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# Symplectic homology II: a general construction

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# Symplectic homology II

## A general construction

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### 1 Introduction

In [11] the second and third author introduced a symplectic homology theory. By means of a general construction, given real numbers $a < b$ and an integer $k$ they assigned to each open set $U$ of $\mathbb{C}^n$ a group $S_k^{[a,b)}(U)$ and studied its properties. In the present paper which continues the work in [11], we show how this construction can be carried over to more general manifolds. We assume the reader to be familiar with [11], since there are many constructions which we recall here in a more general set up without giving a detailed proof. In fact, the arguments given in [11] work under more general circumstances at least if some topological assumptions are met. Only in the case that there is a considerable difference we give complete details.

For applications we refer the reader to [13] and the forthcoming paper [14]. For motivation of the present construction we refer the reader to [11]. However we recall that the crucial observation made [2] is that periodic orbits for Hamiltonian systems can be used to construct many new symplectic invariants. Our construction in [11] and the present paper precisely exploits the same aspects of this observation. We also would like to point the attention to [3, 4, 5] for other applications of this "philosophy".

* Andreas Floer died on May 15th, 1991.

## 2 A general construction

In the following we describe the construction under suitable hypotheses on the symplectic manifold $(M,\omega)$. Other assumptions allow similar constructions and we outline some generalizations later on.

We assume $(M,\omega)$ is a compact symplectic manifold with or without boundary, such that $[\omega]$ vanishes on $\pi_2(M)$. Moreover we assume that the first Chern class $c_1$ for pullback bundles $u^*TM \to \mathbb{S}^2$ and $[u] \in \pi_2(M)$ vanishes, where $TM \to M$ carries the structure of a complex vectorbundle induced by a $\omega$-calibrated almost complex structure $J$. ("$\omega$-calibrated" means that $\omega \circ (J \times id)$ is a Riemannian metric. The space of such structures is contractible.)

If $\partial M \neq \emptyset$, we assume that $\partial M$ is of contact type. This means that there exists an outward pointing transversal vector field $\eta$ defined on an open neighborhood of $\partial M$ in $M$ such that the Lie derivative satisfies $L_\eta \omega = \omega$. Equivalently ( set $\lambda = i_\eta \omega$), there exists a 1-form $\lambda$ on a neighborhood of $\partial M$ such that $d\lambda = \omega$ and $\lambda \wedge (d\lambda)^{n-1}|_{\partial M}$ is a volume form determining the orientation on $\partial M$ induced from the orientation $\omega^n$ on $M$.

We call a smooth Hamiltonian $H : S^1 \times M \to \mathbb{R}$ **admissible** if

$$
\begin{cases}
\bullet \text{ There exists a nonempty open subset } U \text{ of } M \text{ with} \\
\qquad \mathrm{cl}(U) \subset M \setminus \partial M \quad \text{and} \quad H|(\mathbb{S}^1 \times \mathrm{cl}(U)) < 0. \\
\bullet \text{ If } \partial M = \emptyset, \text{ all 1-periodic solutions are nondegenerate .} \\
\bullet \text{ If } \partial M \neq \emptyset, \text{ there exists a constant } m = m(H) > 0, \\
\text{such that } H(t,x) = m \text{ for all } (t,x) \in \mathbb{S}^1 \times W, \text{ where } W \\
\text{is a neighborhood of } \partial M. \text{ Moreover, every 1-periodic} \\
\text{solution } x : \mathbb{S}^1 \to M \text{ satisfying } \int_0^1 H(t,x(t))dt < m \text{ is} \\
\text{nondegenerate. Further, } H(t,x) \leq m \text{ for all } (t,x) \in \\
\mathbb{S}^1 \times M \text{ and the set of 1-periodic solutions } x \text{ with} \\
\int_0^1 H(t,x(t))dt < m \text{ is finite.}
\end{cases}
\tag{1}
$$

We denote by $C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ the set of all smooth contractible loops in $M$. Given $x \in C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ we denote by $\bar{x} : D \to M$ a smooth extension of $x$ to the disk. We define the action of $x$ denoted by $A(x)$ via

$$
A(x) = -\int_D \bar{x}^* \omega. \tag{2}
$$

This is well defined in view of the assumption $[w]|\pi_2(M) = 0$. Given a smooth arc in $C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$ say $(\tau, t) \to x_\tau(t)$ is smooth, the map $\tau \to A(x_\tau)$ is of class $C^\infty$. Let $x_0 = x$, and $\xi(t) = \frac{d}{d\tau} x_\tau(t)|_{\tau=0}$. We compute

$$\begin{aligned}\tfrac{d}{d\tau}|_{\tau=0}A(x_\tau) &= -\int_0^1 \omega(\xi(t),\dot{x}(t))dt \\ &= \int_0^1 \omega(\dot{x}(t),\xi(t))dt.\end{aligned}$$

We define $\Phi_H$ by

$$\Phi_H(x) = A(x) - \int_0^1 H(t,x(t))dt$$

and observe that

$$d\Phi_H(x)\xi = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)),\xi(t))dt$$

for all smooth vector fields $\xi$ along the contractible curve $x \in C^\infty_{\mathrm{cntr}}(\mathbb{S}^1, M)$. Here $X_{H_t}$ is the vector field defined by $i_{X_{H_t}}\omega = dH_t$. Hence

$$d\Phi_H(x) = 0 \iff \dot{x}(t) = X_{H_t}(x(t)), \quad t \in \mathbb{S}^1$$

If $H$ is admissible and $x$ is a contractible 1-periodic solution of the Hamiltonian system associated to $H$ satisfying $\Phi_H(x) > -m(H)$, we infer that $\int_0^1 H(t,x(t)) < m(H)$. In fact, if we have equality $x$ has to be constant and $\Phi_H(x) = -m(H)$. Hence $x$ has to be nondegenerate.

We denote for $a \in \mathbb{R}, a > -m(H)$ by $\mathcal{P}_a(H)$ the finite set consisting of all contractible 1-periodic solutions satisfying $\Phi_H(x) \geq a$. We call $\Phi_H(x)$ the energy of $x$. The energy will be one of the important numerical values attached to a contractible 1-periodic solution. Another invariant is the Conley-Zehnder index, [1]. The version we need is described [19]. Let $x$ be as above and linearize the Hamiltonian vector field along $x$. The linearized system defines a symplectic loop $X(t)$ satisfying

$$X(t) : T_{x(0)}M \longrightarrow T_{x(t)}M$$

is symplectic and

$$X(0) = Id, \quad X(1) : T_{x(0)} \longrightarrow T_{x(0)}$$

since $x(0) = x(1)$. We infer that $1 \notin \mathrm{spec}(X(1))$ since $x$ is nondegenerate. Take an extension $\bar{x} : D \to M$ of $x$ and symplectically trivialize $\bar{x}^*TM \to D$. In view of the condition on $c_1$ the trivializations induced for $x^*TM \to \mathbb{S}^1$ are homotopic and independent of the choice of the extension $\bar{x}$. Hence take such a trivialization

$$\Psi : x^*TM \xrightarrow{\simeq} \mathbb{S}^1 \times \mathbb{C}^n.$$

Let us write $\Psi(t)$ for the induced map

$$\Psi(t) : T_{x(t)}M \longrightarrow \mathbb{C}^n.$$

We consider the arc $\Gamma$ given by $t \to \Psi(t)X(t)\Psi(0)^{-1}$. Then $\Gamma(0) = \mathrm{Id}$ and $\Gamma(1) = \Psi(0)X(1)\Psi(0)^{-1}$ so that $\mathrm{spec}(\Gamma(1)) = \mathrm{spec}\ (X(1))$. Hence $1 \notin$

spec($\Gamma(1)$). According to [19], $\Gamma$ has a Conley-Zehnder index $\mathrm{ind}_{CZ} \in \mathbb{Z}$, which is independent of the choice of $\Psi$ (if $\Psi$ is as described above). Hence $\mathrm{ind}_{CZ}(x) := \mathrm{ind}_{CZ}(\Gamma) \in \mathbb{Z}$ is a well defined invariant of a contractible nondegenerate 1-periodic solution.

