$ of positive action. Then a small tubular neighborhood of L is of restricted contact type and d_H-close to S.
However, this example seems quite artificial. It can be ruled out by a simple topological hypothesis. Let $Hyp^0(\mathbf{R}^{2n})$ be the space of all connected compact hypersurfaces $S \subset \mathbf{R}^{2n}$ such that 0 lies in the bounded component of $\mathbf{R}^{2n} \setminus S$. Then we have

Corollary 1. *A hypersurface $S \in Hyp^0(\mathbf{R}^{2n})$ as in Theorem 1 cannot be approximated in the Hausdorff metric by hypersurfaces in $Hyp^0(\mathbf{R}^{2n})$ of restricted contact type.*

We also get a different criterion in terms of closed characteristics. For a closed characteristic x on a hypersurface S consider its *linear Poincaré map*, i.e. the linearization of the Poincaré return map on a transverse section to x. The characteristic x is called *nondegenerate* if 1 is not an eigenvalue of the linear Poincaré map. It is called *linearly stable* (cf. [GL]) if its linear Poincaré map is symplectically conjugate to a diagonal matrix $diag(e^{i\alpha_1}, \dots, e^{i\alpha_{n-1}}) \in \mathbf{C}^{(n-1)\times(n-1)}$ with $\alpha_1, \dots, \alpha_{n-1} \in \mathbf{R}$.

Corollary 2. *Suppose that a hypersurface $S \in Hyp^0(\mathbf{R}^{2n})$ carries a nondegenerate linearly stable closed characteristic x of negative action. Then S cannot be approximated in the Hausdorff metric by hypersurfaces in $Hyp^0(\mathbf{R}^{2n})$ of restricted contact type.*

Remark. In other words, a necessary condition for S to be approximable by hypersurfaces of restricted contact type is the absence of nondegenerate linearly stable closed characteristics of negative action. This condition is definitively not sufficient. Combining Corollary 2 with the construction in [Ci] we find hypersurfaces $S \in Hyp^0(\mathbf{R}^{2n})$ which carry no closed characteristic of negative action, but still cannot be approximated by hypersurfaces of restricted contact type.

Following Y. Eliashberg and M. Gromov ([EG]), let us call an *open* (i.e. without compact connected components) symplectic manifold (M, ω) *convexly exhaustible* if it admits an exhaustion $U_1 \subset U_2 \subset \dots \subset M$, $\cup_{i\in\mathbf{N}} U_i = M$, by compact subsets U_i with smooth ω-convex boundaries (cooriented by outward pointing normal vector fields). We call (M, ω) *exact convexly exhaustible* if the boundaries ∂U_i are exact convex, i.e. the positive contact forms on ∂U_i extend to M as primitives of ω. It was shown in [EG] that the complement of a small closed ball in a symplectic manifold is not convexly exhaustible. However, this cannot be applied, e.g., to find nonconvex symplectic structures on the open $2n$-ball. The existence of such structures follows from the following corollary, choosing a hypersurface S which bounds a ball.

Corollary 3. *For a hypersurface S as in Corollary 1 or 2, the bounded component of $\mathbf{R}^{2n} \setminus S$ is not exact convexly exhaustible.*

More generally, we have

Corollary 4. *Every open manifold M of dimension $2n \geq 4$ which admits a symplectic structure also admits a symplectic structure which is not exact convexly exhaustible.*

Remarks. 1. Corollary 3 answers a question in [EG]: The exact convex exhaustibility of the interior of a domain $\Omega \subset \mathbf{R}^{2n}$ with smooth boundary $\partial\Omega$ implies certain convexity of $\partial\Omega$. For example, $\partial\Omega$ cannot have a shape as described in Theorem 1 and Fig. 1.
2. The additional question in [EG] whether the actions of all invariant measures on $\partial\Omega$ are nonnegative if the interior is convexly exhaustible must be answered in the negative: By the construction in [Ci] one can always introduce closed characteristics on $\partial\Omega$ with negative action without changing symplectically the interior of Ω.

This paper is organized as follows:
In Sect. 2 we will reduce Theorem 1 to a statement about symplectic homology of hypersurfaces in $\mathbf{R}^{2n}$ (Theorem 2).
Theorem 2 depends on a version of the Monotonicity Lemma for pseudo-holomorphic curves with a Hamiltonian term which will be proved in Sect. 3.
In Sect. 4 we will prove Theorem 2 as well as Corollaries 1–4.

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2 Localization of symplectic homology and proof of Theorem 1

We will first define the version of symplectic homology which we will use to prove Theorem 1. For details we refer the reader to [FH], [FHW], [CFH] and [CFHW]. We shall describe the construction on the standard symplectic space $(\mathbf{R}^{2n}, \omega_{2n} = d\lambda_{2n})$, although it works for any exact symplectic manifold which is convexly exhaustible.
Consider a 1-periodic time-dependent Hamiltonian $H : S^1 \times \mathbf{R}^{2n} \to \mathbf{R}$ such that

$$H(t,x) = 0 \quad \text{for } |x| \text{ large.}$$

To a smooth loop $x : S^1 \to \mathbf{R}^{2n}$ we associate its *Hamiltonian action*

$$A_H(x) := -\int_0^1 x^* \lambda_{2n} - \int_0^1 H(t, x(t)) dt \, .$$

Critical points of A_H are precisely the 1-periodic solutions $x : S^1 \to \mathbf{R}^{2n}$ of

$$\dot{x}(t) = X_H(t, x(t)) \, ,$$

where X_H is the Hamiltonian vector field defined by

$$dH_t(x) = \omega_{2n}\Big(X_H(t, x), \cdot\Big) \, ,$$

and $H_t = H(t, \cdot)$.
Fix an interval $[a, b)$ not containing 0. We call H a *regular Hamiltonian* if all 1-periodic solutions of $\dot{x} = X_H(t, x)$ with $A_H(x) \in [a, b)$ are nondegenerate. Observe that the degenerate constant solutions in the region $\{H \equiv 0\}$ have action $0 \notin [a, b)$.

More generally, we will consider Hamiltonians $H : \mathbf{R} \times S^1 \times \mathbf{R}^{2n} \to \mathbf{R}$ satisfying

$$\frac{\partial H}{\partial s}(s, t, x) \leq 0 \text{ for all } (s, t, x) \, , \tag{$H1$}$$

$$H(s, t, x) = \begin{cases} H_1(t, x) & \text{for } s \leq -s_0; \\ H_2(t, x) & \text{for } s \geq s_0, \end{cases} \tag{$H2$}$$

$$H(s, t, x) = 0 \text{ for } |x| \text{ large.} \tag{$H3$}$$

Let J be an (s, t)-dependent almost complex structure on $\mathbf{R}^{2n}$ such that

$$\omega_{2n}\Big(J(s, t)\cdot, \cdot\Big) \text{ is a Riemannian metric for all } (s, t) \, , \tag{$J1$}$$

$$J(s, t, x) = \begin{cases} J_1(t, x) & \text{for } s \leq -s_0; \\ J_2(t, x) & \text{for } s \geq s_0, \end{cases} \tag{$J2$}$$

$$J(s, t, x) = i \text{ for } |x| \text{ large.} \tag{$J3$}$$

Consider smooth maps $\hat{u} : Z = \mathbf{R} \times S^1 \to \mathbf{R}^{2n}$ satisfying

$$\hat{u}_s + J(s, t, \hat{u})\hat{u}_t + \nabla_{J_{s,t}} H(s, t, \hat{u}) = 0 \, , \tag{$u1$}$$

$$\hat{u}(s, \cdot) \longrightarrow \begin{cases} x_1 & \text{as } s \to -\infty; \\ x_2 & \text{as } s \to +\infty, \end{cases} \tag{$u2$}$$

where x_i are 1-periodic solutions of $\dot{x}_i(t) = X_{H_i}(t, x_i(t))$, and $\nabla_{J_{s,t}}$ denotes the gradient with respect to the metric $\omega_{2n}(J(s, t)\cdot, \cdot)$.

An s-independent pair (H, J) satisfying (H1–3) and (J1–3) is called a *regular pair* if H is a regular Hamiltonian, and 0 is a regular value of the Fredholm operator

$$\hat{u} \mapsto \hat{u}_s + J(t, \hat{u})\hat{u}_t + \nabla_{J_t} H(t, \hat{u})$$

defined on maps $\hat{u}$ satisfying (u2) with $A_H(x_i) \in [a, b)$. An s-dependent pair (H, J) satisfying (H1–3) and (J1–3) is called a *regular monotone homotopy* between the regular pairs (H_1, J_1) and (H_2, J_2) if 0 is a regular value of the Fredholm operator

$$\hat{u} \mapsto \hat{u}_s + J(s, t, \hat{u})\hat{u}_t + \nabla_{J_{s,t}} H(s, t, \hat{u})$$

defined on maps $\hat{u}$ satisfying (u2) with $A_{H_i}(x_i) \in [a, b)$. In these cases the spaces

$$\mathcal{M}(x_1, x_2, H, J)$$

of solutions of $(u1 - 2)$ are finite dimensional manifolds.
For a regular pair (H, J) consider the finite-dimensional vector space over $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$,

$$C^{[a,b)}(H, J) := \{\sum \alpha_i x_i \mid \alpha_i \in \mathbf{Z}_2, x_i \text{ 1-periodic solutions of } \dot{x}_i = X_H(t, x_i) \text{ with } A_H(x_i) \in [a, b)\}\,.$$

We define a linear operator

$$\partial : C^{[a,b)}(H, J) \to C^{[a,b)}(H, J)\,,$$

$$\partial x := \sum_y \langle x, y\rangle_1 \cdot y\,,$$

where $\langle x, y\rangle_1$ is the number mod 2 of 1-dimensional components of $\mathcal{M}(x, y, H, J)$. The operator ∂ satisfies $\partial^2 = 0$. The homology group

$$FH^{[a,b)}(H) := \ker(\partial)/\operatorname{im}(\partial)$$

is called the *Floer homology* in the action interval $[a, b)$. It is independent of J, but it does depend on H, as we will describe now.
To a regular monotone homotopy (H, J) between regular pairs (H_1, J_1), (H_2, J_2) we associate a linear map

$$\sigma(H, J) : C^{[a,b)}(H_1, J_1) \to C^{[a,b)}(H_2, J_2)\,,$$

$$\sigma(H, J)x := \sum_y \langle x, y\rangle_0 \cdot y\,,$$

where $\langle x, y\rangle_0$ denotes the number mod 2 of 0-dimensional components of $\mathcal{M}(x, y, H, J)$. It turns out that $\sigma(H, J)$ is a chain map. The induced map on Floer homology does not depend on the choice of (H, J). We denote it by

$$\sigma(H_1, H_2) : FH^{[a,b)}(H_1) \to FH^{[a,b)}(H_2)\,.$$

For 3 regular Hamiltonians $H_1 \geq H_2 \geq H_3$ the corresponding maps satisfy the composition law

$$\sigma(H_2, H_3) \circ \sigma(H_1, H_2) = \sigma(H_1, H_3)\,.$$

Next let a compact cooriented hypersurface $S \subset \mathbf{R}^{2n}$ be given. We do not require S to be connected, so $\mathbf{R}^{2n} \setminus S$ may have more than two connected components. However, we suppose that $\mathbf{R}^{2n} = B(S) \cup U(S)$, where $B(S)$, $U(S)$ are (not necessarily connected) components, $B(S)$ is bounded, and $U(S)$ is unbounded. Moreover, we assume that the coorientation of S is defined by a normal vector field ν which points everywhere into $U(S)$.
Fix a number $r > 0$. We call a Hamiltonian $H(s, t, x)$ *adapted to* (S, r) if it satisfies (H1–3) and

$$H(s, t, x) = 0 \text{ for } x \text{ outside some compact subset of } B(S)\,,$$

$$-r < H(s, t, x) \leq 0 \quad \text{for all } (s, t, x)$$

(see Fig. 2). The set $Ad_{reg}(S, r)$ of s-independent regular Hamiltonians adapted to (S, r) is a partially ordered set via

$$H_1 \geq H_2 :\Longleftrightarrow H_1(t, x) \geq H_2(t, x) \text{ for all } (t, x)\,.$$

