Symplectic homology II

A general construction

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1 Introduction

In [11] the second and third author introduced a symplectic homology theory. By means of a general construction, given real numbers $a < b$ and an integer k they assigned to each open set U of $\mathbb{C}^n$ a group $S_k^{[a,b)}(U)$ and studied its properties. In the present paper which continues the work in [11], we show how this construction can be carried over to more general manifolds. We assume the reader to be familiar with [11], since there are many constructions which we recall here in a more general set up without giving a detailed proof. In fact, the arguments given in [11] work under more general circumstances at least if some topological assumptions are met. Only in the case that there is a considerable difference we give complete details.

For applications we refer the reader to [13] and the forthcoming paper [14]. For motivation of the present construction we refer the reader to [11]. However we recall that the crucial observation made [2] is that periodic orbits for Hamiltonian systems can be used to construct many new symplectic invariants. Our construction in [11] and the present paper precisely exploits the same aspects of this observation. We also would like to point the attention to [3, 4, 5] for other applications of this "philosophy".

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* Andreas Floer died on May 15th, 1991.

2 A general construction

In the following we describe the construction under suitable hypotheses on the symplectic manifold (M, ω). Other assumptions allow similar constructions and we outline some generalizations later on.

We assume (M, ω) is a compact symplectic manifold with or without boundary, such that $[\omega]$ vanishes on $\pi_2(M)$. Moreover we assume that the first Chern class c_1 for pullback bundles $u^*TM \to \mathbb{S}^2$ and $[u] \in \pi_2(M)$ vanishes, where $TM \to M$ carries the structure of a complex vectorbundle induced by a ω-calibrated almost complex structure J. ("ω-calibrated" means that $\omega \circ (J \times id)$ is a Riemannian metric. The space of such structures is contractible.)

If $\partial M \neq \emptyset$, we assume that ∂M is of contact type. This means that there exists an outward pointing transversal vector field η defined on an open neighborhood of ∂M in M such that the Lie derivative satisfies $L_\eta \omega = \omega$. Equivalently (set $\lambda = i_\eta \omega$), there exists a 1-form λ on a neighborhood of ∂M such that $d\lambda = \omega$ and $\lambda \wedge (d\lambda)^{n-1}|_{\partial M}$ is a volume form determining the orientation on ∂M induced from the orientation ω^n on M.

We call a smooth Hamiltonian $H : S^1 \times M \to \mathbb{R}$ **admissible** if

- There exists a nonempty open subset U of M with
$$\mathrm{cl}(U) \subset M \setminus \partial M \quad \text{and} \quad H|(\mathbb{S}^1 \times \mathrm{cl}(U)) < 0.$$
- If $\partial M = \emptyset$, all 1-periodic solutions are nondegenerate .
- If $\partial M \neq \emptyset$, there exists a constant $m = m(H) > 0$, such that $H(t,x) = m$ for all $(t,x) \in \mathbb{S}^1 \times W$, where W is a neighborhood of ∂M. Moreover, every 1-periodic solution $x : \mathbb{S}^1 \to M$ satisfying $\int_0^1 H(t,x(t))dt < m$ is nondegenerate. Further, $H(t,x) \leq m$ for all $(t,x) \in \mathbb{S}^1 \times M$ and the set of 1-periodic solutions x with $\int_0^1 H(t,x(t))dt < m$ is finite. (1)

We denote by $C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ the set of all smooth contractible loops in M. Given $x \in C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ we denote by $\bar{x} : D \to M$ a smooth extension of x to the disk. We define the action of x denoted by $A(x)$ via

$$A(x) = -\int_D \bar{x}^* \omega. \tag{2}$$

This is well defined in view of the assumption $[w]|\pi_2(M) = 0$. Given a smooth arc in $C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ say $(\tau, t) \to x_\tau(t)$ is smooth, the map $\tau \to A(x_\tau)$ is of class C^∞. Let $x_0 = x$, and $\xi(t) = \frac{d}{d\tau} x_\tau(t)|_{\tau=0}$. We compute

$$
\begin{aligned}
\frac{d}{d\tau}\Big|_{\tau=0} A(x_\tau) &= -\int_0^1 \omega(\xi(t), \dot{x}(t))dt \\
&= \int_0^1 \omega(\dot{x}(t), \xi(t))dt.
\end{aligned}
$$

We define Φ_H by

$$
\Phi_H(x) = A(x) - \int_0^1 H(t, x(t))dt
$$

and observe that

$$
d\Phi_H(x)\xi = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), \xi(t))dt
$$

for all smooth vector fields ξ along the contractible curve $x \in C^\infty_{\text{cntr}}(\mathbb{S}^1, M)$. Here X_{H_t} is the vector field defined by $i_{X_{H_t}}\omega = dH_t$. Hence

$$
d\Phi_H(x) = 0 \iff \dot{x}(t) = X_{H_t}(x(t)), \quad t \in \mathbb{S}^1
$$

If H is admissible and x is a contractible 1-periodic solution of the Hamiltonian system associated to H satisfying $\Phi_H(x) > -m(H)$, we infer that $\int_0^1 H(t, x(t)) < m(H)$. In fact, if we have equality x has to be constant and $\Phi_H(x) = -m(H)$. Hence x has to be nondegenerate.

We denote for $a \in \mathbb{R}, a > -m(H)$ by $\mathcal{P}_a(H)$ the finite set consisting of all contractible 1-periodic solutions satisfying $\Phi_H(x) \geq a$. We call $\Phi_H(x)$ the energy of x. The energy will be one of the important numerical values attached to a contractible 1-periodic solution. Another invariant is the Conley-Zehnder index, [1]. The version we need is described [19]. Let x be as above and linearize the Hamiltonian vector field along x. The linearized system defines a symplectic loop $X(t)$ satisfying

$$
X(t) : T_{x(0)}M \longrightarrow T_{x(t)}M
$$

is symplectic and

$$
X(0) = Id, \quad X(1) : T_{x(0)} \longrightarrow T_{x(0)}
$$

since $x(0) = x(1)$. We infer that $1 \notin \text{spec}(X(1))$ since x is nondegenerate. Take an extension $\bar{x} : D \to M$ of x and symplectically trivialize $\bar{x}^*TM \to D$. In view of the condition on c_1 the trivializations induced for $x^*TM \to \mathbb{S}^1$ are homotopic and independent of the choice of the extension $\bar{x}$. Hence take such a trivialization

$$
\Psi : x^*TM \xrightarrow{\simeq} \mathbb{S}^1 \times \mathbb{C}^n.
$$

Let us write $\Psi(t)$ for the induced map

$$
\Psi(t) : T_{x(t)}M \longrightarrow \mathbb{C}^n.
$$

We consider the arc Γ given by $t \to \Psi(t)X(t)\Psi(0)^{-1}$. Then $\Gamma(0) = \text{Id}$ and $\Gamma(1) = \Psi(0)X(1)\Psi(0)^{-1}$ so that $\text{spec}(\Gamma(1)) = \text{spec }(X(1))$. Hence $1 \notin$

spec($\Gamma(1)$). According to [19], Γ has a Conley-Zehnder index $\mathrm{ind}_{CZ} \in \mathbb{Z}$, which is independent of the choice of Ψ (if Ψ is as described above). Hence $\mathrm{ind}_{CZ}(x) := \mathrm{ind}_{CZ}(\Gamma) \in \mathbb{Z}$ is a well defined invariant of a contractible nondegenerate 1-periodic solution.