Summing up we have a map $\mathcal{P}_a(H) \to \mathbb{R} \times \mathbb{Z}$ associating to $x \in \mathcal{P}_a(H)$ the "local invariants" $(\Phi_H(x), \mathrm{ind}_{CZ}(x))$, where $a > -m(H)$ and $H$ is admissible. Then for every $x \in \mathcal{P}_a(H)$ we have $x(\mathbb{S}^1) \subset M \setminus \partial M$.

A smooth time-dependent almost complex structure $J : S^1 \times TM \to TM$ is called $\omega$-calibrated if $\omega \circ (J_t \times id)$ is a time-dependent Riemannian metric on $M$, where $J_t = J(t, \cdot)$. If $\partial M \neq \emptyset$ we further assume the following: There exists an outward pointing transversal vector field $\eta$ near $\partial M$ with $L_\eta \omega = \omega$. The flow $(\psi_t)_{-\varepsilon < t \leq 0}$ of $\eta$ yields a diffeomorphism $\psi : (-\varepsilon, 0] \times \partial M \to W$ onto some neighbourhood $W$ of $\partial M$ in $M$. Let $W_s := \psi_s(\partial M)$. The restriction of $\lambda = i_\eta \omega$ to $W_s$ is a contact form with contact bundle $\xi = ker(\lambda \mid_{W_s})$ and Reeb vector field $X \in TW_s$ defined by $i_X d\lambda \mid_{W_s} = 0$, $i_X \lambda = 1$. We assume that on $W, J$ is time-independent, leaves $\xi$ invariant and maps $X$ onto $\eta$. Denote by $\mathcal{J}$ the set of all such $J$.

We claim that $\mathcal{J}$ is nonempty, and any two elements of $\mathcal{J}$ can be joined by a smooth path $(J_\tau)_{0 \leq \tau \leq 1}$ in $\mathcal{J}$.

To see this observe that the existence of a $W, \eta, \lambda, \xi, X$ as described above follows directly from our assumption that $\partial M$ is of contact type. Now it is well-known that for every symplectic vector bundle $(E, \omega)$ the space of almost complex structures on $E$ for which $\omega \circ (J \times id)$ is a bundle metric is nonempty and contractible. So we find an almost complex structure on $\xi$ such that $\omega \mid_\xi \circ (J \times id)$ is a bundle metric on $\xi$. Extending it first to the bundle $TM \mid_W$ via $J(X) = \eta$, $J(\eta) = -X$, and then to the whole bundle $TM \to M$, we obtain a $J \in \mathcal{J}$. For the second part let $J^0, J^1 \in \mathcal{J}$ be given with associated vectorfields $\eta_0, \eta_1$ near $\partial M$. Since both $\eta_0$ and $\eta_1$ are transversal to $\partial M$ and outward pointing, the same is true for all $\eta_\tau := \tau \eta_1 + (1 - \tau)\eta_0$, $0 \leq \tau \leq 1$. Moreover, $L_{\eta_\tau} \omega = \omega$ for all $\tau$. Arguing as above we can associate to every $\eta_\tau$ a $\tilde{J}^\tau \in \mathcal{J}$, depending smoothly on $\tau$. We can also achieve that $\tilde{J}^0 = J^0$, $\tilde{J}^1 = J^1$, and the claim is proved.

Now consider pairs $(J, H)$, where $H$ is an admissible Hamiltonian and $J \in \mathcal{J}$.

Given $x, y \in \mathcal{P}_a(H), a > -m(H)$, we consider the set $\mathcal{M}(x, y; J, H)$ defined by

$$\mathcal{M}(x, y; J, H) = \{u : Z \to M \mid u \text{ is smooth and satisfies (3) below}\},$$

where $Z = \mathbb{R} \times \mathbb{S}^1, (s, t) \in Z$, and

$$\begin{aligned} &u_s + J(t, u)u_t + \nabla_{J_t} H(t, u) = 0 \\ &u(s, *) \longrightarrow x \text{ in } C^\infty \text{ for } s \longrightarrow -\infty \\ &u(s, *) \longrightarrow y \text{ in } C^\infty \text{ for } s \longrightarrow +\infty. \end{aligned} \tag{3}$$

Here $\nabla_{J_t}$ is the gradient taken with respect to the Riemannian metric $\omega \circ (J_t \times id)$. If follows from [18], Lemma 2.4, that if $W$ is a foliated neighborhood of $\partial M$ as described above and $H \equiv m(H)$ on $W$, then $u(Z) \subset M \setminus W$.

By $C^\infty$-modifying $J$ (see [12]) we can achieve that the new pair $(\widetilde{J},H)$ still satisfies all the hypotheses above, and for every pair $(x,y)$, $\mathcal{M}(x,y;\widetilde{J},H)$ is a finite dimensional manifold. This is due to the fact that $\mathcal{M}(x,y;\widetilde{J},H)$ can be considered as the zero set of a regular Fredholm section. We call such a pair $(\widetilde{J},H)$ **admissible**.

Let us denote for simplicity by $Ad(M)$ the collection of all admissible pairs $(J,H)$. Following [11], we define for $(J,H) \in Ad(M)$ and $a > -m(H)$ the graded free Abelian group $C_a(J,H)$:

$$\begin{aligned} C_a(J,H) &= \oplus_k C_a^k(J,H) \\ C_a^k(J,H) &= \oplus_{x\in\mathcal{P}_a(H)^k}\mathbb{Z}x \\ \mathcal{P}_a(H)^k &= \{x\in\mathcal{P}_a(H)\mid \operatorname{ind}_{CZ}(x)=k\}. \end{aligned} \tag{4}$$

We define $\delta_k : C_a^k(J,H) \to C_a^{k+1}(J,H)$ by

$$\delta_k(x) = \sum_{y\in\mathcal{P}_a(H)^{k+1}} \left( \sum_{\hat{u}\in\widehat{\mathcal{M}}(x,y;J,H)} \tau(\hat{u}) \right) y \ ,$$

where $\widehat{\mathcal{M}}(x,y;J,H) = \mathcal{M}(x,y;J,H)/\mathbb{R}$ and $\tau(\hat{u}) \in \{1,-1\}$ are suitable "orientations" attached to the points in $\widehat{\mathcal{M}}$, see [10]. We recall that $\dim \mathcal{M}(x,y;J,H)$ is given by the difference $\operatorname{ind}_{CZ}(y) - \operatorname{ind}_{CZ}(x)$, see [19]. The crucial point is that $\delta_{k+1}\circ\delta_k = 0$. Hence $(C_a(J,H),\delta)$ is a co-chain complex. Moreover, for $b \geq a > -m(H)$ we have the commutative diagram.

$$\begin{array}{ccc} C_a(J,H) & \xrightarrow{\delta} & C_a(J,H) \\ \uparrow & & \uparrow \\ C_b(J,H) & \xrightarrow{\delta} & C_b(J,H) \end{array}$$

where the vertical arrows are given by inclusion.