This partial ordering and the homomorphisms $\sigma(H_1, H_2)$ turn the set of Floer homology groups

$$\left(HF^{[a,b)}(H)\right)_{H \in Ad_{reg}(S,r)}$$

into a directed system. We define the *symplectic homology of* (S, r) as the direct limit of this system as H decreases,

$$SH^{[a,b)}(S, r) := \lim_{\longrightarrow} HF^{[a,b)}(H)\,.$$

Geometrically, decreasing sequences in $Ad_{reg}(S, r)$ tend (pointwise) to $-r\chi_{B(S)}$, where $\chi_{B(S)}$ is the characteristic function of the bounded component $B(S)$.
An inclusion $S_1 \subset B(S_2)$ of two hypersurfaces induces a natural inclusion $Ad_{reg}(S_1, r) \subset Ad_{reg}(S_2, r)$, and thus a homomorphism

$$\rho(S_1, S_2) : SH^{[a,b)}(S_1, r) \to SH^{[a,b)}(S_2, r)\,.$$

For three hypersurfaces satisfying $S_1 \subset B(S_2)$, $S_2 \subset B(S_3)$ we have

$$\rho(S_2, S_3) \circ \rho(S_1, S_2) = \rho(S_1, S_3)\,.$$

Theorem 1 will be an easy consequence of the following result about the nontriviality of certain symplectic homology groups.

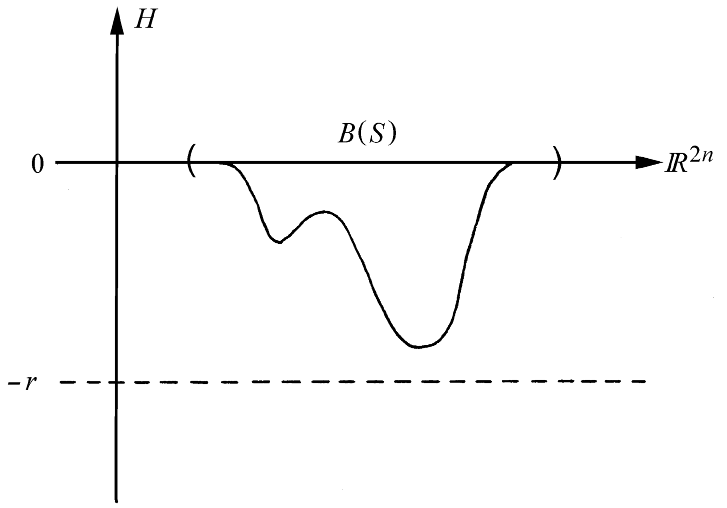

Fig. 2.

Theorem 2 (localization of symplectic homology). *For $n \geq 2$ and $c \neq 0$ there exist positive numbers a, b, δ, r (which can be chosen arbitrarily small) and an embedding $f : U := [-4b, 4b] \times S^1 \times B^{2n-2}(4b) \hookrightarrow \mathbf{R}^{2n}$ with the following property:*
Let $S_1, S_2 \subset \mathbf{R}^{2n}$ be disjoint compact, cooriented, not necessarily connected hypersurfaces such that $S_1 \subset B(S_2)$. Suppose that

$$S_1 \cap f(U) = f\Big(\{-a\} \times S^1 \times B^{2n-2}(4b)\Big) ,$$

$$S_2 \cap f(U) = f\Big(\{+a\} \times S^1 \times B^{2n-2}(4b)\Big) ,$$

and the coorientations agree on these intersections. Then the closed characteristics $y_1 = f\Big(\{-a\} \times S^1 \times \{0\}\Big)$ of S_1 and $y_2 = f\Big(\{+a\} \times S^1 \times \{0\}\Big)$ of S_2 give rise to nontrivial elements

$$0 \neq [y_i^+], [y_i^-] \in HS^{[c-\delta, c+\delta)}(S_i, r)$$

in symplectic homology, and the inclusion induced homomorphism

$$\rho(S_1, S_2) : HS^{[c-\delta,c+\delta)}(S_1, r) \to HS^{[c-\delta,c+\delta)}(S_2, r)$$

maps $[y_1^+]$ onto $[y_2^+]$ and $[y_1^-]$ onto $[y_2^-]$.

Remark. The first part of this theorem can be interpreted as follows: The 'germ of a hypersurface' $f\Big(\{a\} \times S^1 \times B^{2n-2}(4b)\Big)$ around the closed characteristic y_1 carries nontrivial local symplectic homology which persists

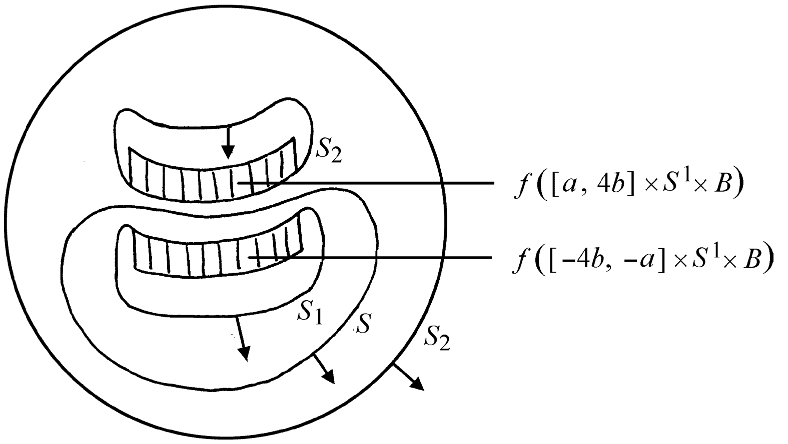

Fig. 3.

in the global symplectic homology of every closed hypersurface containing this germ.

Proof of Theorem 1 assuming Theorem 2
Let c, a, b, δ, r and $f : U \hookrightarrow \mathbf{R}^{2n}$ be as in Theorem 2 such that $c + \delta < 0$. Let $S \subset \mathbf{R}^{2n}$ be a hypersurface as in Theorem 1. Define a hypersurface $S_1 \subset \mathbf{R}^{2n}$ by smoothing the corners of the boundary of $f\Big([-4b, -a] \times S^1 \times B^{2n-2}(4b)\Big)$. Choose the coorientation of S_1 such that it agrees with the coorientation on $f\Big(\{-a\} \times S^1 \times B^{2n-2}(4b)\Big)$. Let the hypersurface S_2 be the disjoint union of the smoothening of the boundary of $f\Big([a, 4b] \times S^1 \times B^{2n-2}(4b)\Big)$ with a large sphere enclosing S and $f(U)$ (see Fig. 3). Choose the coorientation of S_2 to agree with the coorientation on $f\Big(\{a\} \times S^1 \times B^{2n-2}(4b)\Big)$ and to be outward pointing on the large sphere. Then the hypersurfaces S_1 and S_2 satisfy the hypotheses of Theorem 2, so the inclusion induced homomorphism

$$\rho(S_1, S_2) : SH^{[c-\delta, c+\delta)}(S_1, r) \to SH^{[c-\delta, c+\delta)}(S_2, r)$$

between their symplectic homologies is nontrivial. Since

$$\rho(S_1, S_2) = \rho(S, S_2) \circ \rho(S_1, S)\,,$$

this implies

$$SH^{[c-\delta, c+\delta)}(S, r) \neq \{0\}\,.$$

On the other hand, if S is of restricted contact type with 1-form λ, then it can only be ω-convex because

$$\int_S \lambda \wedge (d\lambda)^{n-1} = \int_{B(S)} \omega^n > 0\,.$$

In this case all closed characteristics on S have positive action. Moreover, by the stability property mentioned in the introduction, S possesses a tubular neighborhood $\phi([-\epsilon, \epsilon] \times S)$ such that all closed characteristics on all the hypersurfaces $S_\rho = \phi(\{\rho\} \times S)$ have positive action. Thus for adapted Hamiltonians which have the S_ρ as level surfaces and are close to $-r$ respectively 0 outside the tubular neighborhood, all 1-periodic orbits have positive action. This shows that the symplectic homology groups of S are trivial in all negative action intervals. In particular,

$$HS^{[c-\delta, c+\delta)}(S, r) = \{0\}\,,$$

and we have a contradiction. Hence S cannot be of restricted contact type. □

3 A version of the Monotonicity Lemma

We will start with a local model for the characteristic flow near a closed characteristic. Let

$$U := [-4b, 4b] \times S^1 \times B^{2n-2}(4b)$$

be as in Theorem 2. Denote coordinates on U by (ρ, θ, z) with $z = (z_1, \dots \dots, z_{n-1}) \in \mathbf{C}^{n-1} = \mathbf{R}^{2n-2}$, $z_j = x_j + iy_j$. We equip U with the symplectic form

$$\begin{aligned} \omega &:= d\theta \wedge d\rho + d\theta \wedge \sum_{j=1}^{n-1} \alpha_j (x_j\, dx_j + y_j\, dy_j) + \omega_{2n-2}\,, \\ &= d\theta \wedge d\rho + d\theta \wedge d\Big(\frac{1}{2}\sum_{j=1}^{n-1} \alpha_j |z_j|^2\Big) + \omega_{2n-2}\,, \end{aligned}$$

where $\omega_{2n-2} = \sum_{j=1}^{n-1} dx_j \wedge dy_j$ is the standard symplectic form on $\mathbf{C}^n$, and

$$\alpha_1, \dots, \alpha_{n-1} \in \mathbf{R}^+ \setminus 2\pi\mathbf{Z}$$

real numbers which are not multiples of 2π.