Summing up we have a map $\mathcal{P}_a(H) \to \mathbb{R} \times \mathbb{Z}$ associating to $x \in \mathcal{P}_a(H)$ the "local invariants" $(\Phi_H(x), \mathrm{ind}_{CZ}(x))$, where $a > -m(H)$ and H is admissible. Then for every $x \in \mathcal{P}_a(H)$ we have $x(\mathbb{S}^1) \subset M \backslash \partial M$.

A smooth time-dependent almost complex structure $J : S^1 \times TM \to TM$ is called ω-calibrated if $\omega \circ (J_t \times id)$ is a time-dependent Riemannian metric on M, where $J_t = J(t, \cdot)$. If $\partial M \neq \emptyset$ we further assume the following: There exists an outward pointing transversal vector field η near ∂M with $L_\eta \omega = \omega$. The flow $(\psi_t)_{-\varepsilon < t \leq 0}$ of η yields a diffeomorphism $\psi : (-\varepsilon, 0] \times \partial M \to W$ onto some neighbourhood W of ∂M in M. Let $W_s := \psi_s(\partial M)$. The restriction of $\lambda = i_\eta \omega$ to W_s is a contact form with contact bundle $\xi = ker(\lambda\,|_{W_s})$ and Reeb vector field $X \in TW_s$ defined by $i_X d\lambda\,|_{W_s} = 0$, $i_X \lambda = 1$. We assume that on W, J is time-independent, leaves ξ invariant and maps X onto η. Denote by $\mathcal{J}$ the set of all such J.

We claim that $\mathcal{J}$ is nonempty, and any two elements of $\mathcal{J}$ can be joined by a smooth path $(J_\tau)_{0 \leq \tau \leq 1}$ in $\mathcal{J}$.

To see this observe that the existence of a W, η, λ, ξ, X as described above follows directly from our assumption that ∂M is of contact type. Now it is well-known that for every symplectic vector bundle (E, ω) the space of almost complex structures on E for which $\omega \circ (J \times id)$ is a bundle metric is nonempty and contractible. So we find an almost complex structure on ξ such that $\omega\,|_\xi \circ (J \times id)$ is a bundle metric on ξ. Extending it first to the bundle $TM\,|_W$ via $J(X) = \eta$, $J(\eta) = -X$, and then to the whole bundle $TM \to M$, we obtain a $J \in \mathcal{J}$. For the second part let $J^0, J^1 \in \mathcal{J}$ be given with associated vectorfields η_0, η_1 near ∂M. Since both η_0 and η_1 are transversal to ∂M and outward pointing, the same is true for all $\eta_\tau := \tau\eta_1 + (1 - \tau)\eta_0$, $0 \leq \tau \leq 1$. Moreover, $L_{\eta_\tau}\omega = \omega$ for all τ. Arguing as above we can associate to every η_τ a $\tilde{J}^\tau \in \mathcal{J}$, depending smoothly on τ. We can also achieve that $\tilde{J}^0 = J^0$, $\tilde{J}^1 = J^1$, and the claim is proved.

Now consider pairs (J, H), where H is an admissible Hamiltonian and $J \in \mathcal{J}$.

Given $x, y \in \mathcal{P}_a(H), a > -m(H)$, we consider the set $\mathcal{M}(x, y; J, H)$ defined by

$$\mathcal{M}(x, y; J, H) = \{u : Z \to M \mid u \text{ is smooth and satisfies (3) below}\},$$

where $Z = \mathbb{R} \times \mathbb{S}^1, (s, t) \in Z$, and

$$\begin{aligned} u_s + J(t, u)u_t + \nabla_{J_t} H(t, u) &= 0 \\ u(s, *) \longrightarrow x \text{ in } C^\infty \text{ for } s &\longrightarrow -\infty \\ u(s, *) \longrightarrow y \text{ in } C^\infty \text{ for } s &\longrightarrow +\infty. \end{aligned} \tag{3}$$

Here ∇_{J_t} is the gradient taken with respect to the Riemannian metric $\omega \circ (J_t \times id)$. If follows from [18], Lemma 2.4, that if W is a foliated neighborhood of ∂M as described above and $H \equiv m(H)$ on W, then $u(Z) \subset M \backslash W$.

By C^∞-modifying J (see [12]) we can achieve that the new pair $(\widetilde{J}, H)$ still satisfies all the hypotheses above, and for every pair $(x,y), \mathcal{M}(x,y;\widetilde{J},H)$ is a finite dimensional manifold. This is due to the fact that $\mathcal{M}(x,y;\widetilde{J},H)$ can be considered as the zero set of a regular Fredholm section. We call such a pair $(\widetilde{J}, H)$ **admissible**.

Let us denote for simplicity by $Ad(M)$ the collection of all admissible pairs (J,H). Following [11], we define for $(J,H) \in Ad(M)$ and $a > -m(H)$ the graded free Abelian group $C_a(J,H)$:

$$\begin{aligned} C_a(J,H) &= \oplus_k C_a^k(J,H) \\ C_a^k(J,H) &= \oplus_{x \in \mathcal{P}_a(H)^k} \mathbb{Z}x \\ \mathcal{P}_a(H)^k &= \{x \in \mathcal{P}_a(H) \mid \operatorname{ind}_{CZ}(x) = k\}. \end{aligned} \tag{4}$$

We define $\delta_k : C_a^k(J,H) \to C_a^{k+1}(J,H)$ by

$$\delta_k(x) = \sum_{y \in \mathcal{P}_a(H)^{k+1}} \left(\sum_{\hat{u} \in \widehat{\mathcal{M}}(x,y;J,H)} \tau(\hat{u}) \right) y \ ,$$

where $\widetilde{\mathcal{M}}(x,y;J,H) = \mathcal{M}(x,y;J,H)/\mathbb{R}$ and $\tau(\hat{u}) \in \{1,-1\}$ are suitable "orientations" attached to the points in $\widehat{\mathcal{M}}$, see [10]. We recall that $\dim \mathcal{M}\ (x,y;J,H)$ is given by the difference $\operatorname{ind}_{CZ}(y) - \operatorname{ind}_{CZ}(x)$, see [19]. The crucial point is that $\delta_{k+1} \circ \delta_k = 0$. Hence $(C_a(J,H), \delta)$ is a co-chain complex. Moreover, for $b \geq a > -m(H)$ we have the commutative diagram.

$$\begin{array}{ccc} C_a(J,H) & \xrightarrow{\delta} & C_a(J,H) \\ \uparrow & & \uparrow \\ C_b(J,H) & \xrightarrow{\delta} & C_b(J,H) \end{array}$$

where the vertical arrows are given by inclusion.

Next we have to investigate the dependence of the co-chain complex on J and H. To do this we define a partial ordering on $Ad(M)$ by

$$(J_1,H_1) \leq (J_2,H_2) :\Longleftrightarrow H_1(t,z) \leq H_2(t,z) \ \text{ for all } \ (t,z). \tag{5}$$

We would like to associate, in analogy to [11], to the above situation a natural chain homotopy class

$$\sigma(J_2,H_2;J_1,H_1) : C_a(J_2,H_2) \to C_a(J_1,H_1).$$

To this purpose we take a **monotone homotopy**, i.e. a smooth pair $(J,H) = (J(s,t,z), H(s,t,z))$, $(s,t,z) \in \mathbb{R} \times S^1 \times M$ such that

$$
\begin{aligned}
(J(s,\cdot), H(s,\cdot)) &= (J_1(\cdot), H_1(\cdot)) \qquad \text{for } s \geq s_0 \\
(J(s,\cdot), H(s,\cdot)) &= (J_2(\cdot), H_2(\cdot)) \qquad \text{for } s \leq -s_0 \\
J(s,\cdot) &\in \mathcal{J} \qquad \text{for all } s \in \mathbb{R} \\
H(s,t,z) &\equiv H(s) \qquad \text{for } z \text{ near } \partial M
\end{aligned}
\tag{6}
$$

and

$$
\partial_s H(s,t,z) \leq 0 \qquad \text{for all } (s,t,z) \tag{7}
$$

The existence of such a homotopy follows from the discussion after the definition of $\mathcal{J}$.