Next we have to investigate the dependence of the co-chain complex on $J$ and $H$. To do this we define a partial ordering on $Ad(M)$ by

$$(J_1,H_1) \leq (J_2,H_2) :\Longleftrightarrow H_1(t,z) \leq H_2(t,z) \text{ for all } (t,z). \tag{5}$$

We would like to associate, in analogy to [11], to the above situation a natural chain homotopy class

$$\sigma(J_2,H_2;J_1,H_1) : C_a(J_2,H_2) \to C_a(J_1,H_1).$$

To this purpose we take a **monotone homotopy**, i.e. a smooth pair $(J,H) = (J(s,t,z),H(s,t,z))$, $(s,t,z)\in\mathbb{R}\times S^1\times M$ such that

$$\begin{aligned} (J(s,\cdot),H(s,\cdot)) &= (J_1(\cdot),H_1(\cdot)) \qquad \text{for } s \geq s_0 \\ (J(s,\cdot),H(s,\cdot)) &= (J_2(\cdot),H_2(\cdot)) \qquad \text{for } s \leq -s_0 \end{aligned} \tag{6}$$

$$\begin{aligned} J(s,\cdot) &\in \mathcal{J} \qquad \text{for all } s \in \mathbb{R} \\ H(s,t,z) &\equiv H(s) \qquad \text{for } z \text{ near } \partial M \end{aligned}$$

and

$$\partial_s H(s,t,z) \leq 0 \qquad \text{for all } (s,t,z) \tag{7}$$

The existence of such a homotopy follows from the discussion after the definition of $\mathcal{J}$.

For $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$, $a > -m(H_1)$ define $\mathcal{M}(x_2,x_1;J,H)$ as before. In order to get compactness results as in [11] we must ensure that the elements of $\mathcal{M}(x_2,x_1;J,H)$ are bounded away uniformly from $\partial M$. We can no longer apply Lemma 2.4 from [18] since $J$ may depend on $s$ near $\partial M$.

**Lemma.** *Given a monotone homotopy $(J,H)$ and $a > -m(H_1)$, there exist constants $c_0, \delta > 0$ depending on $J,H$ and $a$ such that for all $c \geq c_0$, $(J^c(s,\cdot),H^c(s,\cdot)) = (J(\frac{s}{c},\cdot),H(\frac{s}{c},\cdot))$, $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2,x_1;J^c,H^c)$ we have (with respect to some fixed metric on $M$):*

$$dist(u(Z),\partial M) \geq \delta$$

*We call such a $(J^c,H^c)$ a* **slow monotone homotopy**.

**Proof.** Choose an $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ we have diffeomorphisms $\psi(s) : (-2\varepsilon,0] \times \partial M \xrightarrow{\cong} W(s)$ induced by the vector field $\eta(s,\cdot)$ near $\partial M$. We may assume that $H(s,\cdot) \equiv H(s)$ on $\bigcup_{s\in\mathbb{R}} W(s)$. Take a smooth function $f : \mathbb{R} \times M \to [-\frac{3}{2}\varepsilon,0]$ satisfying $f \equiv -\frac{3}{2}\varepsilon$ on $M \setminus \bigcup W(s)$, $f(s,z) = \tau$ for $x \in \psi(s)(\{\tau\} \times \partial M)$, $\tau \in [-\varepsilon,0]$, and $|df(s,\cdot)| \leq 1$ where $|\cdot|$ is taken with respect to the metric $\omega \circ (J(s,\cdot) \times \mathrm{id})$. Pick an open neighbourhood $W$ of $\partial M$ contained in $\bigcap_{s\in\mathbb{R}} \psi(s)([-\varepsilon,0] \times \partial M)$ and having smooth boundary. Since $\bar{W}$ is compact, there is a $\rho > 0$ such that every closed curve in $W$ with length $< \rho$ is contractible in $W$. Let $a > -m(H_1)$ be given and define

$$\begin{aligned} b &:= \sup_{\mathbb{R}\times(M\setminus W)} f \qquad < 0 \\ \Delta &:= \max\{\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \mid x_1 \in P_a(H_1), x_2 \in P_a(H_2)\} \end{aligned}$$

Take an $l > 0$ large enough such that

$$\sqrt{\frac{\Delta}{l}} < \min\{\rho, -\frac{b}{2}, a + m(H_1)\}$$

and let

$$\mu := -\frac{2b}{l^2} > 0$$

By passing from $(J,H)$ to $(J^c,H^c)$ for some $c$ large enough we may assume that on $W$

$$|\partial_s^2 f| + \frac{1}{2}|\partial_s \lambda|^2 + \frac{1}{2}|d(\partial_s f)|^2 < \mu$$

where $\lambda = i_\eta \omega$ and norms are taken with respect to the metric $\omega \circ (J_s \times \mathrm{id})$.

Now let $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2, x_1; J, H)$ be given and consider the function $g(s,t) := f(s, u(s,t))$ on the cylinder $Z = \mathbb{R} \times S^1$. We will show that $g \leq \frac{b}{4} < 0$, from which the lemma follows.

From $\partial_s H \leq 0$ it follows that

$$\begin{aligned} \int_Z |u_s|^2 ds\, dt &\leq \Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \\ &\leq \triangle \end{aligned}$$

So there exist $s_k$, $k \in \mathbb{Z}$, with $|s_{k+1} - s_k| \leq l$ and $\int_0^1 |u_s(s_k, t)|^2 dt \leq \frac{\triangle}{l}$. For $k \in \mathbb{Z}$ let $x_k(t) := u(s_k, t)$.

**Case 1:** $x_k(t) \in W$ for all $t \in S^1$. Using that on $W$ $u$ satisfies the equation $u_s + J(s,u)u_t = 0$ we calculate

$$\begin{aligned} \mathrm{length}(x_k) &\leq \left(\int_0^1 |\dot{x}_k|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 |u_s(s_k, t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{\triangle}{l}} \\ &< \rho \end{aligned}$$

By definition of $\rho$ this implies that $x_k$ is contractible in $W$. Let $\overline{x_k} : D \to W$ be a smooth extension of $x_k$ to the closed unit disk. Using $\omega = d\lambda(s_k, \cdot)$ on $W$ we get

$$\begin{aligned} |A(x_k)| &\equiv \left|\int_D \overline{x_k}^* w\right| \\ &= \left|\int_{S^1} x_k{}^* \lambda(s_k, \cdot)\right| \\ &= \left|\int_0^1 \omega(\eta(s_k, x_k), \dot{x}_k) dt\right| \\ &\leq \int_0^1 |\dot{x}_k| dt \\ &\leq \sqrt{\frac{\triangle}{l}} \end{aligned}$$

Hence

$$\begin{aligned} \Phi_{H(s_k,\cdot)}(x_k) &\equiv A(x_k) - \int_0^1 H(s_k, x_k) dt \\ &\leq \sqrt{\frac{\triangle}{l}} - m(H_1) \\ &< a \end{aligned}$$

in contradiction to $a \leq \Phi_{H_2}(x_2) \leq \Phi_{H(s_k,\cdot)}(x_k)$. So case 1 does not occur, and for all $k \in \mathbb{Z}$ we are in

**Case 2:** There is a $t_k \in S^1$ such that $x_k(t_k) \notin W$. Then for any $t \in S^1$ we find a $t_0 \in S^1$, $t - 1 < t_0 \leq t$, such that $x_k(t_0) \notin W$ and $x_k(t') \in W$ for all $t_0 < t' \leq t$. This yields

$$\begin{aligned} g(s_k,t) &\leq g(s_k,t_0) + \int_{t_0}^{t} |\partial_t g(s_k,t')| dt' \\ &\leq b + \int_{t_0}^{t} |df(s_k,x_k)\cdot \dot{x}_k| dt' \\ &\leq b + \int_{t_0}^{t} |\dot{x}_k| dt' \\ &\leq b + \sqrt{\frac{\triangle}{l}} \\ &\leq \frac{b}{2} \end{aligned}$$

where the last but one inequality follows as in case 1.