The Hamiltonian vector field of the function $\rho : U \to \mathbf{R}$ is given by

$$X := \frac{\partial}{\partial\theta} - \sum_{j=1}^{n-1} \alpha_j (y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j})\,,$$

as the following calculation shows:

$$
\begin{aligned}
i_X\omega = d\rho + \sum_{j=1}^{n-1} [(\alpha_j)^2(x_jy_j - y_jx_j)d\theta + \alpha_j(x_jdx_j + y_jdy_j) \\
-\alpha_j(y_jdy_j + x_jdx_j)] \\
= d\rho\,.
\end{aligned}
$$

The components of X can also be written as

$$X = (0, 1, i\alpha_1 z_1, \dots, i\alpha_{n-1} z_{n-1}) \in \mathbf{R} \times \mathbf{R} \times \mathbf{C}^{n-1}\,.$$

Consider the germs of hypersurfaces

$$U_\rho := \{\rho\} \times S^1 \times B^{2n-2}(4b)\,.$$

If they are cooriented by the gradient of the function ρ, the Hamiltonian vector field X is a positive (i.e. $\omega(X, \nabla\rho) > 0$) generator of the characteristic foliations on all the hypersurfaces U_ρ. Every U_ρ carries the closed characteristic

$$x_\rho(t) = (\rho, t, 0),\ t \in [0, 1]\,.$$

The linearization at $(\rho, \theta_0, 0)$ of the Poincaré return map of X on the transverse section $\{0\} \times \{\theta_0\} \times B^{2n-2}(4b)$ to x_ρ in U_ρ is the linear map of $\mathbf{C}^{n-1}$ given by the diagonal matrix

$$\Delta := \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_{n-1}})\,.$$

Since the α_j are not multiples of 2π, the matrix Δ does not have 1 in its spectrum. So the x_ρ are nondegenerate closed characteristics.
For a given $c \neq 0$ choose a 1-form λ_c on U satisfying

$$
\begin{aligned}
d\lambda_c = 0\omega\,, \\
-\int_{x_0} \lambda_c = c\,.
\end{aligned}
$$

Define the action of a loop $x : S^1 \to U$ by

$$A(x) := -\int_x \lambda_c\,.$$

In particular, the closed characteristic x_0 has action c.

The exact symplectic manifold $(U, d\lambda_c)$ can easily be exact symplectically embedded in $(\mathbf{R}^{2n}, d\lambda_{2n})$: Let $y_0 : S^1 \to \mathbf{R}^{2n}$ be an embedded loop with $-\int_{y_0} \lambda_{2n} = c$. By the symplectic neighborhood theorem (see [We1]) there exists an embedding $f : U = [-4b, 4b] \times S^1 \times B^{2n-2}(4b) \hookrightarrow \mathbf{R}^{2n}$ such that $f(x_0(t)) = y_0(t)$, and $f^*\omega_{2n} = \omega$ with the form ω defined above.

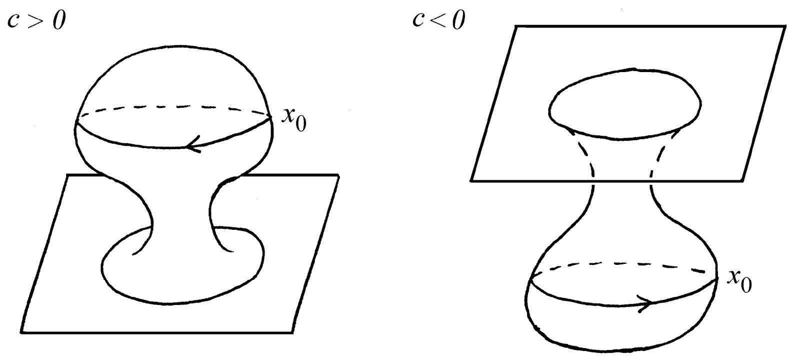

Fig. 4.

So on U we have $d(f^*\lambda_{2n}) = d\lambda_c$ and $\int_{x_0} f^*\lambda_{2n} = \int_{x_0} \lambda_c$. Since x_0 generates the first homology of U, this implies that $f^*\lambda_{2n}$ and λ_c differ by an exact 1-form.

For later use let us give a more explicit embedding of U in $\mathbf{R}^{2n}$.

Lemma 1. *For every* $c \neq 0$ *and* $n \geq 2$ *there exists an embedding* $f : [-4b, 4b] \times \mathbf{R}^{2n-1} \hookrightarrow \mathbf{R}^{2n}$ *and a tubular neighborhood* $S^1 \times B^{2n-2}(4b) \cong V \subset \mathbf{R}^{2n-1}$ *of an embedded loop such that*
(i) $f = \mathrm{id}$ *outside some compact subset,*
(ii) $f : \left(U = [-4b, 4b] \times V, d\lambda_c\right) \hookrightarrow (\mathbf{R}^{2n}, d\lambda_{2n})$ *is exact symplectic.*

Remark. In particular, the hypersurface $f\left(\{0\} \times \mathbf{R}^{2n-1}\right)$ contains a closed characteristic x_0 of action c near which the hypersurface looks like U_0 above (see Fig. 4).

Proof. For $c > 0$ consider the function

$$\rho(z) := c - \frac{1}{2}\sum_{j=1}^{n} \alpha_j |z_j|^2$$

on $\mathbf{C}^n$, where $\alpha_n = 2\pi$, and $\alpha_1, \ldots, \alpha_{n-1}$ are the positive numbers from above. Fix c, and pick $b > 0$ small enough such that the level sets

$$E_a := \{z \in \mathbf{C}^n \mid \rho(z) = a\},$$

$a \in [-4b, 4b]$, form a family of ellipsoids around the origin. Each ellipsoid E_a carries the closed characteristic

$$x_a(t) = (0, \ldots, 0, r_a e^{2\pi i t}), \quad t \in [0, 1],$$

where r_a is determined by the equation

$$c - \frac{1}{2} r_a^2 = a.$$

The action of the characteristic x_0 is

$$\begin{aligned} A(x_0) &= -\int_{x_0} \lambda_{2n} \\ &= -\frac{1}{2}\int_0^1 r_0^2(\cos^2 t + \sin^2 t)dt \\ &= -\frac{1}{2}r_0^2 \\ &= -c\,. \end{aligned}$$

In a tubular neighborhood U of all the characteristics x_a we can choose polar coordinates $z_n = re^{2\pi i\theta}$ in the last component. The form ω_{2n} becomes

$$\omega_{2n}|_U = 2\pi r\, dr \wedge d\theta + \omega_{2n-2}\,.$$

Now we take ρ as a new independent coordinate and eliminate r. Pick U of the form $U = [-4b, 4b] \times S^1 \times B^{2n-2}(4b)$ in the coordinates $(\rho, \theta, z_1, \dots$ $\dots, z_{n-1})$. From

$$d\rho = -2\pi r\, dr - \sum_{j=1}^{n-1} \alpha_j(x_j dx_j + y_j dy_j)$$

we find the expression for ω_{2n} in the new coordinates $(\rho, \theta, z_1, \dots, z_{n-1})$,

$$\omega_{2n}|_U = d\theta \wedge d\rho + d\theta \wedge \sum_{j=1}^{n-1} \alpha_j(x_j dx_j + y_j dy_j) + \omega_{2n-2}\,,$$

which is the form ω from above. This shows that the 1-form $\lambda_{2n}|_U - \lambda_{-c}$ is closed. Since $A(x_0) = -c$, it vanishes on the generator x_0 of $H_1(U)$, so it is exact.

So far we have constructed a foliated family of ellipsoids $(E_a)_{a\in[-4b,4b]} \subset \mathbf{R}^{2n}$ containing a set $U = [-4b, 4b] \times S^1 \times B^{2n-2}(4b)$ which satisfies (ii) of the lemma. Now translate each ellipsoid E_a in the negative x_1-direction (by the same amount for all a) until it lies below the hypersurface $\{x_1 = a\} \subset \mathbf{R}^{2n}$. Cut out from E_a a small $(2n-1)$-ball around the north pole (the point with the maximal value of x_1), and cut out from the hypersurface $\{x_1 = a\}$ a small ball around $(a, 0, \dots, 0)$. Connect the two hypersurfaces along the holes as shown in Fig. 4. Since increasing values of a correspond to smaller ellipsoids, this connected sum can be performed simultaneously for all the ellipsoids E_a as a foliated family.

This finishes the proof in the case of negative action $-c$. In the case of positive action c replace the function ρ by $-\rho$. The positively oriented closed characteristic x_0 then has action $+c$. Now increasing values of a correspond to bigger ellipsoids. So if we translate them in the positive x_1-direction, then we can connect them to the hyperplanes $\{x_1 = a\}$ as before, cutting out small balls around the south pole. □

The proof of Theorem 2 depends on an estimate of the energy of pseudo-holomorphic curves in $U = [-4b, 4b] \times S^1 \times B^{2n-2}(4b)$. We will identify U with its image $f(U) \subset \mathbf{R}^{2n}$ constructed in Lemma 1. Let $H : \mathbf{R} \times S^1 \times \mathbf{R}^{2n} \to \mathbf{R}$ be a smooth Hamiltonian satisfying (H1–2). Moreover, suppose that for $x = (\rho, \theta, z) \in U$ we have $H(s, t, x) = h(s, \rho)$, where $h : \mathbf{R} \times [-4b, 4b] \to \mathbf{R}$ is a smooth function with

$$\left\{ \begin{array}{l} -\dfrac{|c|}{4} \leq h(s,\rho) \leq 0 \text{ for all } (s,\rho); \\ h(s,\rho) = h^{\pm}(s) \text{ for } \pm\rho \in [2b, 4b]\,. \end{array} \right\} \qquad (H4)$$

Let J be an (s, t)-dependent almost complex structure on $\mathbf{R}^{2n}$ satisfying (J1–2), and such that for $x \in U$, $J(s, t, x) : T_xU \to T_xU$ is given by

$$\left\{ \begin{array}{l} J(s,t,x) \cdot \frac{\partial}{\partial\rho} = X(x)\,, \\ J(s,t,x) \cdot X(x) = -\frac{\partial}{\partial\rho}\,, \\ J(s,t,x) \cdot w = -iw \text{ for } w \in \{0\} \times \{0\} \times \mathbf{C}^{n-1}\,. \end{array} \right\} \qquad (J4)$$

Consider smooth maps $\hat{u} : Z = \mathbf{R} \times S^1 \to \mathbf{R}^{2n}$ satisfying (u1-2), where x_i are 1-periodic solutions of $\dot{x}_i(t) = X_{H_i}(t, x_i(t))$. Moreover, suppose that

$$x_1(t) = (\rho_1, t, 0) \in U \text{ for some } \rho_1 \in [-2b, 2b]\,; \qquad (u3)$$

$$x_2 \text{ does not meet } U\,; \qquad (u4)$$

$$A_H(x_2) - A_H(x_1) \leq \frac{|c|}{4}\,. \qquad (u5)$$

Proposition 1 ('Hamiltonian Monotonicity Lemma'). *There exists a constant κ depending only on the numbers $\alpha_1, \ldots, \alpha_{n-1} \in \mathbf{R}^+ \setminus 2\pi\mathbf{Z}$ such that for all sufficiently small $b > 0$ the following holds:*
For all H, J and $\hat{u}$ satisfying (H1–2), (H4), (J1–2), (J4) and (u1-5),

$$\int_Z |\hat{u}_s|^2 ds\, dt \geq \kappa b^2\,.$$

The statement remains true if the roles of x_1 and x_2 are interchanged in (u3-5).

Remarks. 1. The corresponding statement for $H \equiv 0$ is the classical Monotonicity Lemma for pseudo-holomorphic curves ([Gr], inequality (14); [Hu], Chapter II, Theorem 1.3). However, this case is excluded by the hypothesis of Proposition 1 on the nonconstant periodic solution x_1.

2. It would be interesting to know under which weaker hypotheses on the Hamiltonian flow near x_1 the conclusion of Proposition 1 remains true. It should definitely be sufficient that x_1 is nondegenerate and *stable* in the sense that the forward and backward orbit of any point which is δ-close to x_1 remains ϵ-close to x_1. In view of the classical Monotonicity Lemma it seems plausible that the nondegeneracy assumption can be removed.
Question: Does Proposition 1 remain valid without any assumption on H near x_1?