For $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$, $a > -m(H_1)$ define $\mathcal{M}(x_2, x_1; J, H)$ as before. In order to get compactness results as in [11] we must ensure that the elements of $\mathcal{M}(x_2, x_1; J, H)$ are bounded away uniformly from ∂M. We can no longer apply Lemma 2.4 from [18] since J may depend on s near ∂M.

Lemma. *Given a monotone homotopy (J,H) and $a > -m(H_1)$, there exist constants $c_0, \delta > 0$ depending on J, H and a such that for all $c \geq c_0$, $(J^c(s,\cdot), H^c(s,\cdot)) = (J(\frac{s}{c},\cdot), H(\frac{s}{c},\cdot))$, $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2, x_1; J^c, H^c)$ we have (with respect to some fixed metric on M):*

$$
dist(u(Z), \partial M) \geq \delta
$$

We call such a (J^c, H^c) a **slow monotone homotopy**.

Proof. Choose an $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ we have diffeomorphisms $\psi(s) : (-2\varepsilon, 0] \times \partial M \xrightarrow{\cong} W(s)$ induced by the vector field $\eta(s,\cdot)$ near ∂M. We may assume that $H(s,\cdot) \equiv H(s)$ on $\bigcup_{s\in\mathbb{R}} W(s)$. Take a smooth function $f : \mathbb{R} \times M \to [-\frac{3}{2}\varepsilon, 0]$ satisfying $f \equiv -\frac{3}{2}\varepsilon$ on $M \setminus \bigcup W(s)$, $f(s,z) = \tau$ for $x \in \psi(s)(\{\tau\} \times \partial M)$, $\tau \in [-\varepsilon, 0]$, and $|df(s,\cdot)| \leq 1$ where $|\cdot|$ is taken with respect to the metric $\omega \circ (J(s,\cdot) \times \mathrm{id})$. Pick an open neighbourhood W of ∂M contained in $\bigcap_{s\in\mathbb{R}} \psi(s)([-\varepsilon, 0] \times \partial M)$ and having smooth boundary. Since $\bar{W}$ is compact, there is a $\rho > 0$ such that every closed curve in W with length $< \rho$ is contractible in W. Let $a > -m(H_1)$ be given and define

$$
\begin{aligned}
b &:= \sup_{\mathbb{R}\times(M\setminus W)} f \quad < 0 \\
\triangle &:= \max\{\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \mid x_1 \in P_a(H_1), x_2 \in P_a(H_2)\}
\end{aligned}
$$

Take an $l > 0$ large enough such that

$$
\sqrt{\frac{\triangle}{l}} < \min\{\rho, -\frac{b}{2}, a + m(H_1)\}
$$

and let

$$
\mu := -\frac{2b}{l^2} > 0
$$

By passing from (J,H) to (J^c, H^c) for some c large enough we may assume that on W

$$|\partial_s^2 f| + \frac{1}{2}|\partial_s\lambda|^2 + \frac{1}{2}|d(\partial_s f)|^2 < \mu$$

where $\lambda = i_\eta\omega$ and norms are taken with respect to the metric $\omega \circ (J_s \times \mathrm{id})$.

Now let $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2, x_1; J, H)$ be given and consider the function $g(s,t) := f(s, u(s,t))$ on the cylinder $Z = \mathbb{R} \times S^1$. We will show that $g \leq \frac{b}{4} < 0$, from which the lemma follows.

From $\partial_s H \leq 0$ it follows that

$$\begin{aligned} \int_Z |u_s|^2 ds\, dt &\leq \Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \\ &\leq \triangle \end{aligned}$$

So there exist s_k, $k \in \mathbb{Z}$, with $|s_{k+1} - s_k| \leq l$ and $\int_0^1 |u_s(s_k,t)|^2 dt \leq \frac{\triangle}{l}$. For $k \in \mathbb{Z}$ let $x_k(t) := u(s_k, t)$.

Case 1: $x_k(t) \in W$ for all $t \in S^1$. Using that on W u satisfies the equation $u_s + J(s,u)u_t = 0$ we calculate

$$\begin{aligned} \mathrm{length}(x_k) &\leq \left(\int_0^1 |\dot{x}_k|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 |u_s(s_k,t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{\triangle}{l}} \\ &< \rho \end{aligned}$$

By definition of ρ this implies that x_k is contractible in W. Let $\overline{x_k} : D \to W$ be a smooth extension of x_k to the closed unit disk. Using $\omega = d\lambda(s_k, \cdot)$ on W we get

$$\begin{aligned} |A(x_k)| &\equiv \left|\int_D \overline{x_k}^* w\right| \\ &= \left|\int_{S^1} x_k^* \lambda(s_k, \cdot)\right| \\ &= \left|\int_0^1 \omega(\eta(s_k, x_k), \dot{x}_k) dt\right| \\ &\leq \int_0^1 |\dot{x}_k| dt \\ &\leq \sqrt{\frac{\triangle}{l}} \end{aligned}$$

Hence

$$\begin{aligned} \Phi_{H(s_k,\cdot)}(x_k) &\equiv A(x_k) - \int_0^1 H(s_k, x_k) dt \\ &\leq \sqrt{\frac{\triangle}{l}} - m(H_1) \\ &< a \end{aligned}$$

in contradiction to $a \leq \Phi_{H_2}(x_2) \leq \Phi_{H(s_k,\cdot)}(x_k)$. So case 1 does not occur, and for all $k \in \mathbb{Z}$ we are in

Case 2: There is a $t_k \in S^1$ such that $x_k(t_k) \notin W$. Then for any $t \in S^1$ we find a $t_0 \in S^1$, $t-1 < t_0 \leq t$, such that $x_k(t_0) \notin W$ and $x_k(t') \in W$ for all $t_0 < t' \leq t$. This yields

$$\begin{aligned} g(s_k, t) &\leq g(s_k, t_0) + \int_{t_0}^{t} |\partial_t g(s_k, t')| dt' \\ &\leq b + \int_{t_0}^{t} |df(s_k, x_k) \cdot \dot{x}_k| dt' \\ &\leq b + \int_{t_0}^{t} |\dot{x}_k| dt' \\ &\leq b + \sqrt{\frac{\triangle}{l}} \\ &\leq \frac{b}{2} \end{aligned}$$

where the last but one inequality follows as in case 1.

So for the finite cylinders $Z_k := [s_k, s_{k+1}] \times S^1$ we have shown that $g \leq \frac{b}{2} < 0$ on ∂Z_k. It remains to estimate g in the interior of Z_k. If $(s,t) \in Z_k$ and $g(s,t) > b$ then $u(s,t) \in W$, and we can compute

$$\begin{aligned} \partial_s(\lambda(s,u) \cdot u_t) - \partial_t(\lambda(s,u) \cdot u_s) &= \partial_s \lambda(s,u) \cdot u_t + u^* d\lambda(\partial_s, \partial_t) \\ &= \partial_s \lambda(s,u) \cdot u_t + \omega(u_s, u_t) \\ &= \partial_s \lambda(s,u) \cdot u_t - \frac{1}{2}(|u_s|^2 + |u_t|^2) \end{aligned}$$

Using $df \circ J = \lambda$ on W we obtain

$$\begin{aligned} g_s(s,t) &= \partial_s f(s,u) - \lambda(s,u) \cdot u_t \\ g_t(s,t) &= \lambda(s,u) \cdot u_s \end{aligned}$$

and hence

$$\begin{aligned} \triangle g &= g_{ss} + g_{tt} \\ &= \partial_s^2 f + d(\partial_s f) \cdot u_s - \partial_s \lambda \cdot u_t + \frac{1}{2}(|u_s|^2 + |u_t|^2) \\ &\geq -|\partial_s^2 f| - \frac{1}{2}|d(\partial_s f)|^2 - \frac{1}{2}|\partial_s \lambda|^2 \\ &> -\mu \end{aligned}$$

for all $(s,t) \in Z_k$ with $g(s,t) > b$.