So for the finite cylinders $Z_k := [s_k, s_{k+1}] \times S^1$ we have shown that $g \leq \frac{b}{2} < 0$ on $\partial Z_k$. It remains to estimate $g$ in the interior of $Z_k$. If $(s,t) \in Z_k$ and $g(s,t) > b$ then $u(s,t) \in W$, and we can compute

$$\begin{aligned} \partial_s(\lambda(s,u)\cdot u_t) - \partial_t(\lambda(s,u)\cdot u_s) &= \partial_s\lambda(s,u)\cdot u_t + u^* d\lambda(\partial_s,\partial_t) \\ &= \partial_s\lambda(s,u)\cdot u_t + \omega(u_s,u_t) \\ &= \partial_s\lambda(s,u)\cdot u_t - \frac{1}{2}(|u_s|^2 + |u_t|^2) \end{aligned}$$

Using $df \circ J = \lambda$ on $W$ we obtain

$$\begin{aligned} g_s(s,t) &= \partial_s f(s,u) - \lambda(s,u)\cdot u_t \\ g_t(s,t) &= \lambda(s,u)\cdot u_s \end{aligned}$$

and hence

$$\begin{aligned} \triangle g &= g_{ss} + g_{tt} \\ &= \partial_s^2 f + d(\partial_s f)\cdot u_s - \partial_s\lambda\cdot u_t + \frac{1}{2}(|u_s|^2 + |u_t|^2) \\ &\geq -|\partial_s^2 f| - \frac{1}{2}|d(\partial_s f)|^2 - \frac{1}{2}|\partial_s\lambda|^2 \\ &> -\mu \end{aligned}$$

for all $(s,t) \in Z_k$ with $g(s,t) > b$.

Define

$$\hat{g}(s,t) := g(s,t) + \frac{\mu}{2}\left(s - \frac{s_k + s_{k+1}}{2}\right)^2$$

For $(s,t) \in Z_k$ with $\hat{g}(s,t) > \frac{3}{4}b$ we have $g(s,t) > b$ (remember $\mu = -\frac{2b}{l^2}$ and $|s_{k+1} - s_k| \leq l$ ) and therefore

$$\triangle \hat{g} = \triangle g + \mu > 0$$

So $\hat{g}$ cannot have an interior maximum $> \frac{3}{4}b$ in $Z_k$. On the other hand, $g \leq \frac{b}{2}$ on $\partial Z_k$ implies $\hat{g} \leq \frac{b}{4}$ on $\partial Z_k$, thus $g(s,t) \leq \hat{g}(s,t) \leq \frac{b}{4}$ for all $(s,t) \in Z_k$. Since $k \in \mathbb{Z}$ was arbitrary, the lemma is proved. ∎

This lemma ensures that for slow monotone homotopies $(J,H)$ the solution spaces $\mathcal{M}(x_2,x_1;J,H)$ have the usual compactness properties. Now we can proceed as in [11]: For admissible pairs $(J_1,H_1) \leq (J_2,H_2)$, regular slow monotone homotopies between $(J_1,H_1)$ and $(J_2,H_2)$ induce a unique chain homotopy class

$$\sigma(J_2,H_2;J_1,H_1) : C_a(J_2,H_2) \longrightarrow C_a(J_1,H_1)$$

such that the following properties hold:

$$\sigma(J,H;J,H) = id \tag{8}$$

For $(J_1,H_1) \leq (J_2,H_2) \leq (J_3,H_3)$ we have

$$\sigma(J_2,H_2,J_1,H_1) \circ \sigma(J_3,H_3,J_2,H_2) = \sigma(J_3,H_3,J_1,H_1) \tag{9}$$

For $b \geq a > -m(H_1)$ the following diagram commutes:

$$\begin{array}{ccc} C_a(J_2,H_2) & \xrightarrow{\sigma} & C_a(J_1,H_1) \\ \uparrow & & \uparrow \\ C_b(J_2,H_2) & \xrightarrow{\sigma} & C_b(J_1,H_1)\,. \end{array} \tag{10}$$

Note in (9) that if two slow monotone homotopies between $(J_1,H_1)$ and $(J_2,H_2)$ respectively $(J_2,H_2)$ and $(J_3,H_3)$ are sufficiently slow then their composition is a slow monotone homotopy between $(J_1,H_1)$ and $(J_3,H_3)$. Then use uniqueness of $\sigma$.

Now let $R$ be any commutative ring. We apply the functor $\mathrm{Hom}(*,R)$ to the quotient co-chain complexes $C_a/C_b$, where $b \geq a > -m(H)$ to obtain the chain complexes

$$D^{[a,b)}(J,H) := \mathrm{Hom}(C_a(J,H)/C_b(J,H);R) \tag{11}$$

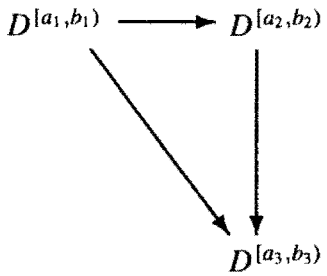
with boundary operator $\partial$ induced by $\delta$. For numbers $b \geq a > -m(H)$ and $b' \geq a' > -m(H)$, such that $b \geq b'$ and $a \geq a'$ we have obviously induced homotopy classes of maps

$$C_a/C_b \longrightarrow C_{a'}/C_{b'}$$

and consequently

$$D^{[a',b')}(J,H) \longrightarrow D^{[a,b)}(J,H)\,. \tag{12}$$

These maps behave functorially, i.e. for $b_i \geq a_i > -m(H), i = 1,2,3$, and $b_3 \geq b_2 \geq b_1$, $a_3 \geq a_2 \geq a_1$ we have




(13)

If $(J,H) \leq (\widetilde{J},\widetilde{H})$ then the following diagram is also commutative

$$\begin{array}{ccc} D^{[a,b)}(J,H) & \xrightarrow{\sigma} & D^{[a,b)}(\widetilde{J},\widetilde{H}) \\ \uparrow & & \uparrow \\ D^{[a',b')}(J,H) & \xrightarrow{\sigma} & D^{[a',b')}(\widetilde{J},\widetilde{H}) \end{array} \tag{14}$$

where $\sigma$ is the dual of $\sigma(\widetilde{J},\widetilde{H};J,H)$ and the vertical maps are those in (12). Here $a,b,a',b'$ are as described above.

Now we pass to homology obtaining the symplectic homology groups $S^{[a,b)}(J,H)$ of the admissible pair $(J,H) \in Ad(M)$.

These homology groups are independent of $J$ (by the discussion above), but they depend on $H$. In order to obtain invariants of the manifold, we have to pass to a direct limit over the Hamiltonians. This is done as follows:

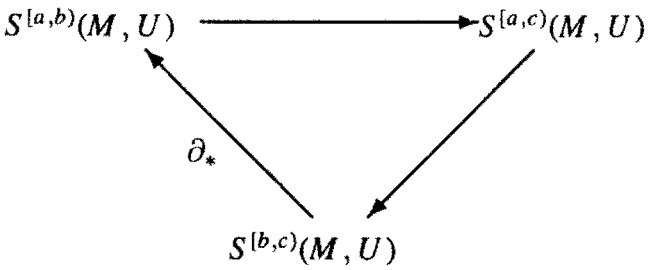
Let $U \subset M$ be open (possibly empty) with $\mathrm{cl}(U) \subset M \backslash \partial M$. Let $Ad(M,U)$ be the subset of $Ad(M)$ consisting of those pairs $(J,H)$ for which $H|_{\mathbb{S}^1 \times \mathrm{cl}(U)} < 0$. The density results on admissible pairs (see [12], [19]) imply that $(Ad(M,U), \leq)$ is a directed set. Thus we may take the direct limit over $Ad(M,U)$ to obtain the symplectic homology groups

$$S^{[a,b)}(M,U) = \varinjlim S^{[a,b)}(J,H)\,. \tag{15}$$

Note that $S^{[a,b)}(J,H)$ is well defined if $b \geq a > -m(H)$; omit in the direct limit all those $(J,H)$ for which this not true. /parbox Given numbers $c \geq b \geq a$ the exact sequence of chain complexes

$$0 \longrightarrow D^{[a,b)}(J,H) \longrightarrow D^{[a,c)}(J,H) \longrightarrow D^{[b,c)}(J,H) \longrightarrow 0 \tag{16}$$

for $(J,H) \in Ad(M,U), a > -m(H)$, gives, since direct limits preserve exactness, the long exact homology sequence




(17)

where the connecting homomorphism $\partial_*$ has degree $-1$. We shall denote the diagram (17) by $\triangle_{a,b,c}(M, U)$ and call it the exact triangle. If $a \leq b \leq c$ and $a' \leq b' \leq c'$ with $a' \leq a, b' \leq b$ and $c' \leq c$, the maps from (12) induce a morphism between triangles

$$\triangle_{a',b',c'}(M, U) \longrightarrow \triangle_{a,b,c}(M, U). \tag{18}$$

which is functorial.