Idea of the proof. As the proof of this proposition will occupy the remainder of this section, let us first describe the idea. By the asymptotic conditions on $\hat{u}$, there exist values of s for which the loops $y(t) := \hat{u}(s,t)$ meet the 'annulus' $\{b \le |z| \le 2b\} \subset U$. Once we have a uniform estimate from below,

$$\int_0^1 |\dot{y} - X_H(y)| dt \ge \kappa b \,, \qquad (*)$$

for the deviation of such y from a periodic orbit, the energy estimate will follow by integration over s. To get the estimate (*), we have to consider two different cases. Either the loop y stays in U all the time; then the estimate follows from the the nondegeneracy assumption on x_1 which implies that the only periodic orbits in U are contained in the set $\{z = 0\}$. Or the loop y leaves U; then (*) follows from the stability assumption on x_1 which implies that no orbit leaves the set U.
Let us first focus our attention to the local hypersurface

$$S^1 \times B^{2n-2}(3b) \cong \{0\} \times S^1 \times B^{2n-2}(3b) \subset U \,,$$

where we have dropped the variable $\rho = 0$ from the notation. For $x \in S^1 \times B^{2n-2}(3b)$ denote by

$$\pi_x : T_x\Big((S^1 \times B^{2n-2}(3b)\Big) \cong \mathbf{R} \times \mathbf{C}^{n-1} \to \mathbf{C}^{n-1}$$

the projection along $X(x)$. In complex coordinates we have $x = (\theta, z_1, \dots \dots, z_{n-1})$, the components of X are

$$X(x) = (1, i\alpha_1 z_1, \dots, i\alpha_{n-1} z_{n-1}) \,,$$

and the projection π is given explicitly by

$$\pi_x(v, w_1, \dots, w_{n-1}) = (w_1 - iv\alpha_1 z_1, \dots, w_{n-1} - iv\alpha_{n-1} z_{n-1}) \,.$$

Lemma 2. *Let*

$$x, y : [0,d] \to S^1 \times B^{2n-2}(3b) \,,$$
$$x(t) = (\theta(t), z(t)) \,,$$

$$y(t) = (\theta(t), w(t))\,,$$

be two smooth curves, having the same θ-component, such that y satisfies

$$\pi_{y(t)}\dot{y}(t) = 0 \textit{ for all } t \in [0, d]\,.$$

Then

$$\left|\frac{d}{dt}|z(t) - w(t)|\right| \le |\pi_{x(t)}\dot{x}(t)|$$

for every $t \in [0, d]$ with $|z(t) - w(t)| > 0$. In particular,

$$\|z - w\|_{C^0} \le \int_0^d |\pi_{x(t)}\dot{x}(t)|dt + |z(0) - w(0)|.$$

Proof. The equation $\pi_{y(t)}\dot{y}(t) = 0$ is given explicitly by

$$\dot{w}_j(t) - i\dot{\theta}(t)\alpha_j w_j(t) = 0 \text{ for } j = 1, \ldots, n-1\,.$$

Using this, we can calculate

$$\begin{aligned}
\frac{d}{dt}|z_j(t) - w_j(t)|^2 &= 2\langle z_j(t) - w_j(t), \dot{z}_j(t) - \dot{w}_j(t)\rangle \\
&= 2\langle z_j(t) - w_j(t), \dot{z}_j(t) - i\dot{\theta}(t)\alpha_j z_j(t)\rangle \\
&\quad +2\langle z_j(t) - w_j(t), i\dot{\theta}(t)\alpha_j z_j(t) - i\dot{\theta}(t)\alpha_j w_j(t)\rangle \\
&= 2\langle z_j(t) - w_j(t), \dot{z}_j(t) - i\dot{\theta}(t)\alpha_j z_j(t)\rangle\,,
\end{aligned}$$

since $\langle v, irv\rangle = 0$ for any $v \in \mathbf{C}$ and $r \in \mathbf{R}$. Summation over j yields

$$\begin{aligned}
\left|\frac{d}{dt}|z(t) - w(t)|^2\right| &= \left|2\langle z(t) - w(t), \pi_{x(t)}\dot{x}(t)\rangle\right| \\
&\le 2|z(t) - w(t)|\,|\pi_{x(t)}\dot{x}(t)|\,,
\end{aligned}$$

from which the first statement follows. The second statement follows by integration over t. □

Recall that the action of a loop $x : S^1 \to U$ is given by $A(x) = -\int_x \lambda_c$, for a 1-form λ_c on U with $d\lambda_c = \omega$ and $-\int_{x_0} \lambda_c = c$, where $x_0(t) = (0, t, 0)$, $t \in [0, 1]$.

Lemma 3. *There exists a constant $\kappa > 0$ depending only on the numbers $\alpha_1, \ldots, \alpha_{n-1}$ such that for all sufficiently small $b > 0$ the following holds: Suppose that $x = (\theta, z) : [0, d] \to S^1 \times B^{2n-2}(3b)$ is a smooth curve satisfying*

$$b \le |z(0)| \le 2b$$

and one of the following two conditions:
(a) $|z(d)| = 3b$, or

(b) x is closed, and $|A(x) - A(x_0)| \leq \frac{3|c|}{4}$.
Then

$$\int_0^d |\pi_{x(t)}\dot{x}(t)|dt \geq \kappa b\,.$$

Proof. (a) In Case a we have $|z(d)| = 3b$ and $|z(0)| \leq 2b$. So Lemma 2 applied to the curves $x(t)$ and $y(t) = (\theta(t), 0)$ yields

$$3b \leq \int_0^d |\pi_{x(t)}\dot{x}(t)|dt + 2b\,,$$

thus

$$\int_0^d |\pi_{x(t)}\dot{x}(t)|dt \geq b\,.$$

(b) Now suppose that x is closed and satisfies (b). Arguing by contradiction, let us assume that

$$\int_{S^1} |\pi_{x(t)}\dot{x}(t)|dt < \kappa b, \qquad (**)$$

where κ will be chosen later.
After a shift in t we may assume that $\theta(0) = 0$. Consider the universal covering $\mathbf{R} \to S^1 = \mathbf{R}/\mathbf{Z}$, and let $\hat{\theta} : [0,1] \to \mathbf{R}$ be the lift of θ with $\hat{\theta}(0) = 0$. Then

$$\hat{\theta}(1) = l$$

for some integer l which is called the *winding number* of θ.

Claim. For b sufficiently small, θ has winding number $l = 1$.

Proof. Define $u : [0,1] \times S^1 \to S^1 \times B^{2n-2}(3b)$,

$$u(s,t) := (\theta(t), s\, z(t))\,,$$

and compute the action difference

$$\begin{aligned} A(x) - A(u(0,\cdot)) &= -\int_{[0,1]\times S^1} u^*\omega \\ &= -\int_0^1 \int_0^1 \omega(u)\Big((0,z), (\dot{\theta}, s\dot{z})\Big) ds\, dt \\ &= -\int_0^1 \int_0^1 \Big[-\sum_{j=1}^{n-1} \alpha_j \dot{\theta}(s x_j^2 + s y_j^2) \\ &\quad +\omega_{2n-2}(z, s\dot{z})\Big] ds\, dt \\ &= \frac{1}{2}\int_0^1 \Big[\sum_{j=1}^{n-1} \alpha_j \dot{\theta} |z_j|^2 - \omega_{2n-2}(z,\dot{z})\Big] dt \end{aligned}$$

$$= \frac{1}{2}\int_0^1 \sum_{j=1}^{n-1}\Big[\alpha_j\dot\theta|z_j|^2 - dx_j\wedge dy_j(z_j,\dot z_j - i\alpha_j\dot\theta z_j)$$
$$-dx_j\wedge dy_j(z_j, i\alpha_j\dot\theta z_j)\Big]dt$$
$$= -\frac{1}{2}\int_0^1 \sum_{j=1}^{n-1} dx_j\wedge dy_j(z_j,\dot z_j - i\alpha_j\dot\theta z_j)dt$$
$$= -\frac{1}{2}\int_0^1 \omega_{2n-2}(z,\pi_x\dot x)\,dt\,.$$

Using assumption (**) we get

$$\begin{aligned}|A(x) - A(u(0,\cdot))| &\le \frac{1}{2}\int_0^1 |z|\,|\pi_x\dot x|dt\\ &< \frac{1}{2}\cdot 3b\cdot\kappa b\\ &< \frac{|c|}{4}\end{aligned}$$

for b sufficiently small. On the other hand,

$$A(u(0,\cdot)) = l\,A(x_0) = lc\,,$$

and therefore

$$\begin{aligned}\frac{|c|}{4} &> |A(x) - lc|\\ &\ge |l-1|\,|c| - |A(x) - c|\\ &\ge |l-1|\,|c| - \frac{3|c|}{4}\end{aligned}$$

by hypothesis (b). Dividing by $|c| > 0$ yields $|l-1| < 1$, thus $l = 1$, and the claim is proved.

Now let $y : [0,1] \to S^1 \times B^{2n-2}(3b)$ be the curve satisfying

$$\left\{\begin{array}{l} y(t) = (\theta(t), w(t))\,,\\ \pi_{y(t)}\dot y(t) = 0\,,\\ y(0) = x(0)\,.\end{array}\right\}$$

The hypothesis that $\alpha_1,\dots,\alpha_{n-1}$ are not mupliples of 2π implies that $S^1\times\{0\}$ is the only closed characteristic in $S^1\times B^{2n-2}(3b)$ of winding number 1. Consequently, since θ has winding number 1 and $|w(0)|\ge b$,

$$|w(1) - w(0)| \ge \kappa b$$

for some constant κ depending only on $\alpha_1, \ldots, \alpha_{n-1}$. By the choice of y we have $z(0) = w(0)$, so from assumption (**) and Lemma 2 we get

$$\begin{aligned} |z(1) - z(0)| &\geq |w(1) - w(0)| - |z(1) - w(1)| \\ &> \kappa b - \kappa b = 0 \, . \end{aligned}$$

But this contradicts the hypothesis that x is a closed curve, and Lemma 3 is proved. □

Now let H, J, and $\hat{u}$ satisfy the hypotheses (H1–2), (H4), (J1–2), (J4) and (u1–5). Let $\Omega \subset Z$ be the set of all z for which $\hat{u}(z)$ lies in the region

$$\Big([-4b, -2b] \cup [2b, 4b]\Big) \times S^1 \times B^{2n-2}(4b) \, .$$

By (H4) the gradient of H vanishes in this region. So the restriction $\hat{u}|_\Omega$ is J-holomorphic, and its area equals

$$\int_\Omega |\hat{u}_s|^2 ds \, dt \, .$$

By the Monotonicity Lemma (see [Hu], Chapter II, Theorem 1.3) there exists a constant D depending only on $\alpha_1, \ldots, \alpha_{n-1}$ such that if $\hat{u}|_\Omega$ meets the set

$$A := \{\pm 3b\} \times S^1 \times B^{2n-2}(3b) \, ,$$

then its area is at least

$$\int_\Omega |\hat{u}_s|^2 ds \, dt \geq Db^2 \, ,$$

and Proposition 1 follows. So let us suppose from now on that $\hat{u}$ does not meet the set A.
Define a continuous function $d : \mathbf{R}^{2n} \setminus A \to [0, 3b]$ by

$$d(x) := \begin{cases} |z| & \text{for } x = (\rho, \theta, z) \in (-3b, 3b) \times S^1 \times B^{2n-2}(3b); \\ 3b & \text{otherwise} \end{cases}$$

(see Fig. 5). Given $\hat{u} : Z \to \mathbf{R}^{2n}$ as above, define $B : \mathbf{R} \to [0, 3b]$,

$$B(s) := \left(\int_0^1 d(\hat{u}(s,t))^2 dt \right)^{\frac{1}{2}} \, .$$

By the asymptotic conditions (u2), (u3) and (u4) we have

$$\begin{aligned} B(s) &\to 0 \quad \text{as } s \to -\infty \, , \\ B(s) &\to 3b \quad \text{as } s \to +\infty \, . \end{aligned}$$

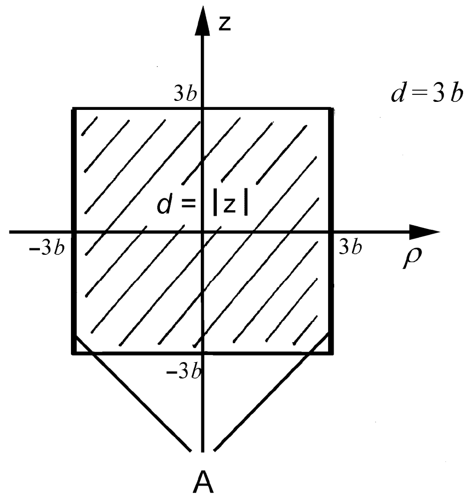

Fig. 5.