Define

$$\hat{g}(s,t) := g(s,t) + \frac{\mu}{2}\left(s - \frac{s_k + s_{k+1}}{2}\right)^2$$

For $(s,t) \in Z_k$ with $\hat{g}(s,t) > \frac{3}{4}b$ we have $g(s,t) > b$ (remember $\mu = -\frac{2b}{l^2}$ and $|s_{k+1} - s_k| \leq l$) and therefore

$$\Delta\hat{g} = \Delta g + \mu > 0$$

So $\hat{g}$ cannot have an interior maximum $> \frac{3}{4}b$ in Z_k. On the other hand, $g \leq \frac{b}{2}$ on ∂Z_k implies $\hat{g} \leq \frac{b}{4}$ on ∂Z_k, thus $g(s,t) \leq \hat{g}(s,t) \leq \frac{b}{4}$ for all $(s,t) \in Z_k$. Since $k \in \mathbb{Z}$ was arbitrary, the lemma is proved. ∎

This lemma ensures that for slow monotone homotopies (J,H) the solution spaces $\mathcal{M}(x_2, x_1; J, H)$ have the usual compactness properties. Now we can proceed as in [11]: For admissible pairs $(J_1, H_1) \leq (J_2, H_2)$, regular slow monotone homotopies between (J_1, H_1) and (J_2, H_2) induce a unique chain homotopy class

$$\sigma(J_2, H_2; J_1, H_1) : C_a(J_2, H_2) \longrightarrow C_a(J_1, H_1)$$

such that the following properties hold:

$$\sigma(J, H; J, H) \; = \; id \tag{8}$$

For $(J_1, H_1) \leq (J_2, H_2) \leq (J_3, H_3)$ we have

$$\sigma(J_2, H_2, J_1, H_1) \circ \sigma(J_3, H_3, J_2, H_2) = \sigma(J_3, H_3, J_1, H_1) \tag{9}$$

For $b \geq a > -m(H_1)$ the following diagram commutes:

$$\begin{array}{ccc} C_a(J_2, H_2) & \xrightarrow{\sigma} & C_a(J_1, H_1) \\ \uparrow & & \uparrow \\ C_b(J_2, H_2) & \xrightarrow{\sigma} & C_b(J_1, H_1)\,. \end{array} \tag{10}$$

Note in (9) that if two slow monotone homotopies between (J_1, H_1) and (J_2, H_2) respectively (J_2, H_2) and (J_3, H_3) are sufficiently slow then their composition is a slow monotone homotopy between (J_1, H_1) and (J_3, H_3). Then use uniqueness of σ.

Now let R be any commutative ring. We apply the functor $\mathrm{Hom}(*, R)$ to the quotient co-chain complexes C_a/C_b, where $b \geq a > -m(H)$ to obtain the chain complexes

$$D^{[a,b)}(J, H) \; := \mathrm{Hom}(C_a(J, H)/C_b(J, H); R) \tag{11}$$

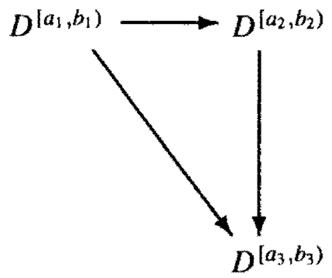
with boundary operator ∂ induced by δ. For numbers $b \geq a > -m(H)$ and $b' \geq a' > -m(H)$, such that $b \geq b'$ and $a \geq a'$ we have obviously induced homotopy classes of maps

$$C_a/C_b \longrightarrow C_{a'}/C_{b'}$$

and consequently

$$D^{[a',b')}(J, H) \longrightarrow D^{[a,b)}(J, H)\,. \tag{12}$$

These maps behave functorially, i.e. for $b_i \geq a_i > -m(H), i = 1,2,3$, and $b_3 \geq b_2 \geq b_1$, $a_3 \geq a_2 \geq a_1$ we have



(13)

If $(J,H) \le (\widetilde{J},\widetilde{H})$ then the following diagram is also commutative

$$\begin{array}{ccc} D^{[a,b)}(J,H) & \xrightarrow{\sigma} & D^{[a,b)}(\widetilde{J},\widetilde{H}) \\ \uparrow & & \uparrow \\ D^{[a',b')}(J,H) & \xrightarrow{\sigma} & D^{[a',b')}(\widetilde{J},\widetilde{H}) \end{array} \tag{14}$$

where σ is the dual of $\sigma(\widetilde{J},\widetilde{H};J,H)$ and the vertical maps are those in (12). Here a,b,a',b' are as described above.

Now we pass to homology obtaining the symplectic homology groups $S^{[a,b)}(J,H)$ of the admissible pair $(J,H) \in Ad(M)$.

These homology groups are independent of J (by the discussion above), but they depend on H. In order to obtain invariants of the manifold, we have to pass to a direct limit over the Hamiltonians. This is done as follows:

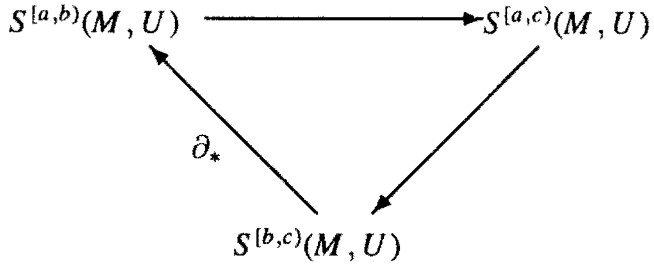
Let $U \subset M$ be open (possibly empty) with $\mathrm{cl}(U) \subset M\backslash\partial M$. Let $Ad(M,U)$ be the subset of $Ad(M)$ consisting of those pairs (J,H) for which $H|_{\mathbb{S}^1\times \mathrm{cl}(U)} < 0$. The density results on admissible pairs (see [12], [19]) imply that $(Ad(M,U),\le)$ is a directed set. Thus we may take the direct limit over $Ad(M,U)$ to obtain the symplectic homology groups

$$S^{[a,b)}(M,U) = \varinjlim S^{[a,b)}(J,H). \tag{15}$$

Note that $S^{[a,b)}(J,H)$ is well defined if $b \ge a > -m(H)$; omit in the direct limit all those (J,H) for which this not true. /parbox Given numbers $c \ge b \ge a$ the exact sequence of chain complexes

$$0 \longrightarrow D^{[a,b)}(J,H) \longrightarrow D^{[a,c)}(J,H) \longrightarrow D^{[b,c)}(J,H) \longrightarrow 0 \tag{16}$$

for $(J,H) \in Ad(M,U), a > -m(H)$, gives, since direct limits preserve exactness, the long exact homology sequence



(17)

where the connecting homomorphism ∂_* has degree -1. We shall denote the diagram (17) by $\triangle_{a,b,c}(M, U)$ and call it the exact triangle. If $a \leq b \leq c$ and $a' \leq b' \leq c'$ with $a' \leq a, b' \leq b$ and $c' \leq c$, the maps from (12) induce a morphism between triangles

$$\triangle_{a',b',c'}(M, U) \longrightarrow \triangle_{a,b,c}(M, U). \tag{18}$$

which is functorial.