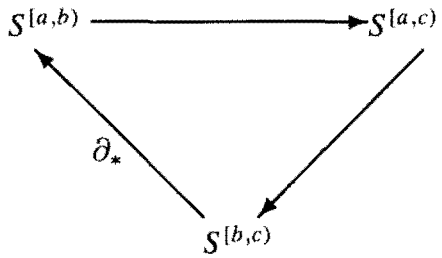
This construction has many nice properties, which we shall study in the next section.

## 3 Elementary properties of symplectic homology

Let us first recall the properties already outlined in the previous section. Given $b \geq a$ and $b' \geq a'$, such that $a \geq a'$ and $b \geq b'$ we have natural transformations.

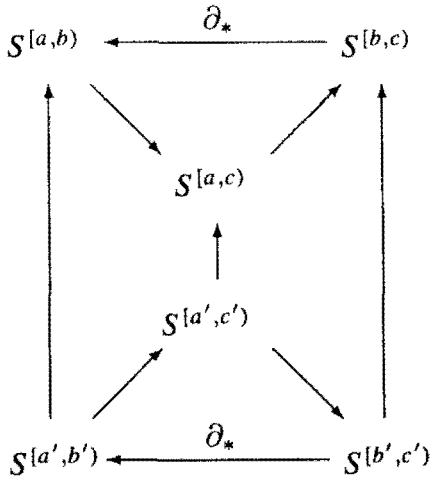
$$S^{[a',b')} \longrightarrow S^{[a,b)}. \tag{19}$$

There is another natural transformation $\partial_*$ of degree $-1$, which together with (19) yields for $a \leq b \leq c$ the exact triangle $\triangle_{a,b,c}$.




(20)

The natural transformations in (19) induce morphisms between exact triangles. In order to be more precise, assume $a \leq b \leq c, a' \leq b' \leq c'$ and $a' \leq a, b' \leq b, c' \leq c$. Then the following diagram is commutative




(21)

In short we have $\triangle_{a',b',c'} \longrightarrow \triangle_{a,b,c}$ which is functorial in the numbers $a, b$, e.t.c.

Next assume $U \subset V$ are open with $\mathrm{cl}(V) \subset M \setminus \partial M$. Then $Ad(M, V) \subset Ad(M, U)$, hence we obtain a natural map

$$S^{[a,b)}(M, V) \longrightarrow S^{[a,b)}(M, U) \tag{22}$$

inducing a morphism between exact triangles and being compatible with all other natural transformations.

Next assume $\Psi : M \xrightarrow{\simeq} M$ is a symplectic diffeomorphism mapping $U$ into $V$, where $U$ and $V$ are open subsets of $M$ with their closures contained in $M \setminus \partial M$. We shall write

$$\Psi : (M, U) \xrightarrow{s} (M, V). \tag{23}$$

If $(J, H) \in Ad(M, V)$ then $H_\Psi$ defined by $H_\Psi(t, x) = H(t, \Psi(x))$, satisfies $H_\Psi|_{(\mathbb{S}^1 \times \bar{U})} < 0$. We define further $J_\Psi$ by

$$J_\Psi(t, x)h = T\Psi(x)^{-1} J(t, \Psi(x)) T\Psi(x)h \ .$$

Then $(J_\Psi, H_\Psi) \in Ad(M, U)$.

If $u \in \mathcal{M}(x, y; J, H)$ then $\Psi^{-1}(u)$ is in $\mathcal{M}(\Psi^{-1}(x), \Psi^{-1}(y); J_\Psi, H_\Psi)$. So, the new connecting orbit spaces are obtained by applying $\Psi^{-1}$. Consequently we obtain a co-chain map $\Psi^{\#} : C_a(J, H) \to C_a(J_\Psi, H_\Psi)$ by $x \to \Psi^{-1}(x)$ inducing

$$\Psi_{\#} : D^{[a,b)}(J_\Psi, H_\Psi) \longrightarrow D^{[a,b)}(J, H) \tag{24}$$

and consequently the isomorphism

$$\Psi_{\#\#} : S^{[a,b)}(J_\Psi, H_\Psi) \longrightarrow S^{[a,b)}(J, H) \ . \tag{25}$$

Passing to the direct limit induces an isomorphism

$$\widehat{\Psi} : S^{[a,b)}(M, \Psi^{-1}(V)) \xrightarrow{\simeq} S^{[a,b)}(M, V). \tag{26}$$

Combining $\widehat{\Psi}^{-1}$ with the natural map (22), namely since $U \subset \Psi^{-1}(V)$:

$$S^{[a,b)}(M, \Psi^{-1}(V)) \longrightarrow S^{[a,b)}(M, U) \tag{27}$$

gives a morphism $\Psi^*$, i.e

$$\begin{array}{ccc} S^{[a,b)}(M, V) & \xrightarrow{\Psi^*} & S^{[a,b)}(M, U) \\ \Big\downarrow{\scriptstyle \widehat{\Psi}^{-1}} & \nearrow & \\ S^{[a,b)}(M, \Psi^{-1}(V)) & & \end{array} \tag{28}$$

It follows immediately from the construction that $\Psi^*$ gives a morphism between the exact triangles $\triangle_{a,b,c}(M,V)$ and $\triangle_{a,b,c}(M,U)$. Moreover, it is compatible with all the introduced natural transformations. Further, it is clear that $(\Phi\Psi)^* = \Psi^*\Phi^*$ and Id : $(M,U) \to (M,U)$ induces Id. (22) is in fact induced by Id : $(M,U) \to (M,V)$ for $U \subset V$.

The key point will be that an isotopy of symplectic maps $\Psi_s : (M,U) \to (M,V)$ with $s \in [0,1]$ will induce maps $(\Psi_s)^*$ which are independent of $s \in [0,1]$. This will be shown in the next section.

## 4 Isotopy invariance

We follow the ideas of [11]. Assume $(M,\omega)$ is a symplectic manifold with the properties described before. Let $H$ be an admissible Hamiltonian and fix numbers $b \geq a > -m(H)$. Define

$$g(H,[a,b)) := \min\Big\{ \inf_{\Phi_H(x)<a} (a - \Phi_H(x)), \quad \inf_{\Phi_H(x)<b} (b - \Phi_H(x))\Big\}, \tag{29}$$

where the infima are taken over all contractible 1-periodic solutions of $\dot{x} = X_{H_t}(x)$ with $\Phi_H(x) > -m(H)$. Since the set of such solutions is finite, we have $g(H,[a,b)) > 0$. We shall call $g(H,[a,b))$ the "gap".