Thus there exist values $s_1 < s_2$ such that

$$B(s_1) = b\,,$$

$$B(s_2) = 2b\,, \text{ and}$$

$$b \leq B(s) \leq 2b \text{ for all } s \in [s_1, s_2]\,.$$

Lemma 4. *For each $t \in S^1$,*

$$d(\hat{u}(s_2,t)) - d(\hat{u}(s_1,t)) \leq \int_{s_1}^{s_2} |\hat{u}_s(s,t)| ds\,.$$

Proof. Consider the smaller region

$$U' := [-3b, 3b] \times S^1 \times B^{2n-2}(3b) \subset U\,.$$

We will distinguish several cases, for $t \in S^1$ fixed.
Case 1: $\hat{u}(s,t) \in U'$ for all $s \in [s_1, s_2]$.
On U', the metric $\langle \cdot, \cdot \rangle = \omega(J\cdot, \cdot)$ is given by

$$\begin{aligned}
&\langle u\frac{\partial}{\partial\rho} + v\,X + w, u'\frac{\partial}{\partial\rho} + v'X + w' \rangle \\
&= \omega\left(-v\frac{\partial}{\partial\rho} + u\,X - iw, u'\frac{\partial}{\partial\rho} + v'X + w'\right) \\
&= vv' + uu' + \langle w, w' \rangle_{2n-2}\,,
\end{aligned}$$

where $u, v \in \mathbf{R}$, $w \in \{0\} \times \{0\} \times \mathbf{C}^{n-1}$, and $\langle \cdot, \cdot \rangle_{2n-2}$ is the Euclidean metric on $\mathbf{C}^{n-1}$.
Writing $\hat{u}$ in U' as

$$\hat{u} = (\rho, u) = (\rho, \theta, z)\,,$$

we obtain

$$\begin{aligned}\hat{u}_s &= \rho_s \frac{\partial}{\partial \rho} + u_s \\ &= \rho_s \frac{\partial}{\partial \rho} + \theta_s X(u) + \pi_u u_s\,,\end{aligned}$$

and therefore

$$\begin{aligned}|\hat{u}_s|^2 &= |\rho_s|^2 + |\theta_s|^2 + |\pi u_s|^2 \\ &\geq |\pi u_s|^2\,.\end{aligned}$$

Using Lemma 2 we infer

$$\begin{aligned}d(\hat{u}(s_2,t)) - d(\hat{u}(s_1,t)) &= |z(s_2,t)| - |z(s_1,t)| \\ &\leq \int_{s_1}^{s_2} \left| \frac{\partial}{\partial s} |z(s,t)| \right| ds \\ &\leq \int_{s_1}^{s_2} |\pi u_s(s,t)|\, ds \\ &\leq \int_{s_1}^{s_2} |\hat{u}_s(s,t)|\, ds\,.\end{aligned}$$

Case 2: $\hat{u}(s_1,t), \hat{u}(s_2,t) \in U'$, but $\hat{u}(s,t) \notin U'$ for some $s \in [s_1, s_2]$.
Since $\hat{u}$ avoids the set A, it must leave and enter U' through

$$B := [-3b, 3b] \times S^1 \times \partial B^{2n-2}(3b)\,.$$

Choose $s_1 \leq s_1' \leq s_2' \leq s_2$ such that

$$\hat{u}(s_1',t), \hat{u}(s_2',t) \in B\,,$$

$$\hat{u}(s,t) \in U' \text{ for all } s \in [s_1, s_1'] \cup [s_2', s_2]\,.$$

Then by Case 1

$$\begin{aligned}&d\left(\hat{u}(s_2,t)\right) - d\left(\hat{u}(s_1,t)\right) \\ &= d\left(\hat{u}(s_2,t)\right) - d\left(\hat{u}(s_2',t)\right) + d\left(\hat{u}(s_1',t)\right) - d\left(\hat{u}(s_1,t)\right) \\ &\leq \int_{s_1}^{s_1'} |\hat{u}_s(s,t)| ds + \int_{s_2'}^{s_2} |\hat{u}_s(s,t)| ds \\ &\leq \int_{s_1}^{s_2} |\hat{u}_s(s,t)| ds\,.\end{aligned}$$

Case 3: $\hat{u}(s_1, t) \in U'$, and $\hat{u}(s_2, t) \notin U'$.
Choose a number $s_1 \le s_1' \le s_2$ such that

$$\hat{u}(s_1', t) \in B,$$
$$\hat{u}(s, t) \in U' \text{ for all } s \in [s_1, s_1'] .$$

Then

$$\begin{aligned} d(\hat{u}(s_2, t)) - d(\hat{u}(s_1, t)) &= d(\hat{u}(s_1', t)) - d(\hat{u}(s_1, t)) \\ &\le \int_{s_1}^{s_2} |\hat{u}_s(s, t)| ds . \end{aligned}$$

Case 4: If $\hat{u}(s_1, t), \hat{u}(s_2, t) \notin U'$, then $d(\hat{u}(s_2, t)) - d(\hat{u}(s_1, t)) = 0$.
Up to interchanging the roles of s_1 and s_2, these are all the cases, and the lemma is proved. □

Proof of Proposition 1
The differential equation (u1) for $\hat{u}$ yields

$$\begin{aligned} \frac{d}{ds} A_H\Big(\hat{u}(s, \cdot)\Big) &= \int_{S^1} |\hat{u}_s(s, t)|^2 dt - \int_{S^1} H_s(s, t, \hat{u}) dt \\ &\ge 0 \end{aligned}$$

by (H1). So the hypothesis (u5) implies

$$|A_H\Big(\hat{u}(s, \cdot)\Big) - A_H(x_1)| \le \frac{|c|}{4}$$

for all $s \in \mathbf{R}$.
Suppose that for some $s \in \mathbf{R}$ we have $\hat{u}(s, t) \in U$ for all $t \in S^1$. Let $u_0(s, t) := \Big(0, u(s, t)\Big)$, where $\hat{u}(s, t) = \Big(\rho(s, t), u(s, t)\Big)$. Recall that $x_0(t) = (0, t, 0) \in U$. With these notations,

$$\begin{aligned} |A\Big(u_0(s)\Big) - A(x_0)| &\le \Big|A\Big(\hat{u}(s)\Big) - A(x_1)\Big| + \Big|A\Big(\hat{u}(s)\Big) - A\Big(u_0(s)\Big)\Big| \\ &\quad + |A(x_1) - A(x_0)| \\ &\le \Big|A_H\Big(\hat{u}(s)\Big) - A_H(x_1)\Big| \\ &\quad + \int_0^1 \Big|H\Big(s, t, \hat{u}(s, t)\Big) - H_1\Big(t, x_1(t)\Big)\Big| \, dt \\ &\quad + 3b + 3b \\ &\le \frac{|c|}{4} + \frac{|c|}{4} + 3b + 3b \\ &\le \frac{3|c|}{4} \end{aligned}$$

for b sufficiently small, where we have used the estimate above and (H4).

By the choice of s_1 and s_2 preceding Lemma 4, for each $s \in [s_1, s_2]$ there exists a $t(s) \in S^1$ such that

$$b \le d\Big(\hat{u}(s, t(s))\Big) \le 2b\,,$$

i.e. $\hat{u}\Big(s, t(s)\Big) = (\rho, \theta, z) \in U'$ and $b \le |z| \le 2b$.
If moreover $\hat{u}(s,t) \in U'$ for all $t \in S^1$, then by the computation above its projection $u_0(s, \cdot)$ satisfies hypothesis (b) of Lemma 3.
If the curve $\hat{u}(s, \cdot)$ leaves U', then since it avoids the set A it must pass through $B = [-3b, 3b] \times S^1 \times \partial B^{2n-2}(3b)$. So in this case $\hat{u}(s, \cdot)$ satisfies (a) of Lemma 3.
Hence by Lemma 3 for each $s \in [s_1, s_2]$

$$\int_{I(s)} |\pi u_t| dt \ge \kappa b\,,$$

where $I(s) := \{t \in S^1 \mid \hat{u}(s,t) \in U'\}$. On the other hand, on U' the Hamiltonian vector field of H is given by

$$\begin{aligned} X_H(\rho, x) &= J(\rho, x) \nabla_J H(s, \rho, x) \\ &= J(\rho, x)\, h_\rho(s, \rho)\, \frac{\partial}{\partial \rho} \\ &= h_\rho(s, \rho)\, X(x)\,. \end{aligned}$$

Hence for $t \in I(s)$,

$$\begin{aligned} |\hat{u}_t - X_H(\hat{u})|^2 &= |\rho_t|^2 + |\theta_t - h_\rho(s, \rho)|^2 + |\pi u_t|^2 \\ &\ge |\pi u_t|^2\,. \end{aligned}$$

Integrating over t we obtain

$$\|\hat{u}_t(s, \cdot) - X_H(\hat{u}(s, \cdot))\|_{L^1(dt)} \ge \kappa b \qquad (*)$$

for all $s \in [s_1, s_2]$. So we can estimate

$$\begin{aligned} &\int_Z |\hat{u}_s(s,t)|^2 ds\, dt \\ &\ge \int_{s_1}^{s_2} \int_0^1 |\hat{u}_s(s,t)|^2 ds\, dt \\ &= \int_{s_1}^{s_2} \|\hat{u}_s(s, \cdot)\|_{L^2(dt)}\, \|\hat{u}_t(s, \cdot) - X_H(\hat{u}(s, \cdot))\|_{L^2(dt)} ds \\ &\ge \int_{s_1}^{s_2} \|\hat{u}_s(s, \cdot)\|_{L^2(dt)}\, \|\hat{u}_t(s, \cdot) - X_H(\hat{u}(s, \cdot))\|_{L^1(dt)} ds \end{aligned}$$

$$
\begin{aligned}
&\geq \kappa b \int_{s_1}^{s_2} \|\hat{u}_s(s,\cdot)\|_{L^2(dt)} ds \\
&\geq \kappa b \left\| \int_{s_1}^{s_2} |\hat{u}_s(s,\cdot)| ds \right\|_{L^2(dt)} \\
&\geq \kappa b \Big\| d(\hat{u}(s_2,\cdot)) - d(\hat{u}(s_1,\cdot)) \Big\|_{L^2(dt)} \\
&\geq \kappa b \Big(\|d(\hat{u}(s_2,\cdot))\|_{L^2(dt)} - \|d(\hat{u}(s_1,\cdot))\|_{L^2(dt)} \Big) \\
&= \kappa b \Big(B(s_2) - B(s_1) \Big) \\
&= \kappa b^2 .
\end{aligned}
$$

If the roles of x_1 and x_2 are interchanged we obtain the same estimate, and Proposition 1 is proved. □

4 Proof of Theorem 2 and Corollaries 1–4

Let $c \neq 0$ be given. Consider Hamiltonians H which satisfy (H1–4) of Sect. 3, as well as almost complex structures J satisfying (J1–4).