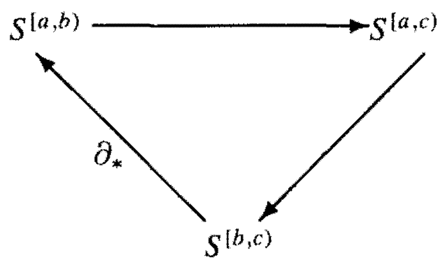
This construction has many nice properties, which we shall study in the next section.

3 Elementary properties of symplectic homology

Let us first recall the properties already outlined in the previous section. Given $b \geq a$ and $b' \geq a'$, such that $a \geq a'$ and $b \geq b'$ we have natural transformations.

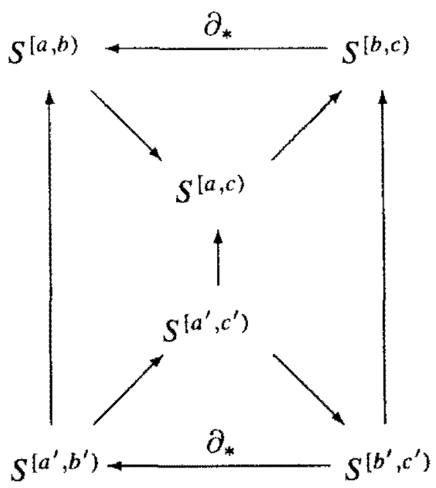
$$S^{[a',b')} \longrightarrow S^{[a,b)}. \tag{19}$$

There is another natural transformation ∂_* of degree -1, which together with (19) yields for $a \leq b \leq c$ the exact triangle $\triangle_{a,b,c}$.



(20)

The natural transformations in (19) induce morphisms between exact triangles. In order to be more precise, assume $a \leq b \leq c, a' \leq b' \leq c'$ and $a' \leq a, b' \leq b, c' \leq c$. Then the following diagram is commutative



(21)

In short we have $\triangle_{a',b',c'} \to \triangle_{a,b,c}$ which is functorial in the numbers a, b, e.t.c.

Next assume $U \subset V$ are open with $\mathrm{cl}(V) \subset M \backslash \partial M$. Then $Ad(M, V) \subset Ad(M, U)$, hence we obtain a natural map

$$S^{[a,b)}(M, V) \longrightarrow S^{[a,b)}(M, U) \tag{22}$$

inducing a morphism between exact triangles and being compatible with all other natural transformations.

Next assume $\Psi : M \xrightarrow{\simeq} M$ is a symplectic diffeomorphism mapping U into V, where U and V are open subsets of M with their closures contained in $M \backslash \partial M$. We shall write

$$\Psi : (M, U) \xrightarrow{s} (M, V). \tag{23}$$

If $(J, H) \in Ad(M, V)$ then H_Ψ defined by $H_\Psi(t, x) = H(t, \Psi(x))$, satisfies $H_\Psi|_{(\mathbb{S}^1 \times \bar{U})} < 0$. We define further J_Ψ by

$$J_\Psi(t, x)h = T\Psi(x)^{-1} J(t, \Psi(x)) T\Psi(x)h .$$

Then $(J_\Psi, H_\Psi) \in Ad(M, U)$.

If $u \in \mathcal{M}(x, y; J, H)$ then $\Psi^{-1}(u)$ is in $\mathcal{M}(\Psi^{-1}(x), \Psi^{-1}(y); J_\Psi, H_\Psi)$. So, the new connecting orbit spaces are obtained by applying Ψ^{-1}. Consequently we obtain a co-chain map $\Psi^\# : C_a(J, H) \to C_a(J_\Psi, H_\Psi)$ by $x \to \Psi^{-1}(x)$ inducing

$$\Psi_\# : D^{[a,b)}(J_\Psi, H_\Psi) \longrightarrow D^{[a,b)}(J, H) \tag{24}$$

and consequently the isomorphism

$$\Psi_{\#\#} : S^{[a,b)}(J_\Psi, H_\Psi) \longrightarrow S^{[a,b)}(J, H) . \tag{25}$$

Passing to the direct limit induces an isomorphism

$$\widehat{\Psi} : S^{[a,b)}(M, \Psi^{-1}(V)) \xrightarrow{\simeq} S^{[a,b)}(M, V). \tag{26}$$

Combining $\widehat{\Psi}^{-1}$ with the natural map (22), namely since $U \subset \Psi^{-1}(V)$:

$$S^{[a,b)}(M, \Psi^{-1}(V)) \longrightarrow S^{[a,b)}(M, U) \tag{27}$$

gives a morphism Ψ^*, i.e

$$\begin{array}{ccc} S^{[a,b)}(M, V) & \xrightarrow{\quad \Psi^* \quad} & S^{[a,b)}(M, U) \\ \Big\downarrow {\scriptstyle \widehat{\Psi}^{-1}} & \nearrow & \\ S^{[a,b)}(M, \Psi^{-1}(V)) & & \end{array} \tag{28}$$

It follows immediately from the construction that Ψ^* gives a morphism between the exact triangles $\triangle_{a,b,c}(M,V)$ and $\triangle_{a,b,c}(M,U)$. Moreover, it is compatible with all the introduced natural transformations. Further, it is clear that $(\Phi\Psi)^* = \Psi^*\Phi^*$ and $\mathrm{Id} : (M,U) \to (M,U)$ induces Id. (22) is in fact induced by $\mathrm{Id} : (M,U) \to (M,V)$ for $U \subset V$.

The key point will be that an isotopy of symplectic maps $\Psi_s : (M,U) \to (M,V)$ with $s \in [0,1]$ will induce maps $(\Psi_s)^*$ which are independent of $s \in [0,1]$. This will be shown in the next section.

4 Isotopy invariance

We follow the ideas of [11]. Assume (M,ω) is a symplectic manifold with the properties described before. Let H be an admissible Hamiltonian and fix numbers $b \geq a > -m(H)$. Define

$$g(H,[a,b)) := \min\Big\{ \inf_{\Phi_H(x)<a} (a - \Phi_H(x)),\ \inf_{\Phi_H(x)<b} (b - \Phi_H(x)) \Big\}, \tag{29}$$

where the infima are taken over all contractible 1-periodic solutions of $\dot{x} = X_{H_t}(x)$ with $\Phi_H(x) > -m(H)$. Since the set of such solutions is finite, we have $g(H,[a,b)) > 0$. We shall call $g(H,[a,b))$ the "gap".

Given admissible Hamiltonians H and K we define their distance $d(H,K)$ as follows. Consider a smooth path $\tau \to H_\tau$ of Hamiltonians such that each H_τ is constant close to ∂M, $\tau \in [\tau_1,\tau_2]$ and $H_{\tau_1} = H, H_{\tau_2} = K$. Take the integral $d((H_\tau))$ defined by

$$d((H_\tau)) = \int_{\tau_1}^{\tau_2} \Big[\max_{(t,x)} \Big| \frac{\partial H_\tau}{\partial \tau}(t,x) \Big| \Big] d\tau . \tag{30}$$

Then let $d(H,K)$ be the infimum of all numbers $d((H_\tau))$ taken all smooth paths as just described.