Given admissible Hamiltonians $H$ and $K$ we define their distance $d(H,K)$ as follows. Consider a smooth path $\tau \to H_\tau$ of Hamiltonians such that each $H_\tau$ is constant close to $\partial M, \tau \in [\tau_1,\tau_2]$ and $H_{\tau_1} = H, H_{\tau_2} = K$. Take the integral $d((H_\tau))$ defined by

$$d((H_\tau)) = \int_{\tau_1}^{\tau_2} \Big[\max_{(t,x)} \Big| \frac{\partial H_\tau}{\partial \tau}(t,x)\Big|\Big] d\tau \,. \tag{30}$$

Then let $d(H,K)$ be the infimum of all numbers $d((H_\tau))$ taken all smooth paths as just described.

Now let $(J_1,H_1)$ and $(J_2,H_2)$ be admissible, $a$ and $b$ fixed such that

$$b \geq a > \max\{-m(H_1), -m(H_2)\} \,.$$

Suppose we have the estimate

$$d(H_1,H_2) < \min\Big\{g(H_1,[a,b)), g(H_2,[a,b))\Big\} \,. \tag{31}$$

Then we can take a regular homotopy $(J(s,t,z), H(s,t,z))$ which satisfies (6) and is slow in the sense of the lemma of section 2, but instead of being monotone satisfies the weaker condition

$$d((H_s)) < \min\Big\{g(H_1,[a,b)), g(H_2,[a,b))\Big\} \,. \tag{32}$$

For $u \in \mathcal{M}(x_2, x_1; J, H)$ we calculate

$$\frac{d}{ds}\Phi_{H_s}(u(s)) \;=\; ||\Phi'_{H_s}(u(s))||^2_{L^2(s)} \;-\; \int_0^1 \frac{\partial H}{\partial s}(s,t,u(s,t))dt \;. \tag{33}$$

Here $L^2(s)$ is the $L^2$-section space along loops equipped with the $L^2$-inner product associated to the $\omega$-calibrated $t$-dependent almost complex structure $J(s,*,*)$. Integrating (33) we infer via (32)

$$\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \;>\; -\min\left\{g(H_1,[a,b)), g(H_2,[a,b))\right\} \;. \tag{34}$$

If $x_2 \in P_\mu(H_2), \mu = a, b$, then (34) and the definition of the "gaps" imply

$$\Phi_{H_1}(x_1) \;\geq\; \mu.$$

Now we can proceed as in section 2 to obtain unique chain homotopy classes

$$\sigma(J_2, H_2; J_1, H_1) \;:\; C_\mu(J_2, H_2) \longrightarrow C_\mu(J_1, H_1) \tag{35}$$

for $\mu = a, b$. Furthermore, if $(J_i, H_i), i = 1, 2, 3$, are admissible pairs satisfying

$$\begin{aligned} d(H_1, H_2) &< \frac{1}{2}\min\left\{g(H_1,[a,b)), g(H_2,[a,b))\right\} \\ d(H_2, H_3) &< \frac{1}{2}\min\left\{g(H_2,[a,b)), g(H_3,[a,b))\right\} \end{aligned} \tag{36}$$

then

$$\sigma(J_2, H_2; J_1, H_1) \circ \sigma(J_3, H_3; J_2, H_2) \;=\; \sigma(J_3, H_3; J_1, H_1) \;. \tag{37}$$

Taking $(J_3, H_3) = (J_1, H_1)$ we conclude that in this case $\sigma(J_2, H_2; J_1, H_1)$ induces an isomorphism in homology

$$S^{[a,b)}(J_1, H_1) \underset{sdi}{\overset{\simeq}{\longrightarrow}} S^{[a,b)}(J_2, H_2) \tag{38}$$

with inverse induced by $\sigma(J_1, H_1; J_2, H_2)$. Here "*sdi*" stands for the small distance isomorphism. This uniquely defined isomorphism always exists if the distance between $H_1$ and $H_2$ is small with respect to the "gaps", more precisely if the first equation of (36) holds.

Now we are in the position to prove the isotopy invariance. Assume $U, V$ are open subsets of $M$ such that $\bar{U}, \bar{V} \subset M \setminus \partial M$. Suppose $(\Psi_\tau)$ is a smooth isotopy of symplectic maps such that $\Psi_\tau(U) \subset V, \Psi_\tau : M \overset{\simeq}{\to} M$. Fix $b \geq a$ and choose an admissible pair $(J, H)$ with $H|(\mathbb{S}^1 \times \bar{V}) < 0$ and define $H_\tau : \mathbb{S}^1 \times M \to \mathbb{R}$ by

$$H_\tau(t,x) \;=\; H(t, \Psi_\tau(x))$$

and

$$J_\tau(t,x) \;=\; T\Psi_\tau(x)^{-1} J(t, \Psi_\tau(x)) T\Psi_\tau(x) \;.$$

Clearly $(J_\tau, H_\tau)$ is admissible for all $\tau \in [0,1]$. Since $U \subset \Psi_\tau^{-1}(V)$ for every $\tau \in [0,1]$ we can find an admissible $(\widehat{J}, \widehat{K})$ such that

$$\begin{aligned} \widehat{K} \mid (\mathbb{S}^1 \times \bar{U}) &< 0 \\ (\widehat{J}, \widehat{K}) \geq (J_\tau, H_\tau) \text{ for all } \tau &\in [0,1] \,. \end{aligned} \tag{39}$$

Taking for fixed $\tau$ a slow monotone homotopy we obtain a unique chain homotopy class $\sigma_\tau$

$$C_\mu(\widehat{J}, \widehat{K}) \longrightarrow C_\mu(J_\tau, H_\tau) \tag{40}$$

for every $\mu > -m(H)$. We observe that the distance $d(H_{\tau_0}, H_{\tau_1})$ for fixed $\tau_0$ and $\tau_1$ close to $\tau_0$ is as small as we wish. Consequently, the map in (40) for $\tau = \tau_0$ and $\mu = a, b$ factors through a monotone homotopy to $C_\mu(J_{\tau_1}, H_{\tau_1})$ and the "smallness map" from $C_\mu(J_{\tau_1}, H_{\tau_1})$ to $C_\mu(J_{\tau_0}, H_{\tau_0})$ provided $\tau_1$ is close enough to $\tau_0$. Hence for $\tau_1$ close to $\tau_0$ we have the commutative diagram

$$\begin{array}{ccc} S^{[a,b)}(J_{\tau_0}, H_{\tau_0}) & \longrightarrow & S^{[a,b)}(\widehat{J}, \widehat{K}) \\ \Big\downarrow sdi & \nearrow & \\ S^{[a,b)}(J_{\tau_1}, H_{\tau_1}) & & \end{array} \tag{41}$$

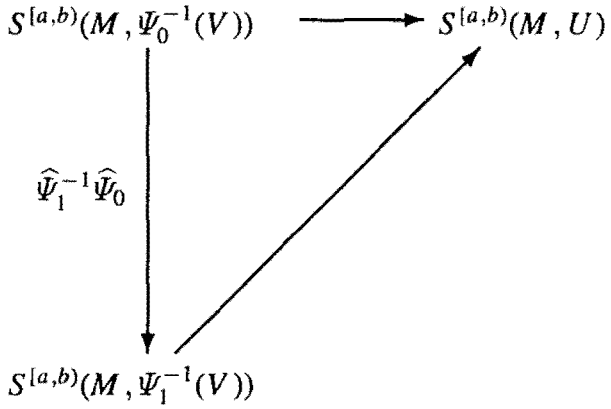
Here the unmarked arrows are coming from slow montone homotopies. If we can show that the "*sdi*"-map in the notation of (25) is given by

$$(\Psi_{\tau_1}^{-1}\Psi_{\tau_0})_{\#\#} \;:\; S^{[a,b)}(J_{\tau_0}, H_{\tau_0}) \xrightarrow{\simeq} S^{[a,b)}(J_{\tau_1}, H_{\tau_1})$$

we have as an easy corollary the isotopy invariance. In fact, in this case we obtain the commutative diagramm