Fix positive numbers $a \leq b$ and $r \leq \frac{|c|}{4}$. By hypothesis (H4) the Hamiltonian is given on the set $U = [-4b, 4b] \times S^1 \times B^{2n-2}(4b)$ by $H(s,t,x) = h(s,\rho)$, $x = (\rho,\theta,z) \in U$. To compute symplectic homology we will choose $h : \mathbf{R} \times [-4b, 4b] \to \mathbf{R}$ of a particular form which is similar to the Hamiltonians in [CFHW]: For every $s \in \mathbf{R}$ there exist numbers $-2a \leq \rho_1 < \rho_2 < \rho_3 < \rho_4 \leq 2a$ such that $k = h(s,\cdot)$ satisfies

$$
\left\{
\begin{array}{l}
k \equiv const > -r \text{ in } [-4b, \rho_1]; \\
0 < k' < 1 \text{ in } (\rho_1,\rho_2) \cup (\rho_3,\rho_4); \\
k'(\rho_2) = 1, k''(\rho_2) > 0, k(\rho_2) < -r + a; \\
k'(\rho_3) = 1, k''(\rho_3) < 0, k(\rho_3) > -a; \\
k' > 1 \text{ in } (\rho_2,\rho_3); \\
k \equiv 0 \text{ in } [\rho_4, 4b]
\end{array}
\right\} \qquad (H5)
$$

(see Fig. 6).

Fix a number $0 < \delta < |c|$ which will be specified later, and consider Floer homology in the action interval $[c-\delta, c+\delta)$. Notice that an s-independent Hamiltonian satisfying (H5) possesses two particular degenerate 1-periodic orbits in U,

$$
\begin{aligned}
x_H(t) &:= (\rho_3, t, 0) \text{ and} \\
\tilde{x}_H(t) &:= (\rho_2, t, 0) \, .
\end{aligned}
$$

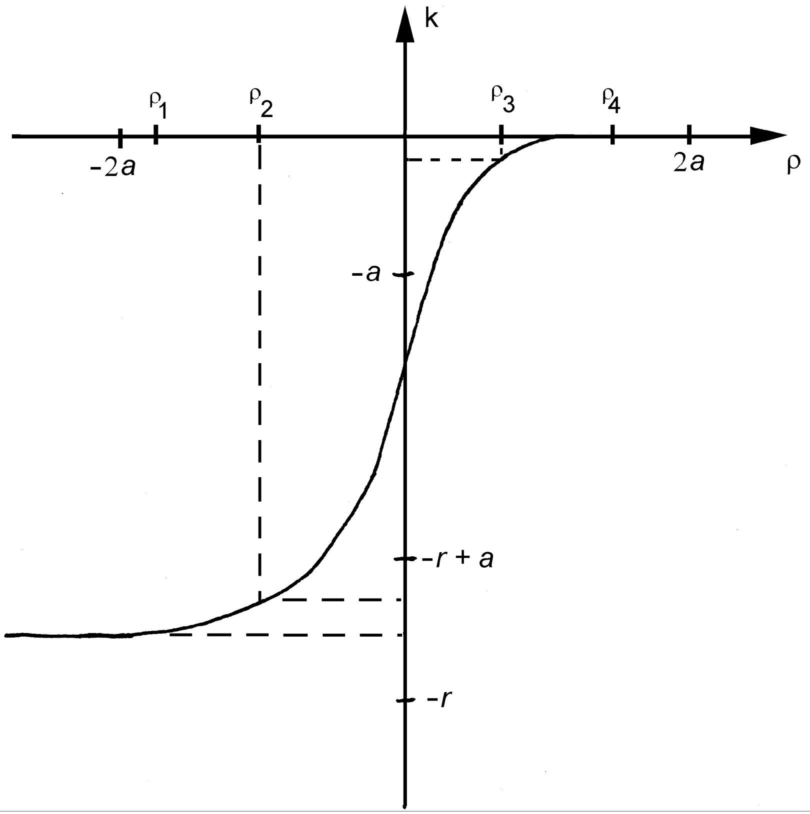

Fig. 6.

The Hamiltonian action of x_H is

$$\begin{aligned} A_H(x_H) &= -\int_{x_H} \lambda_c - \int_0^1 H(t, x_H)dt \\ &= c - \rho_3 - h(\rho_3)\,. \end{aligned}$$

By (H5) this implies $c - 2a \leq A_H(x_H) \leq c + 3a$, hence

$$A_H(x_H) \in (c - \delta, c + \delta)$$

if $3a < \delta$. As x_H is degenerate, an s-independent Hamiltonian H satisfying (H5) cannot be regular with respect to the action interval $[c - \delta, c + \delta)$. However, H can be made regular by a C^∞-small perturbation. We denote by

$$HF^{[c-\delta,c+\delta)}(H_{reg})$$

the Floer homology of a sufficiently small regular perturbation of H. Although the Floer homology may depend on the perturbation, the properties stated in the following proposition are independent of the perturbation.

Proposition 2. *Suppose that* $0 < 4a \leq \delta \leq \frac{r}{2} \leq \frac{|c|}{8}$ *and* $\delta \leq \frac{1}{2}\kappa b^2$*, where* κ *is the constant of Proposition 1. Then for every sufficiently small regular perturbation* H_{reg} *of an* s*-independent Hamiltonian* H *satisfying (H1–5), the 1-periodic orbit* x_H *yields two nontrivial elements*

$$0 \neq [x_H^+], [x_H^-] \in HF^{[c-\delta,c+\delta)}(H_{reg}) .$$

Moreover, for two Hamiltonians $H_1 \geq H_2$ *satisfying (H1–5), the induced homomorphism*

$$\sigma\Big((H_1)_{reg}, (H_2)_{reg}\Big) : \ HF^{[c-\delta,c+\delta)}\Big((H_1)_{reg}\Big) \\ \to HF^{[c-\delta,c+\delta)}\Big((H_2)_{reg}\Big)$$

maps $[x_{H_1}^+]$ *onto* $[x_{H_2}^+]$ *and* $[x_{H_1}^-]$ *onto* $[x_{H_2}^-]$.

Proof. 1. Let H be an s-independent Hamiltonian satisfying (H1–5). Let us first determine all 1-periodic orbits of H in U whose actions lie in the interval $[c-\delta, c+\delta]$.
It has been shown above that the orbit $x_H(t) = (\rho_3, t, 0)$ has action $A_H(x_H) \in (c-\delta, c+\delta)$ for $4a \leq \delta$.
The action of the orbit $\tilde{x}_H(t) = (\rho_2, t, 0)$ is

$$\begin{aligned} A_H(\tilde{x}_H) &= c - \rho_2 - h(\rho_2) \\ &\geq c + r - 3a \\ &> c + \delta \end{aligned}$$

for $4a \leq \delta \leq \frac{r}{2}$.
Multiply covered orbits $x(t) = (\rho, kt, z(t))$, $k \geq 2$, have action

$$\begin{aligned} A_H(x) &\geq k(c-2a) \\ &> c + \delta \end{aligned}$$

if $c > 0$, and

$$\begin{aligned} A_H(x) &\leq k(c+2a) + r \\ &< c - \delta \end{aligned}$$

if $c < 0$.
The constant orbits (r, θ, z) with $\rho \in [\rho_4, 4b]$ have action $-h(\rho) = 0 \notin [c-\delta, c+\delta]$, and the constant orbits with $\rho \in [-4b, \rho_1]$ have action $-h(\rho) \approx r \notin [c-\delta, c+\delta]$.
So x_H is the only 1-periodic orbit in U with action in the interval $[c-\delta, c+\delta]$.

2. Choose an s-independent almost complex structure J satisfying (J1–4). Suppose that y is a 1-periodic orbit outside U, and there exists a 'connecting orbit' $\hat{u}$ satisfying (u1) and

$$\hat{u}(s,\cdot) \longrightarrow \begin{cases} x_H & \text{as } s \to -\infty; \\ y & \text{as } s \to +\infty. \end{cases}$$

By Proposition 1, the action of y is at least

$$\begin{aligned} A_H(y) &\geq A_H(x_H) + \int_Z |\hat{u}_s|^2 ds\, dt \\ &> c - \delta + \kappa b^2 \\ &\geq c + \delta \end{aligned}$$

because $\delta \leq \frac{1}{2}\kappa b^2$. Similarly, if $\hat{u}$ is a 'connecting orbit' with

$$\hat{u}(s,\cdot) \longrightarrow \begin{cases} y & \text{as } s \to -\infty; \\ x_H & \text{as } s \to +\infty, \end{cases}$$

then $A_H(y) < c - \delta$.
So there exists no 'connecting orbit' $\hat{u}$ between x_H and any other 1-periodic solution $y \neq x_H$ with action in $[c-\delta, c+\delta]$. By a compactness argument, this property persists under a small perturbation of (H, J) to a regular pair $(H, J)_{reg}$. Hence the contribution of x_H to

$$HF^{[c-\delta,c+\delta)}(H_{reg}) = HF^{[c-\delta,c+\delta)}\Big((H, J)_{reg}\Big)$$

equals the local Floer homology of x_H, which was shown in [CFHW] to have two generators $[x_H^{\pm}]$ of Conley-Zehnder indices $\mathrm{ind}([x_H^+]) = \mathrm{ind}([x_H^-])+1$.
3. Two s-independent pairs (H_1, J_1), (H_2, J_2) satisfying (H1–5) and (J1–4) with $H_1 \geq H_2$ can be connected by a monotone homotopy (H, J) also satisfying (H1–5) and (J1–4). Arguing as in 2., we conclude from Proposition 1 that there exist no solutions $\hat{u}$ of (u1) connecting one of the orbits $x_{H_1}^{\pm}$, $x_{H_2}^{\pm}$ to any orbit different from them with action in $[c-\delta, c+\delta]$. In particular, the image of $[x_{H_1}^{\pm}]$ under the homomorphism

$$\sigma\Big((H_1)_{reg}, (H_2)_{reg}\Big) = \sigma\Big((H, J)_{reg}\Big)$$

can be computed on the set U.
But during the whole homotopy the only 1-periodic orbits of $H(s,\cdot)$ in U with action in $[c-\delta, c+\delta]$ are $x_{H(s,\cdot)}^{\pm}$, and their actions remain in the open interval $(c-\delta, c+\delta)$. So $\sigma\Big((H, J)_{reg}\Big)\Big|_U$ is a composition of small distance isomorphisms in the sense of [FHW], and therefore an isomorphism. In view of the Conley-Zehnder indices of the orbits $x_{H_i}^{\pm}$ this is only possible if $\sigma\Big((H, J)_{reg}\Big)$ maps $[x_{H_1}^+]$ onto $[x_{H_2}^+]$ and $[x_{H_1}^-]$ onto $[x_{H_2}^-]$. $\square$