Now let (J_1,H_1) and (J_2,H_2) be admissible, a and b fixed such that

$$b \geq a > \max\{-m(H_1), -m(H_2)\} .$$

Suppose we have the estimate

$$d(H_1,H_2) < \min\Big\{ g(H_1,[a,b)), g(H_2,[a,b)) \Big\} . \tag{31}$$

Then we can take a regular homotopy $(J(s,t,z), H(s,t,z))$ which satisfies (6) and is slow in the sense of the lemma of section 2, but instead of being monotone satisfies the weaker condition

$$d((H_s)) < \min\Big\{ g(H_1,[a,b)), g(H_2,[a,b)) \Big\} . \tag{32}$$

For $u \in \mathcal{M}(x_2, x_1; J, H)$ we calculate

$$\frac{d}{ds}\Phi_{H_s}(u(s)) = ||\Phi'_{H_s}(u(s))||^2_{L^2(s)} - \int_0^1 \frac{\partial H}{\partial s}(s,t,u(s,t))dt \ . \tag{33}$$

Here $L^2(s)$ is the L^2-section space along loops equipped with the L^2-inner product associated to the ω-calibrated t-dependent almost complex structure $J(s,*,*)$. Integrating (33) we infer via (32)

$$\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) > -\min\left\{g(H_1,[a,b)), g(H_2,[a,b))\right\} . \tag{34}$$

If $x_2 \in P_\mu(H_2), \mu = a, b$, then (34) and the definition of the "gaps" imply

$$\Phi_{H_1}(x_1) \geq \mu.$$

Now we can proceed as in section 2 to obtain unique chain homotopy classes

$$\sigma(J_2,H_2;J_1,H_1) : C_\mu(J_2,H_2) \longrightarrow C_\mu(J_1,H_1) \tag{35}$$

for $\mu = a, b$. Furthermore, if $(J_i, H_i), i = 1,2,3$, are admissible pairs satisfying

$$\begin{aligned} d(H_1,H_2) &< \frac{1}{2}\min\left\{g(H_1,[a,b)), g(H_2,[a,b))\right\} \\ d(H_2,H_3) &< \frac{1}{2}\min\left\{g(H_2,[a,b)), g(H_3,[a,b))\right\} \end{aligned} \tag{36}$$

then

$$\sigma(J_2,H_2;J_1,H_1) \circ \sigma(J_3,H_3;J_2,H_2) = \sigma(J_3,H_3;J_1,H_1) \ . \tag{37}$$

Taking $(J_3,H_3) = (J_1,H_1)$ we conclude that in this case $\sigma(J_2,H_2;J_1,H_1)$ induces an isomorphism in homology

$$S^{[a,b)}(J_1,H_1) \underset{sdi}{\overset{\simeq}{\longrightarrow}} S^{[a,b)}(J_2,H_2) \tag{38}$$

with inverse induced by $\sigma(J_1,H_1;J_2,H_2)$. Here "sdi" stands for the small distance isomorphism. This uniquely defined isomorphism always exists if the distance between H_1 and H_2 is small with respect to the "gaps", more precisely if the first equation of (36) holds.

Now we are in the position to prove the isotopy invariance. Assume U, V are open subsets of M such that $\bar{U}, \bar{V} \subset M \setminus \partial M$. Suppose (Ψ_τ) is a smooth isotopy of symplectic maps such that $\Psi_\tau(U) \subset V, \Psi_\tau : M \overset{\simeq}{\to} M$. Fix $b \geq a$ and choose an admissible pair (J,H) with $H|(\mathbb{S}^1 \times \bar{V}) < 0$ and define $H_\tau : \mathbb{S}^1 \times M \to \mathbb{R}$ by

$$H_\tau(t,x) = H(t,\Psi_\tau(x))$$

and

$$J_\tau(t,x) = T\Psi_\tau(x)^{-1}J(t,\Psi_\tau(x))T\Psi_\tau(x) \ .$$

Clearly (J_τ, H_τ) is admissible for all $\tau \in [0,1]$. Since $U \subset \Psi_\tau^{-1}(V)$ for every $\tau \in [0,1]$ we can find an admissible $(\widehat{J}, \widehat{K})$ such that

$$\begin{gathered} \widehat{K} \mid (\mathbb{S}^1 \times \bar{U}) < 0 \\ (\widehat{J}, \widehat{K}) \geq (J_\tau, H_\tau) \text{ for all } \tau \in [0,1] \,. \end{gathered} \tag{39}$$

Taking for fixed τ a slow monotone homotopy we obtain a unique chain homotopy class σ_τ

$$C_\mu(\widehat{J}, \widehat{K}) \longrightarrow C_\mu(J_\tau, H_\tau) \tag{40}$$

for every $\mu > -m(H)$. We observe that the distance $d(H_{\tau_0}, H_{\tau_1})$ for fixed τ_0 and τ_1 close to τ_0 is as small as we wish. Consequently, the map in (40) for $\tau = \tau_0$ and $\mu = a, b$ factors through a monotone homotopy to $C_\mu(J_{\tau_1}, H_{\tau_1})$ and the "smallness map" from $C_\mu(J_{\tau_1}, H_{\tau_1})$ to $C_\mu(J_{\tau_0}, H_{\tau_0})$ provided τ_1 is close enough to τ_0. Hence for τ_1 close to τ_0 we have the commutative diagram

$$\begin{array}{ccc} S^{[a,b)}(J_{\tau_0}, H_{\tau_0}) & \longrightarrow & S^{[a,b)}(\widehat{J}, \widehat{K}) \\ {\scriptstyle sdi}\downarrow & \nearrow & \\ S^{[a,b)}(J_{\tau_1}, H_{\tau_1}) & & \end{array} \tag{41}$$

Here the unmarked arrows are coming from slow montone homotopies. If we can show that the "sdi"-map in the notation of (25) is given by

$$(\Psi_{\tau_1}^{-1} \Psi_{\tau_0})_{\#\#} \;:\; S^{[a,b)}(J_{\tau_0}, H_{\tau_0}) \xrightarrow{\simeq} S^{[a,b)}(J_{\tau_1}, H_{\tau_1})$$

we have as an easy corollary the isotopy invariance. In fact, in this case we obtain the commutative diagramm

$$\begin{array}{ccc} S^{[a,b)}(J_0, H_0) & \longrightarrow & S^{[a,b)}(\widehat{J}, \widehat{K}) \\ {\scriptstyle (\Psi_1^{-1}\Psi_0)_{\#\#}}\downarrow & \nearrow & \\ S^{[a,b)}(J_1, H_1) & & \end{array} \tag{42}$$

Here we can pass to the direct limit to obtain

$$\begin{array}{ccc} S^{[a,b)}(M,\Psi_0^{-1}(V)) & \longrightarrow & S^{[a,b)}(M,U) \\ {\scriptstyle \widehat{\Psi}_1^{-1}\widehat{\Psi}_0}\Big\downarrow & \nearrow & \\ S^{[a,b)}(M,\Psi_1^{-1}(V)) & & \end{array} \qquad (43)$$

Now the composites $\sigma \circ \widehat{\Psi}_0^{-1}$ and $\sigma \circ \widehat{\Psi}_1^{-1}$, where σ stands for the appropriate natural map induced by inclusion, define Ψ_0^* and Ψ_1^* respectively. Hence we obtain the commutative diagram

$$\begin{array}{ccccc} & & S^{[a,b)}(M,\Psi_0^{-1}(V)) & & \\ & {\scriptstyle \widehat{\Psi}_0^{-1}}\nearrow & & \searrow & \\ S^{[a,b)}(M,V) & & \xrightarrow{\quad \Psi_0^* = \Psi_1^* \quad} & & S^{[a,b)}(M,U) \\ & {\scriptstyle \widehat{\Psi}_1^{-1}}\searrow & & \nearrow & \\ & & S^{[a,b)}(M,\Psi_1^{-1}(V)) & & \end{array} \qquad (44)$$