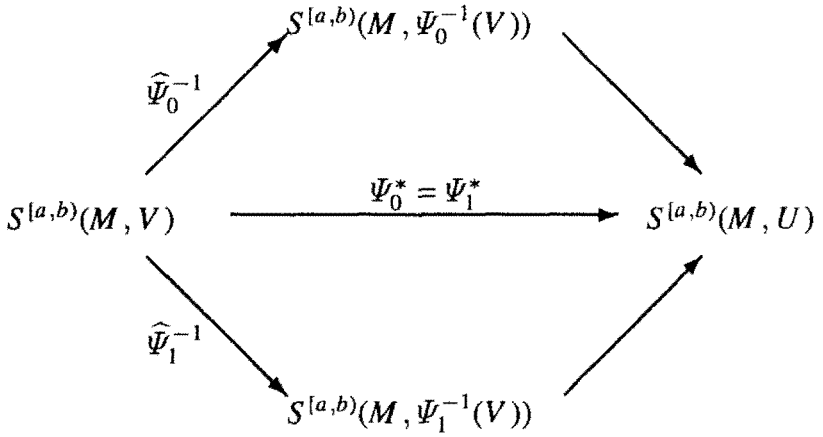
$$\begin{array}{ccc} S^{[a,b)}(J_0, H_0) & \longrightarrow & S^{[a,b)}(\widehat{J}, \widehat{K}) \\ \Big\downarrow (\Psi_1^{-1}\Psi_0)_{\#\#} & \nearrow & \\ S^{[a,b)}(J_1, H_1) & & \end{array} \tag{42}$$

Here we can pass to the direct limit to obtain




(43)

Now the composites $\sigma \circ \widehat{\Psi}_0^{-1}$ and $\sigma \circ \widehat{\Psi}_1^{-1}$, where $\sigma$ stands for the appropriate natural map induced by inclusion, define $\Psi_0^*$ and $\Psi_1^*$ respectively. Hence we obtain the commutative diagram




(44)

So it remains to prove the identity $sdi = (\Psi_{\tau_1}^{-1}\Psi_{\tau_0})_{\#\#}$ for $\tau_1$ close to $\tau_0$. In order to simplify notation we note that the above reduces to the following question:

Assume $\tau \to \Psi_\tau$ is a smooth arc of symplectic maps with $\Psi_0 = Id, (J,H)$ is admissible and, $b \geq a > -m(H)$, is it then true that $(\Psi_\tau)_{\#\#} = sdi$ for $\tau$ close to zero? Let $\beta : \mathbb{R} \to [0,1]$ be a smooth map satisfying $\beta(s) = 0$ for $s \leq 0, \beta'(s) > 0$ for $s \in (0,1)$ and $\beta(s) = 1$ for $s \geq 1$. Define

$$\begin{aligned} H^\varepsilon(s,t,x) &:= H(t,\Psi_{\varepsilon\beta(s)}(x)) \\ J^\varepsilon(s,t,x) &:= T\Psi_{\varepsilon\beta(s)}(x)^{-1} \circ J(t,\Psi_{\varepsilon\beta(s)}(x)) \circ T\Psi_{\varepsilon\beta(s)}(x) \end{aligned} \tag{45}$$

for all $(s,t,x) \in \mathbb{R}\times\mathbb{S}^1\times M$ and $\varepsilon > 0$. Let $u \in \mathcal{M}(x,x^\varepsilon;J^\varepsilon,H^\varepsilon)$, i.e. $u$ satisfies

$$u_s + J^\varepsilon(s,t,u)u_t + \nabla_{J^\varepsilon(s,t)}H^\varepsilon(s,t,u) = 0 \tag{46}$$

and converges for $s \to \pm\infty$ to contractible 1-periodic solutions $x, x^\varepsilon$ of $H, H^\varepsilon$ respectively. Define $v : Z \to M$ by

$$u(s,t) = \Psi_{\varepsilon\beta(s)}^{-1}(v(s,t))$$

and insert this in (46) to obtain

$$\begin{aligned} 0 &= \left(\tfrac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}\right)(v(s,t)) + T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))v_s(s,t) \\ &\quad + J^{\varepsilon}(s,t,\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t)))T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))v_t(s,t) \\ &\quad + \nabla_{J^{\varepsilon}(s,t)}H^{\varepsilon}(s,t,\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))) \qquad (47) \\ &= T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))\Big[v_s(s,t) + J(t,v(s,t))v_t(s,t) \\ &\quad + \nabla_{J(t)}H(t,v(s,t)) + T\Psi_{\varepsilon\beta(s)}(v(s,t))\left(\tfrac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}\right)(v(s,t))\Big] \end{aligned}$$

Now if $\mathrm{ind}_{CZ}(x) = \mathrm{ind}_{CZ}(x^{\varepsilon})$, the only solution of $v_s + J(t,v)v_t + \nabla_{J(t)}H(t,v) = 0$ connecting $x$ and $\Psi_{\varepsilon}^{-1} \circ x^{\varepsilon}$ is $v(s,t) \equiv x(t)$.

The expression $T\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))(\frac{d}{ds}\Psi^{-1}_{\varepsilon\beta(s)}(v(s,t))$ defines a small perturbation (depending on $\varepsilon$) compactly supported in $s$. By the compactness estimates in [7] and the inverse function theorem, for small $\varepsilon$ (47) has a unique solution $\tilde{v}$ close to every $v(s,t) = x(t)$, where $x$ is a contractible 1-periodic solution of $\dot{x} = X_{H_t}(x)$.

Consequently, (46) has for every 1-periodic solution $x$ in $P_a(H)\backslash P_b(H)$ a unique solution $u$ connecting $x$ with $\Psi_{\varepsilon}^{-1}(x)$ provided $\varepsilon$ is small enough. Hence the map from $C_{\mu}(J,H) \to C_{\mu}(J_{\Psi},H_{\Psi})$ for $\mu = a,b$ induced by the homotopy $(J^{\varepsilon},H^{\varepsilon})$ is precisely the map $\Psi_{\varepsilon}^{\#}$.

On the other hand for $\varepsilon$ small , $(J^{\varepsilon},H^{\varepsilon})$ is a regular slow homotopy satisfying the smallness condition (32).

Hence we have proved

**Proposition 1.** *Let $(M,\omega)$ be as previously described and $U,V$ open subsets of $M$ such that $\bar{U},\bar{V} \subset M\backslash\partial M$. Given a smooth isotopy of symplectic maps $\Psi_s : (M,U) \hookrightarrow (M,V)$ we have that $(\Psi_s)^*$ is independent of $s$.*

## 5 Possible extensions

To define symplectic homology groups $S^{[a,b)}(J,H)$ we have introduced the space $\mathcal{J}$ of $\omega$-calibrated almost complex structures on $M$ behaving nicely at the boundary. One can also consider the bigger space $\tilde{\mathcal{J}}$ of $\omega$-calibrated almost complex structures defined on the **interior of** $M$ (not necessarily extendible to the boundary) and satisfying the analogous condition near $\partial M$ (where $\eta, \lambda$ etc. may also not be extendible to $\partial M$). For $J \in \tilde{\mathcal{J}}$ and $H$ admissible, solutions in $\mathcal{M}(x_1,x_2;J,H)$ still stay away from the boundary by [18], Lemma 2.4. Hence $S^{[a,b)}(J,H)$ can be defined as before. It is however not clear if two elements of $\tilde{\mathcal{J}}$ can be joined by a path in $\tilde{\mathcal{J}}$. Therefore it cannot be excluded as before that $S^{[a,b)}(J,H)$ might depend on the choice of $J \in \tilde{\mathcal{J}}$. But if we fix a $J \in \tilde{\mathcal{J}}$, we can still take the direct limit over $H$ to obtain symplectic homology groups $S^{[a,b)}(M,U,J)$ having the same properties as before. These groups will be used in [14] to prove the so-called "stability of the action spectrum".