Proof of Theorem 2
For the given number $c \neq 0$, let a, b, δ, r and $U \subset \mathbf{R}^{2n}$ be as in Proposition 2. Suppose that $S_i \subset \mathbf{R}^{2n}$ are disjoint compact cooriented hypersurfaces intersecting U as in Theorem 2. We will compute the symplectic homology of (S_i, r) using the cofinal system consisting of regular adapted Hamiltonians $H_{reg} \in Ad_{reg}(S_i, r)$, where H satisfies (H1–5). By Proposition 2, the Floer homology of every such Hamiltonian contains two nontrivial elements

$$0 \neq [x_H^{\pm}] \in HF^{[c-\delta, c+\delta)}(H_{reg})$$

corresponding to the closed characteristic y_i of Theorem 2. Moreover for two Hamiltonians $H_1 \geq H_2$ satisfying (H1–5),

$$\sigma\Big((H_1)_{reg}, (H_2)_{reg}\Big) \cdot [x_{H_1}^{\pm}] = [x_{H_2}^{\pm}]\,.$$

It follows from the definition of the direct limit that the $[x_H^{\pm}]$ yield nontrivial elements

$$0 \neq [y_i^{\pm}] \in SH^{[c-\delta, c+\delta)}(S_i, r)\,.$$

If $H_{reg} \in Ad_{reg}(S_1, r)$, where H satisfies (H1–5), then (again by Proposition 2) the elements $[x_H^{\pm}]$ also persist under savethe direct limit over $Ad_{reg}(S_2, r)$, giving rise to the elements $[y_2^{\pm}]$. This proves that the inclusion induced homomorphism

$$\rho(S_1, S_2) : SH^{[c-\delta, c+\delta)}(S_1, r) \to SH^{[c-\delta, c+\delta)}(S_2, r)$$

maps $[y_1^{\pm}]$ onto $[y_2^{\pm}]$. $\square$

To prove Corollary 1, we need some elementary properties of the Hausdorff metric d_H formulated in the following two lemmas.
Given a connected topological space X and two disjoint subsets A_1, A_2 of X, we say that a subset B of X *separates* A_1 from A_2 if $X \setminus B$ has precisely 2 connected components U_1, U_2 with U_i containing A_i for $i = 1, 2$.

Lemma 5. *(a) Let (X, d) be a bounded connected metric space, and equip $[0,1] \times X$ with the metric $d\Big((r, x), (s, y)\Big) := |r - s| + d(x, y)$. Then for any closed subset A of $[0,1] \times X$ which separates $\{0\} \times X$ from $\{1\} \times X$,*

$$d_H(A, \{1\} \times X) = \sup_{a \in A} d(a, \{1\} \times X)\,.$$

Proof. Let $b = (1, x) \in \{1\} \times X$. If the line $[0,1] \times \{x\}$ did not intersect A then it would connect $\{0\} \times X$ with $\{1\} \times X$ in $\Big([0,1] \times X\Big) \setminus A$, in contradiction to the separation property of A. Thus there exists a number $r_0 \in [0,1]$ such that

$$a_0 = (r_0, x) \in A\,.$$

We obtain

$$\begin{aligned} d(b, A) &\leq d(b, a_0) \\ &= |1 - r_0| \\ &= d(a_0, \{1\} \times X) \\ &\leq \sup_{a \in A} d(a, \{1\} \times X)\,. \end{aligned}$$

Taking the supremum over all $b \in \{1\} \times X$ yields

$$\sup_{b \in \{1\} \times X} d(b, A) \leq \sup_{a \in A} d(a, \{1\} \times X)\,,$$

and the lemma follows. $\square$

Next consider a compact connected Riemannian manifold N with connected smooth boundary ∂N. Denote by d the distance on N induced by the Riemannian metric. Let $[0,1] \times \partial N$ be a collar neighborhood of ∂N in N, where ∂N is identified with $\{1\} \times \partial N$.

Lemma 6. *Let $p \in \overset{\circ}{N}$ be a given point. Let $(A_n) \subset \overset{\circ}{N}$ be a sequence of compact hypersurfaces separating p from ∂N such that*

$$\sup_{a \in A_n} d(a, \partial N) \to 0 \text{ as } n \to \infty\,.$$

Then for $n \in \mathbf{N}$ sufficiently large, A_n is contained in the collar neighborhood $[0,1] \times \partial N$, A_n separates $\{0\} \times \partial N$ from $\{1\} \times \partial N$, and

$$d_H(A_n, \partial N) \to 0 \text{ as } n \to \infty\,.$$

Proof. For $n \in \mathbf{N}$ sufficiently large we have $A_n \subset (0,1) \times \partial N$. We claim that A_n separates $\{0\} \times \partial N$ from $\{1\} \times \partial N$.
By hypothesis, $N \setminus A_n$ has precisely 2 connected components U_0', U_1 with $p \in U_0'$ and $\partial N \subset U_1$. Without loss of generality we may assume that $p \notin (0,1] \times \partial N$. Since $N \setminus \Big((0,1] \times \partial N\Big)$ is connected and contains p,

$$N \setminus \Big((0,1] \times \partial N\Big) \subset U_0'\,.$$

Hence U_1 and $U_0 := U_0' \cap [0,1] \times \partial N$ are disjoint open subsets of $[0,1] \times \partial N$,

$$\{i\} \times \partial N \subset U_i \text{ for } i = 0, 1\,,$$

and U_1 is connected. So the claim is proved if we can show that U_0 is connected.
Arguing by contradiction, assume that U_0 is a disjoint union of two open

subsets W_0 and W_1. Since $\{0\} \times \partial N$ is connected, it is contained in one of these sets, say in W_0. But then

$$W_0 \cup \Big[N \setminus \Big((0,1] \times \partial N\Big)\Big]$$

is open in U_0', and we have a contradiction to the connectedness of U_0'. Since the metric induced by N on $[0,1] \times \partial N$ is equivalent to the metric

$$d'\Big((r,x),(s,y)\Big) = |r-s| + d(x,y)\,,$$

Lemma 5 implies

$$d_H(A_n, \partial N) \to 0 \text{ as } n \to \infty\,. \qquad \square$$

Proof of Corollary 1
Suppose that $S \in Hyp^0(\mathbf{R}^{2n})$ is a hypersurface as in Theorem 1, and $(S_k)_{k \in \mathbf{N}} \subset Hyp^0(\mathbf{R}^{2n})$ is a sequence of hypersurfaces converging to S in the Hausdorff metric. By definition of the Hausdorff metric, the S_k are contained in a tubular neighborhood $[-1,1] \times S$ of S for large k. The argument in the proof of Lemma 5 shows that S_k separates $\{-1\} \times S$ from $\{1\} \times S$. If the tubular neighborhood is sufficiently small this implies

$$f\Big([-4b,-a] \times S^1 \times B^{2n-2}(4b)\Big) \subset B(S_k),$$
$$f\Big([a,4b] \times S^1 \times B^{2n-2}(4b)\Big) \subset U(S_k)\,.$$

Hence S_k is not of restricted contact type by Theorem 1. $\square$

The proof of Corollary 2 is based on the following lemma.

Lemma 7. *Let $S \subset (M,\omega)$ and $\tilde{S} \subset (\tilde{M},\tilde{\omega})$ be hypersurfaces in symplectic manifolds of the same dimension $2n$ and $P \subset S$, $\tilde{P} \subset \tilde{S}$ closed characteristics. Then the following two statements are equivalent:*
(i) The linear Poincaré maps of P and $\tilde{P}$ are symplectically conjugate.
(ii) There exists a symplectomorphism $F : (U,\omega) \to (\tilde{U},\tilde{\omega})$ between tubular neighborhoods of $P, \tilde{P}$ in $M, \tilde{M}$ such that $F(P) = \tilde{P}$, and $F(S \cap U)$ is tangent of second order to $\tilde{S} \cap \tilde{U}$ along $\tilde{P}$.

Remark. Note that the statement is not tautological. For instance, it implies that at a critical point p of a 1-periodic time-dependent Hamiltonian h on a symplectic manifold (M,ω) the second derivative $h''(p)$ can be made time-independent by a 1-periodic time-dependent symplectic change of coordinates (apply the lemma to the hypersurface $\{r = h(t,z)\}$ in the extended phase space $\mathbf{R} \times S^1 \times M$).

Proof. 1. Clearly (ii) implies (i). So let us suppose that the linear Poincaré maps are conjugate.
A tubular neighborhood of P in (M, ω) is symplectomorphic to (W, ω_{2n}), where

$$W = [-a, a] \times S^1 \times B^{2n-2}(b)$$

with coordinates (r, t, z), $z = x + iy \in \mathbf{R}^{2n-2} = \mathbf{C}^{n-1}$, and

$$\begin{aligned}\omega_{2n} &= dr \wedge dt + \sum_{j=1}^{n-1} dx_j \wedge dy_j \\ &= dr \wedge dt + \omega_{2n-2} \,.\end{aligned}$$

The closed characteristic P corresponds to $\{0\} \times S^1 \times \{0\}^{2n-2} \in W$. It follows that the hypersurface S corresponds to the graph

$$S \cap W = \{r + H(t, z) = 0\}$$

of a function H satisfying $H(t, 0) = 0$ and $dH(t, 0) = 0$ for all $t \in S^1$. After a symplectic change of coordinates we may moreover assume that $H(0, z) = 0$ for all z.
Let $H_2(t, z)$ be the part of H quadratic in z, extended to $S^1 \times \mathbf{R}^{2n-2}$. Replace $S \cap W$ by the hypersurface

$$S_2 := \{r + H_2(t, z) = 0\} \subset \hat{W} := \mathbf{R} \times S^1 \times \mathbf{R}^{2n-2}$$

which is tangent of second order to $S \cap W$ along P. Let $\tilde{H}$, $\tilde{S}_2$ etc. be the analogous objects for $\tilde{S}$. We will show that S_2 can be mapped onto $\tilde{S}_2$ by a symplectomorphism of $\hat{W}$, which implies (ii).
The restriction of ω_{2n} to S_2 is given in coordinates (t, z) by

$$\omega_{2n}|_{S_2} = dt \wedge d_z H_2 + \omega_{2n-2} \,.$$

Its kernel is generated by the vector field

$$X = \frac{\partial}{\partial t} + Z(t, z) \,,$$

where the time-dependent vector field Z on $\mathbf{R}^{2n-2}$ is determined by the equation

$$\begin{aligned}0 &= i_X(\omega_{2n}|_{S_2}) \\ &= d_z H_2 - d_z H_2(Z) \cdot dt + i_Z \omega_{2n-2} \,,\end{aligned}$$

or equivalently

$$i_Z \omega_{2n-2} + d_z H_2 = 0 \,.$$

We see that for every $t \in S^1$, $Z(t, z)$ is linear in z. Let $\tilde{Z}$ be the corresponding vector field for $\tilde{S}$. Notice that $Z(0, z) = \tilde{Z}(0, z) = 0$ for all z by the assumption $H(0, z) = \tilde{H}(0, z) = 0$.