So it remains to prove the identity $sdi = (\Psi_{\tau_1}^{-1}\Psi_{\tau_0})_{\#\#}$ for τ_1 close to τ_0. In order to simplify notation we note that the above reduces to the following question:

Assume $\tau \longrightarrow \Psi_\tau$ is a smooth arc of symplectic maps with $\Psi_0 = Id, (J,H)$ is admissible and, $b \geq a > -m(H)$, is it then true that $(\Psi_\tau)_{\#\#} = sdi$ for τ close to zero? Let $\beta : \mathbb{R} \longrightarrow [0,1]$ be a smooth map satisfying $\beta(s) = 0$ for $s \leq 0, \beta'(s) > 0$ for $s \in (0,1)$ and $\beta(s) = 1$ for $s \geq 1$. Define

$$\begin{aligned} H^\varepsilon(s,t,x) &:= H(t,\Psi_{\varepsilon\beta(s)}(x)) \\ J^\varepsilon(s,t,x) &:= T\Psi_{\varepsilon\beta(s)}(x)^{-1} \circ J(t,\Psi_{\varepsilon\beta(s)}(x)) \circ T\Psi_{\varepsilon\beta(s)}(x) \end{aligned} \qquad (45)$$

for all $(s,t,x) \in \mathbb{R}\times\mathbb{S}^1\times M$ and $\varepsilon > 0$. Let $u \in \mathcal{M}(x,x^\varepsilon;J^\varepsilon,H^\varepsilon)$, i.e. u satisfies

$$u_s + J^\varepsilon(s,t,u)u_t + \nabla_{J^\varepsilon(s,t)}H^\varepsilon(s,t,u) = 0 \qquad (46)$$

and converges for $s \longrightarrow \pm\infty$ to contractible 1-periodic solutions x, x^ε of H, H^ε respectively. Define $v : Z \longrightarrow M$ by

$$u(s,t) = \Psi_{\varepsilon\beta(s)}^{-1}(v(s,t))$$

and insert this in (46) to obtain

$$\begin{aligned}
0 &= \left(\tfrac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}\right)(v(s,t)) + T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))v_s(s,t) \\
&\quad + J^{\varepsilon}(s,t,\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t)))T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))v_t(s,t) \\
&\quad + \nabla_{J^{\varepsilon}(s,t)}H^{\varepsilon}(s,t,\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))) \qquad (47) \\
&= T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))\Big[v_s(s,t) + J(t,v(s,t))v_t(s,t) \\
&\quad + \nabla_{J(t)}H(t,v(s,t)) + T\Psi_{\varepsilon\beta(s)}(v(s,t))\left(\tfrac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}\right)(v(s,t))\Big]
\end{aligned}$$

Now if $\mathrm{ind}_{CZ}(x) = \mathrm{ind}_{CZ}(x^{\varepsilon})$, the only solution of $v_s + J(t,v)v_t + \nabla_{J(t)}H(t,v) = 0$ connecting x and $\Psi_{\varepsilon}^{-1} \circ x^{\varepsilon}$ is $v(s,t) \equiv x(t)$.

The expression $T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))(\frac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))$ defines a small perturbation (depending on ε) compactly supported in s. By the compactness estimates in [7] and the inverse function theorem, for small ε (47) has a unique solution $\tilde{v}$ close to every $v(s,t) = x(t)$, where x is a contractible 1-periodic solution of $\dot{x} = X_{H_t}(x)$.

Consequently, (46) has for every 1-periodic solution x in $P_a(H)\backslash P_b(H)$ a unique solution u connecting x with $\Psi_{\varepsilon}^{-1}(x)$ provided ε is small enough. Hence the map from $C_{\mu}(J,H) \to C_{\mu}(J_{\Psi}, H_{\Psi})$ for $\mu = a, b$ induced by the homotopy $(J^{\varepsilon}, H^{\varepsilon})$ is precisely the map $\Psi_{\varepsilon}^{\#}$.

On the other hand for ε small , $(J^{\varepsilon}, H^{\varepsilon})$ is a regular slow homotopy satisfying the smallness condition (32).

Hence we have proved

Proposition 1. *Let (M,ω) be as previously described and U, V open subsets of M such that $\bar{U}, \bar{V} \subset M\backslash\partial M$. Given a smooth isotopy of symplectic maps $\Psi_s : (M,U) \hookrightarrow (M,V)$ we have that $(\Psi_s)^*$ is independent of s.*

5 Possible extensions

To define symplectic homology groups $S^{[a,b)}(J,H)$ we have introduced the space $\mathcal{J}$ of ω-calibrated almost complex structures on M behaving nicely at the boundary. One can also consider the bigger space $\tilde{\mathcal{J}}$ of ω-calibrated almost complex structures defined on the **interior of** M (not necessarily extendible to the boundary) and satisfying the analogous condition near ∂M (where η, λ etc. may also not be extendible to ∂M). For $J \in \tilde{\mathcal{J}}$ and H admissible, solutions in $\mathcal{M}(x_1, x_2; J, H)$ still stay away from the boundary by [18], Lemma 2.4. Hence $S^{[a,b)}(J,H)$ can be defined as before. It is however not clear if two elements of $\tilde{\mathcal{J}}$ can be joined by a path in $\tilde{\mathcal{J}}$. Therefore it cannot be excluded as before that $S^{[a,b)}(J,H)$ might depend on the choice of $J \in \tilde{\mathcal{J}}$. But if we fix a $J \in \tilde{\mathcal{J}}$, we can still take the direct limit over H to obtain symplectic homology groups $S^{[a,b)}(M,U,J)$ having the same properties as before. These groups will be used in [14] to prove the so-called "stability of the action spectrum".

Our construction was carried out under the standing assumption that $[\omega]$ and c_1 vanish on $\pi_2(M)$. The assumption on c_1 guarantees that a contractible periodic trajectory has a well defined Morse index: the Conley-Zehnder index. The assumption on $[\omega]$ implies that for a contractible loop x we have a well defined action $\Phi_H(x)$.

It makes of course also sense to look for noncontractible 1-periodic solutions in a given class $\alpha \in [\mathbb{S}^1, M]$. We fix in that case a reference loop $x_0 \in \alpha$. Given another loop $x \in \alpha$ we would like to define a number $\Phi_H(x)$, so that the derivative of Φ_H with respect to x vanishes if and only if x is a 1-periodic solution of the Hamiltonian system $\dot{x} = X_{H_t}(x)$ in the class of α. The only way is the following:

Take a smooth map $\bar{x} : [0,1] \times \mathbb{S}^1 \to M$ such that $\bar{x}(0,t) = x_0(t)$ and $\bar{x}(1,t) = x(t)$ for $t \in \mathbb{S}^1$ and define $\Phi_H(x) := \int \bar{x}^*\omega - \int_0^1 H(t, x(t))$. The difficulty here is that $\Phi_H(x)$ is not well defined. Clearly by Stokes's Theorem we obtain the same value if we replace $\bar{x}$ by any $\widehat{x}$ which is homotopic to $\bar{x}$ with boundary fixed. So the right objects to study are not the 1-periodic solutions in the class α, but the 1-periodic solutions together with their "history", i.e. a homotopy of a given reference curve x_0 to the loop x. Let $C^\infty_\alpha(\mathbb{S}^1, M)$ be the collection of all loops x of class α, and denote by $\widetilde{C}^\infty_{\alpha,x_0}(\mathbb{S}^1, M)$ the universal covering with reference curve x_0. Then finding 1-periodic solutions of class α turns out to be a variational problem on this space. Problems of this kind can be handled at the expense of some nontrivial algebraic machinery, see [16].