Our construction was carried out under the standing assumption that $[\omega]$ and $c_1$ vanish on $\pi_2(M)$. The assumption on $c_1$ guarantees that a contractible periodic trajectory has a well defined Morse index: the Conley-Zehnder index. The assumption on $[\omega]$ implies that for a contractible loop $x$ we have a well defined action $\Phi_H(x)$.

It makes of course also sense to look for noncontractible 1-periodic solutions in a given class $\alpha \in [\mathbb{S}^1, M]$. We fix in that case a reference loop $x_0 \in \alpha$. Given another loop $x \in \alpha$ we would like to define a number $\Phi_H(x)$, so that the derivative of $\Phi_H$ with respect to $x$ vanishes if and only if $x$ is a 1-periodic solution of the Hamiltonian system $\dot{x} = X_{H_t}(x)$ in the class of $\alpha$. The only way is the following:

Take a smooth map $\bar{x} : [0,1] \times \mathbb{S}^1 \to M$ such that $\bar{x}(0,t) = x_0(t)$ and $\bar{x}(1,t) = x(t)$ for $t \in \mathbb{S}^1$ and define $\Phi_H(x) := \int \bar{x}^*\omega - \int_0^1 H(t,x(t))$. The difficulty here is that $\Phi_H(x)$ is not well defined. Clearly by Stokes's Theorem we obtain the same value if we replace $\bar{x}$ by any $\hat{x}$ which is homotopic to $\bar{x}$ with boundary fixed. So the right objects to study are not the 1-periodic solutions in the class $\alpha$, but the 1-periodic solutions together with their "history", i.e. a homotopy of a given reference curve $x_0$ to the loop $x$. Let $C^\infty_\alpha(\mathbb{S}^1, M)$ be the collection of all loops $x$ of class $\alpha$, and denote by $\widetilde{C}^\infty_{\alpha,x_0}(\mathbb{S}^1, M)$ the universal covering with reference curve $x_0$. Then finding 1-periodic solutions of class $\alpha$ turns out to be a variational problem on this space. Problems of this kind can be handled at the expense of some nontrivial algebraic machinery, see [16].

All these difficulties can be of course avoided if the symplectic form $\omega$ is exact, i.e. $\omega = d\lambda$ for some 1-form on $M$. Then we define

$$\Phi_H(x) = -\int_0^1 x^*\lambda - \int_0^1 H(t,x(t))dt$$

for every loop.

The previous discussion solves the problem of finding a real filtration provided the Morse complex can be handled analytically. (The bubbling off of holomorphic spheres has to be controlled.) The other problem is the fixing of a grading. For this one takes the reference curve $x_0$ and considers the pullback bundle $x_0^*TM \to \mathbb{S}^1$, which is a symplectic vectorbundle. Of interest are first order linear differential operators $L_0 : L^2(x_0^*TM) \supset H^{1,2}(x_0^*TM) \to L^2(x_0^*TM)$, which in a suitable symplectic trivialization $\bar{x}^*TM \simeq \mathbb{S}^1 \times \mathbb{C}^n$ look like

$$(\widetilde{L}_0 h)(t) = -ih(t) - A(t)h(t) ,$$

where $t \to A(t)$ is a smooth map into the real linear maps of $\mathbb{C}^n$ such that $A(t)$ is symmetric for the real inner product $< *, * >= \mathrm{Re}(*, *)$ on $\mathbb{C}^n$. We assume $\mathrm{Kern}(\widetilde{L}_0) = \{0\}$ and call $\widetilde{L}_0$ and $L_0$ in this case nondegenerate. $L_0$ is what we call in [10] an asymptotic operator.

Now we consider $x_0$ and $L_0$ as some kind of normalizing data. Given any element of $\widetilde{C}^\infty_{\alpha,x_0}(\mathbb{S}^1, M)$, i.e. a homotopy class of maps $\bar{x} : [0,1] \times \mathbb{S}^1 \to M$ with

$\bar{x}(0,t) = x_0(t)$ and $\bar{x}(1,t) = x(t)$ we define a Morse index for $x$ provided it is a nondegenerate 1-periodic solution of a Hamiltonian system $\dot{x} = X_H(x)$. Namely having fixed a calibrated perhaps $t$-dependent almost complex structure $J$ we can write $\dot{x} = X_{H_t}(x)$ as $-J(t,x(t))\frac{d}{dt}x(t) - \nabla_{J_t}H(t,x(t)) = 0$. Linearizing the latter expression by varying $x$ in the function space we obtain a second asymptotic operator $L = L(H,x)$. (It only depends on $x$, not on $\bar{x}$.)

Now one trivialises $\bar{x}^*TM \to [0,1]\times\mathbb{S}^1$. This determines trivializations for $L_0$ and $L(H,x)$ which depend up to homotopy only on the class of $\bar{x}$ in $\widetilde{C}_{\alpha,x_0}(\mathbb{S}^1,M)$. Now one has obtained asymptotic operators $\widetilde{L}_0$ and $\widetilde{L}(H,x)$. One takes now a Fredholm operator $A$ of the type described in [10] having asymptotic operators $A^- = \widetilde{L}_0$ and $A^+ = \widetilde{L}(H,x)$ and defines the generalized Conley-Zehnder index of $[\bar{x}]$ with respect to the normalizing data $(x_0, L_0)$ to be precisely the Fredholm index of $A$.

It follows from results in [10] that this definition is independent of the constructions involved. Let us write for it $\mathrm{ind}([\bar{x}],H) := \mathrm{ind}_{(x_0,L_0)}([\bar{x}],H)$. If we use this construction for contractible loops under the assumption that $c_1$ vanishes on $\pi_2(M)$ we see that this index map differs from the Conley-Zehnder index only by a fixed constant. Given an admissible pair $(J,H)$, consider the manifold of connecting orbits between two nondegenerate 1-periodic solutions in the class $\alpha$, i.e.

$$\mathcal{M}(x^-,x^+;J,H) = \{u : Z \to M \,|\, u_s + J(t,u)u_t + \nabla_{J_t}H(t,u) = 0,$$
$$u(s,*) \to x^{\pm} \text{ as } s \to \pm\infty\}\,.$$

It turns out as the consequence of the discussions in [10] that the local dimension $\dim_u \mathcal{M}$ close to $u$ satisfies

$$\dim_u \mathcal{M} = \mathrm{ind}([b],H) - \mathrm{ind}([a],H)\,,$$

where $a : [0,1]\times\mathbb{S}^1 \to M$ is any fixed homotopy between $x_0$ and $x^-$, and $b$ is the homotopy obtained by first carrying out $a$ followed by the homotopy described by $u$. Let us write $[b] = [a]\&[u]$. Then the above formula reads

$$\dim_u \mathcal{M} = \mathrm{ind}([a]\&[u],H) - \mathrm{ind}([a],H)\,.$$

We note finally that for manifolds $M$ with contact type boundary which are equidimensional submanifolds of a cotangent bundle we have $c_1|\pi_2(M) = 0$ and $\omega|M = d\lambda$. In that case $\dim_u \mathcal{M}$ is independent of the choice of $u$, and only depends on $x^-$ and $x^+$. One can carry out our homology construction for any class $\alpha$ since in that case no holomorphic spheres exist. The integer grading is fixed by prescribing a pair $(x_0, L_0)$ as described above. The filtration is given naturally through the choice of $\lambda = pdq$. Another case where the construction can be carried out are monotone manifolds in the sense of [9].

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