2. Let $\phi_t, \tilde{\phi}_t : \mathbf{R}^{2n-2} \to \mathbf{R}^{2n-2}$ be the linear flows generated by the time-dependent linear vector fields $Z, \tilde{Z}$. The time-1 maps $\phi_1, \tilde{\phi}_1$ are the linear Poincaré maps of $P, \tilde{P}$ in these coordinates. By hypothesis, there exists a linear symplectic map $B : \mathbf{R}^{2n-2} \to \mathbf{R}^{2n-2}$ such that $\tilde{\phi}_1 = B\phi_1 B^{-1}$. Apply the symplectomorphism

$$\hat{B}(r, t, z) := (r, t, Bz)$$

of $\hat{W}$ to S_2. The kernel of the restriction of ω_{2n} to $\hat{B}(S_2)$ is generated by the vector field

$$\frac{\partial}{\partial t} + B_* Z = \frac{\partial}{\partial t} + BZB^{-1} .$$

Its time-1 map equals $B\phi_1 B^{-1} = \tilde{\phi}_1$. Thus after this transformation we may assume that

$$\phi_1 = \tilde{\phi}_1 .$$

3. Define $\Phi : S^1 \times \mathbf{R}^{2n-2} \to S^1 \times \mathbf{R}^{2n-2}$ by

$$\Phi(t, z) := \Big(t, \tilde{\phi}_t \circ \phi_t^{-1}(z)\Big) .$$

Notice that $\Phi(1, z) = \Big(1, \tilde{\phi}_1^{-1} \circ \phi_1(z)\Big) = (1, z)$ by Step 2, so Φ defines a diffeomorphism of $S^1 \times \mathbf{R}^{2n-2}$. Moreover, since the flows of $X, \tilde{X}$ preserve $\omega_{2n}|_{S_2}$ respectively $\omega_{2n}|_{\tilde{S}_2}$, and $X = \tilde{X}$ at $t = 0$, we have

$$\Phi^*(\omega_{2n}|_{\tilde{S}_2}) = \omega_{2n}|_{S_2} .$$

4. Define $\Psi : \hat{W} \to \hat{W}$,

$$\Psi\Big(r, t, z\Big) := \Big(r + H_2(t, z) - \tilde{H}_2(\Phi(t, z)), \Phi(t, z)\Big) .$$

The diffeomorphism Ψ maps each hypersurface $S^c := \{r + H_2(t, z) = c\}$ onto $\tilde{S}^c := \{r + \tilde{H}_2(t, z) = c\}$. Therefore by Step 3 it satisfies

$$\begin{aligned}
(\Psi^* \omega_{2n})|_{S^c} &= \Psi^*(\omega_{2n}|_{\tilde{S}^c}) \\
&= \Phi^*(\omega_{2n}|_{\tilde{S}_2}) \\
&= \omega_{2n}|_{S_2} \\
&= \omega_{2n}|_{S^c} .
\end{aligned}$$

Moreover,

$$\begin{aligned} i_{\frac{\partial}{\partial r}}(\Psi^* \omega_{2n}) &= i_{(\Psi_*^{-1} \frac{\partial}{\partial r})}(\Psi^* \omega_{2n}) \\ &= \Psi^*(i_{\frac{\partial}{\partial r}} \omega_{2n}) \\ &= \Psi^* dt \\ &= dt \\ &= i_{\frac{\partial}{\partial r}} \omega_{2n} \,. \end{aligned}$$

Hence $\Psi^* \omega_{2n} = \omega_{2n}$, i.e. Ψ is a symplectomorphism mapping S_2 onto $\tilde{S}_2$, and the lemma is proved. □

Proof of Corollary 2
Let x be a nondegenerate linearly stable closed characteristic on the hypersurface S whose linear Poincaré map is symplectically conjugate to the diagonal matrix $diag(e^{i\alpha_1}, \ldots, e^{i\alpha_{n-1}})$. Let (U, ω) be a neighborhood of x in $\mathbf{R}^{2n}$ as constructed at the beginning of Sect. 3, with the same numbers $\alpha_1, \ldots, \alpha_{n-1} \in \mathbf{R} \setminus 2\pi\mathbf{Z}$. By Lemma 7 we can assume that $S \cap U$ is tangent of second order to the hyperplane $\{0\} \times S^1 \times B^{2n-2}(4b)$ along $\{0\} \times S^1 \times \{0\}^{2n-2}$. For every sufficiently small $b > 0$ we take $a := \frac{1}{8}\kappa b^2$, $\delta := \frac{1}{2}\kappa b^2$, such that the hypotheses of Proposition 1 are satisfied. Since a depends quadratically on b and $S \cap U$ is flat up to second order in b, for sufficiently small b we will have

$$S \cap U \subset (-a, a) \times S^1 \times B^{2n-2}(4b) \,,$$

thus S satisfies the hypotheses of Theorem 1. It follows from Theorem 1 that S is not of restricted contact type, and the corollary is proved. □

Proof of Corollary 3
Let $S \in Hyp^0(\mathbf{R}^{2n})$ be as in Corollary 1 or 2, and suppose that $A_1 \subset A_2 \subset \ldots \subset B(S)$ is an exhaustion of the bounded component $B(S)$ by compact sets with smooth boundaries. Passing to connected components, we may assume that the A_k are connected. We have $\sup_{a \in \partial A_k} d(a, S) \to 0$ as $k \to \infty$. Each connected component S_k of ∂A_k divides $\mathbf{R}^n$ into a bounded and an unbounded component, $B(S_k)$ and $U(S_k)$, with $S \subset U(S_k)$. If $A \setminus S_k \subset U(S_k)$ we may replace A_k by $A_k \cup B(S_k)$, thus getting rid of the boundary component S_k. Since we cannot get rid of all boundary components, we can choose for every k a boundary component S_k with $A_k \setminus S_k \subset B(S_k)$. Then S_k separates $0 \in \mathbf{R}^{2n}$ from S for large k. Hence by Lemma 6, $d_H(S_k, S) \to 0$ as $k \to \infty$. So by Corollary 1, S_k is not of contact type for large k. □

For the proof of Corollary 4 we need another lemma.

Lemma 8. *If M is an open manifold of dimension $n \geq 1$, then there exists a subset $N \subset M$, $N \neq M$, with nonempty smooth boundary ∂N such that $N \setminus \partial N$ is diffeomorphic to M.*

Remark. In general, an open manifold is not diffeomorphic to the interior of a *compact* manifold with smooth boundary.

Proof. Without loss of generality assume that M is connected. Let $A_1 \subset A_2 \subset \ldots \subset M, \cup_{i \in \mathbf{N}_0} A_i = M$, be an exhaustion of M by compact subsets. Let $(x_i)_{i \in \mathbf{N}_0}$ be a sequence such that for every i the points $x_i, x_{i+1}, \ldots$ lie in the same path connected component of $M \setminus A_i$. Choose a smooth embedded curve $x : [0, \infty) \to M$ such that $x(i) = x_i$ and $x([i, \infty)) \subset M \setminus A_i$ for all i. The image of x is then a closed submanifold with boundary of M. Pick a Riemannian metric on M for which x is a geodesic parametrized by arclength. This can be done by taking the Euclidean metric on a tubular neighborhood $[0, \infty) \times B^{n-1}(1)$ of the image of x and extending it anyhow to M. Let U be another tubular neighborhood of x obtained as the image under the exponential map of the subset

$$\{(t, v) \in [0, \infty) \times \mathbf{R}^{n-1} \;\big|\; |v|^2 \leq \rho(t)\}$$

of the normal bundle over x, where $\rho : [0, \infty) \to (0, 1)$ is a suitable function. Its boundary in M is given by

$$\partial U = \{(t, v) \in [0, \infty) \times \mathbf{R}^{n-1} \;\big|\; |v|^2 = \rho(t)\} \cup \{0\} \times B^{n-1}(\rho(0)) \,.$$

Rescaling in the fibre yields a diffeomorphism

$$U \cong [0, \infty) \times B^{n-1}(1) \,.$$

Take a smooth monotone function $\phi : [0, \frac{1}{4}) \to [1, \infty)$ with $\phi(0) = 1$ and $\phi(r) \to \infty$ as $r \to \frac{1}{4}$ (see Fig. 7). Let

$$R := \{(t, v) \in [0, \infty) \times B^{n-1}(1) \;\big|\; |v| < \frac{1}{2}, t > \phi(|v|^2)\} \,.$$

Via the diffeomorphism above we can view R as a subset of U and thus of M. Its boundary in M is given by

$$\partial R = \{(t, v) \in [0, \infty) \times B^{n-1}(1) \;\big|\; |v| < \frac{1}{2}, t = \phi(|v|^2)\} \,,$$

due to the choice of U. Now $U \setminus \bar{R}$ is diffeomorphic to U by a diffeomorphism which equals the identity near ∂U. Hence $N := M \setminus R$ is the desired subset. $\square$

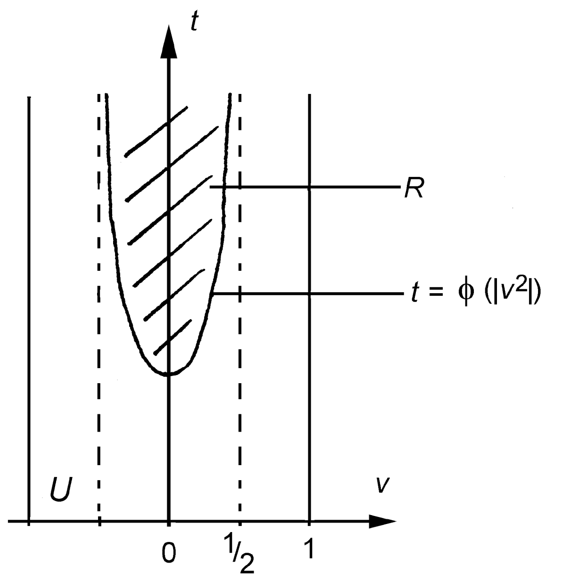

Fig. 7.

Proof of Corollary 4 (sketch)
Let (M, ω) be an open symplectic manifold of dimension $2n \geq 4$. Let $N \subset M$ be the submanifold provided by Lemma 8. In a neighborhood of a point of ∂N the triple (M, N, ω) is symplectomorphic to $(\mathbf{R}^{2n}, \{0\} \times \mathbf{R}^{2n-1}, \omega_{2n})$ near $0 \in \mathbf{R}^{2n}$. So (after rescaling) we may replace ∂N in this neighborhood by the hypersurface $f(\{0\} \times \mathbf{R}^{2n-1})$ constructed in Lemma 1, for some $c < 0$. Denote this new hypersurface by $S \subset M$ and its interior by $B(S)$. Note that $B(S)$ is diffeomorphic to M.

Now suppose that (M, ω) is exact convexly exhaustible, and the same is true for $(B(S), \omega)$ (otherwise there is nothing to show). Let $S_1 \subset B(S)$ be a smooth exact ω-convex compact hypersurface which separates S from the set $f([-4b, -a] \times V)$ of Lemma 1, viewed as a subset of $B(S)$. Let $A \subset M$ be a compact subset with smooth ω-convex boundary which contains $f([-4b, 0] \times V)$ and S_1. Since (A, ω) is a compact exact symplectic manifold with ω-convex boundary, we can define the symplectic homology of compact hypersurfaces of (A, ω), and the analogous statement of Theorem 1 holds. Applied to the hypersurface S_1 this implies that S_1 is not of restricted contact type, and we have a contradiction. □

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