All these difficulties can be of course avoided if the symplectic form ω is exact, i.e. $\omega = d\lambda$ for some 1-form on M. Then we define

$$\Phi_H(x) = -\int_0^1 x^*\lambda - \int_0^1 H(t, x(t))dt$$

for every loop.

The previous discussion solves the problem of finding a real filtration provided the Morse complex can be handled analytically. (The bubbling off of holomorphic spheres has to be controlled.) The other problem is the fixing of a grading. For this one takes the reference curve x_0 and considers the pullback bundle $x_0^*TM \to \mathbb{S}^1$, which is a symplectic vectorbundle. Of interest are first order linear differential operators $L_0 : L^2(x_0^*TM) \supset H^{1,2}(x_0^*TM) \to L^2(x_0^*TM)$, which in a suitable symplectic trivialization $\bar{x}^*TM \simeq \mathbb{S}^1 \times \mathbb{C}^n$ look like

$$(\widetilde{L}_0 h)(t) = -i\dot{h}(t) - A(t)h(t) ,$$

where $t \to A(t)$ is a smooth map into the real linear maps of $\mathbb{C}^n$ such that $A(t)$ is symmetric for the real inner product $< *, * >= \mathrm{Re}(*, *)$ on $\mathbb{C}^n$. We assume $\mathrm{Kern}(\widetilde{L}_0) = \{0\}$ and call $\widetilde{L}_0$ and L_0 in this case nondegenerate. L_0 is what we call in [10] an asymptotic operator.

Now we consider x_0 and L_0 as some kind of normalizing data. Given any element of $\widetilde{C}^\infty_{\alpha,x_0}(\mathbb{S}^1, M)$, i.e. a homotopy class of maps $\bar{x} : [0,1] \times \mathbb{S}^1 \to M$ with

$\bar{x}(0,t) = x_0(t)$ and $\bar{x}(1,t) = x(t)$ we define a Morse index for x provided it is a nondegenerate 1-periodic solution of a Hamiltonian system $\dot{x} = X_H(x)$. Namely having fixed a calibrated perhaps t-dependent almost complex structure J we can write $\dot{x} = X_{H_t}(x)$ as $-J(t,x(t))\frac{d}{dt}x(t) - \nabla_{J_t}H(t,x(t)) = 0$. Linearizing the latter expression by varying x in the function space we obtain a second asymptotic operator $L = L(H,x)$. (It only depends on x, not on $\bar{x}$.)

Now one trivialises $\bar{x}^*TM \to [0,1]\times\mathbb{S}^1$. This determines trivializations for L_0 and $L(H,x)$ which depend up to homotopy only on the class of $\bar{x}$ in $\widetilde{C}_{\alpha,x_0}(\mathbb{S}^1, M)$. Now one has obtained asymptotic operators $\widetilde{L_0}$ and $\widetilde{L}(H,x)$. One takes now a Fredholm operator A of the type described in [10] having asymptotic operators $A^- = \widetilde{L_0}$ and $A^+ = \widetilde{L}(H,x)$ and defines the generalized Conley-Zehnder index of $[\bar{x}]$ with respect to the normalizing data (x_0, L_0) to be precisely the Fredholm index of A.

It follows from results in [10] that this definition is independent of the constructions involved. Let us write for it $\mathrm{ind}([\bar{x}],H) := \mathrm{ind}_{(x_0,L_0)}([\bar{x}],H)$. If we use this construction for contractible loops under the assumption that c_1 vanishes on $\pi_2(M)$ we see that this index map differs from the Conley-Zehnder index only by a fixed constant. Given an admissible pair (J,H), consider the manifold of connecting orbits between two nondegenerate 1-periodic solutions in the class α, i.e.

$$\mathcal{M}(x^-,x^+;J,H) = \{u : Z \to M \,|\, u_s + J(t,u)u_t + \nabla_{J_t}H(t,u) = 0,$$
$$u(s,*) \to x^{\pm} \text{ as } s \to \pm\infty\}.$$

It turns out as the consequence of the discussions in [10] that the local dimension $\dim_u \mathcal{M}$ close to u satisfies

$$\dim_u \mathcal{M} = \mathrm{ind}([b],H) - \mathrm{ind}([a],H),$$

where $a : [0,1] \times \mathbb{S}^1 \to M$ is any fixed homotopy between x_0 and x^-, and b is the homotopy obtained by first carrying out a followed by the homotopy described by u. Let us write $[b] = [a]\&[u]$. Then the above formula reads

$$\dim_u \mathcal{M} = \mathrm{ind}([a]\&[u],H) - \mathrm{ind}([a],H).$$

We note finally that for manifolds M with contact type boundary which are equidimensional submanifolds of a cotangent bundle we have $c_1|\pi_2(M) = 0$ and $\omega|M = d\lambda$. In that case $\dim_u \mathcal{M}$ is independent of the choice of u, and only depends on x^- and x^+. One can carry out our homology construction for any class α since in that case no holomorphic spheres exist. The integer grading is fixed by prescribing a pair (x_0, L_0) as described above. The filtration is given naturally through the choice of $\lambda = pdq$. Another case where the construction can be carried out are monotone manifolds in the sense of [9].

References

1. C. Conley and E. Zehnder. Morse type index theory for flows and periodic solutions of Hamiltonian equations. Comm. Pure Appl. Math. 37 (1984), 207-253.
2. I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics. Math. Zeit. 200 (1990), 355-378.
3. Ya. Eliashberg and H. Hofer. An energy-capacity inequality for the symplectic homology of a flat at infinity hypersurface. Symplectic geometry (ed. by D. Salamon), London Math. Soc. Lecture Note Series 192 (1993), 95-115
4. Ya. Eliashberg and H. Hofer. Towards the definition of a symplectic boundary. Geom. and Funct. Analysis 2 (1992), 211-220.
5. Ya. Eliashberg and H. Hofer. Unseen symplectic boundariès. To appear Proceedings of a Conference in honour of E. Calabi, Pisa 1993.
6. A. Floer. A relative Morse index for the symplectic action. Comm. Pure Appl. Math. 41 (1988), 393-407.
7. A. Floer. The unregularized gradient flow of the symplectic action. Comm. Pure Appl. Math. 41 (1988), 775-813.
8. A. Floer. Morse theory for Lagrangian intersections. J. Diff. Geom. 28 (1988), 513-547.
9. A. Floer. Symplectic fixed points and holomorphic spheres. Comm. Math. Physics 120 (1989), 575-611.
10. A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. Math. Zeit. 212 (1993), 13-38.
11. A. Floer and H. Hofer. Symplectic homology I. Math. Zeit. 215 (1994), 37-88.
12. A. Floer, H. Hofer and D. Salamon. Transversality results in the elliptic Morse theory for the action functional. Preprint.
13. A. Floer, H. Hofer and K. Wysocki. Applications of Symplectic homology I. To appear Math. Zeit.
14. K. Cieliebak, A. Floer, H. Hofer and K. Wysocki. Applications of Sympletic homology II. In preparation.
15. M. Gromov. Pseudo holomorphic curves in symplectic manifolds. Inv. math. 82 (1985), 307-347.
16. H. Hofer, D. Salamon. Floer Homology and Novikov rings. To appear Floer Memorial Volume, Birkhäuser
17. H. Hofer and E. Zehnder. A new capacity for symplectic manifolds. Analysis et cetera. Academic Press (1990), 405-427.
18. D. McDuff. Symplectic manifolds with contact type boundaries. Inv. math. 103 (1991), 651-671.
19. D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. Comm. Pure Appl. Math. 45 (1992), 1303-1360.
20. M. Schwarz. Morse homology. Birkhäuser, Progress in Math. Vol. 111 (1993).