UNIVERSITÄT AUGSBURG
Universitätsbibliothek

OPEN ACCESS
OPUS AUGSBURG

Subcritical Stein manifolds are split

Kai Cieliebak

Angaben zur Veröffentlichung / Publication details:

Cieliebak, Kai. 2002. “Subcritical Stein manifolds are split.” arXiv. Augsburg: Universität Augsburg. https://arxiv.org/abs/math/0204351.

Nutzungsbedingungen / Terms of use:

licgercopyright

Dieses Dokument wird unter folgenden Bedingungen zur Verfügung gestellt: / This document is made available under these conditions:
Deutsches Urheberrecht
Weitere Informationen finden Sie unter: / For more information see:
https://www.uni-augsburg.de/de/organisation/bibliothek/publizieren-zitieren-archivieren/publiz/

arXiv:math/0204351v1 [math.DG] 30 Apr 2002

Subcritical Stein manifolds are split

K. Cieliebak

April 30, 2002

1 Introduction

This note concerns the following little observation.

Theorem 1.1. *Every subcritical Stein manifold (domain) is deformation equivalent to a split one.*

Let me explain the words in the statement.

Stein. A *Stein manifold* is a complex manifold (W, J) without boundary which admits an exhausting plurisubharmonic function $\phi : W \to \mathbb{R}$. Here "exhausting" means proper and bounded from below, and "plurisubharmonic" means that $-d(d\phi \circ J)$ is a positive $(1,1)$ form. A sublevel set $\phi^{-1}(-\infty, c]$ in a Stein manifold is called *Stein domain*. Note that Stein domains are compact, whereas Stein manifolds are noncompact.

Subcritical. It is well-known that the critical points of a plurisubharmonic Morse function ϕ have Morse index $\leq n$, where n is the complex dimension of W. The Stein manifold (domain) is called *subcritical* if it admits an exhausting plurisubharmonic Morse function ϕ all of whose critical points have index $< n$.

Split. A Stein manifold is called *split* if it is of the form $(V \times \mathbb{C}, J_0 \times i)$ for some Stein manifold (V, J_0). An exhausting plurisubharmonic function is given by $\phi = \phi_0 + |z|^2$, where ϕ_0 is an exhausting plurisubharmonic function on V and z is the complex coordinate on $\mathbb{C}$. A Stein domain is called split if it is a sublevel set $\phi^{-1}(-\infty, c]$ in a split Stein manifold. Note that split Stein manifolds (domains) are subcritical.

Deformation equivalence. Two Stein structures J_0, J_1 on the same smooth manifold W are called *Stein homotopic* if there exists a continuous family of Stein structures J_t with exhausting plurisubharmonic functions ϕ_t such that critical points of ϕ_t do not travel to infinity during the homotopy. Two Stein manifolds (W, J) and (W', J') are called *deformation equivalent* if there exists a diffeomorphism $f : W \to W'$ such that J and f^*J' are Stein homotopic on W.

A Stein manifold (W, J, ϕ) carries a canonical symplectic form $-d(d\phi \circ J)$. Deformation equivalence implies symplectomorphism [4], so this is the right notion of

equivalence from the symplectic point of view. Subcritical Stein manifolds have recently received some interest because of their particularly simple symplectic properties ([1],[9],[10],[11]). In view of Theorem 1.1, one can always assume them to be split. This will simplify the study of their symplectic properties, e.g. their symplectic field theory invariants [5].

In the splitting $W = V \times \mathbb{C}$ the homeomorphism type of V may not be unique, as the following example shows.

Example 1.2. Let $W = S^2 \times \mathbb{R}^4$ with an almost complex structure J of first Chern class $c_1(J) = 2k \in H^2(W;\mathbb{Z}) \cong \mathbb{Z}$. By the results in [4], W carries a unique Stein structure in the homotopy class of J for which the function $|x|^2$ on the $\mathbb{R}^4$-factor is plurisubharmonic. Then for every oriented 2-plane bundle $V \to S^2$ whose Euler class $e \in H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ is even and $e \leq -2-2k$, there exists a Stein structure J_0 on V such that $V \times \mathbb{C}$ is deformation equivalent to W.

To see this, consider first $V = T^*S^2$ with its natural Stein structure J_0. It has Chern class $c_1(J_0) = 0$ and Euler class (as a bundle over S^2) $e = -2$. It is obtained from the 4-ball by attaching a 2-handle along the Legendrian unknot with Thurston-Bennequin number $\mathrm{tb} = -1$ and rotation number $r = 0$ (see [6]). Adding l kinks to the knot, the Thurston-Bennequin number becomes $-1-l$, while the rotation number r can be made any of the integers $-l, -l+2, \ldots, l-2, l$. For l even, the resulting Stein surface is the oriented 2-plane bundle over S^2 with Euler class $e = \mathrm{tb} - 1 = -l-2$ and Chern class $c_1(J_0) = r$. So for e even and $e \leq -2k-2$ we can arrange $c_1(J_0) = 2k$. By the arguments below, $V \times \mathbb{C}$ is deformation equivalent to W.

Note that the total spaces of 2-plane bundles over S^2 of different Euler class are not homeomorphic. This can be seen, e.g., from the inverse limits of the fundamental groups of the complements of compact subsets. So there are infinitely many pairwise non-homeomorphic Stein surfaces V such that $V \times \mathbb{C}$ is Stein equivalent to (W, J).

One could also ask whether a subcritical Stein manifold is biholomorphic to a split one. Clearly a necessary condition is the existence of a holomorphic embedding of $\mathbb{C}$ through every point. For example, bounded open domains in $\mathbb{C}^n$ can never be biholomorphically split. The following question grew out of a discussion with R. Hind.

Is every subcritical Stein manifold which has a holomorphic embedding of $\mathbb{C}$ *through every point biholomorphic to a split one?*

Acknowledgement. The question whether every subcritical Stein manifold is split was brought up by K. Mohnke. This note was written during my stay at the Institute for Advanced Study, which I thank for its hospitality.

2 Proof

The proof is based on Eliashberg's theory of Stein manifolds ([3],[4]). Recall first some notation from [3]. Let (W, J) be a Stein domain of complex dimension n. A *handle attaching triple (HAT)* (f, β, γ) is the data for attaching a handle in the catagory of almost complex manifolds. Here $f : S^{k-1} \hookrightarrow \partial W$ is an embedding. It induces injective bundle homomorphisms $df : TS^{k-1} \to f^*T(\partial W)$ and $Df : TS^{k-1} \oplus \underline{\mathbb{R}} \to f^*TW$ by sending the generator of the trivial bundle $\underline{\mathbb{R}}$ to an inward pointing vector field transverse to ∂W. β is a normal framing for f in ∂W, i.e. a bundle isomorphism $\beta : S^{k-1} \times \mathbb{R}^{2n-k} \to \nu_f$ to the normal bundle ν_f to f in ∂W. $\gamma : S^{k-1} \times \mathbb{C}^n \to f^*TW$ is an isomorphism of complex bundles which is homotopic to $Df \oplus \beta$ as an isomorphism of real bundles. Here $TS^{k-1} \oplus \underline{\mathbb{R}} \oplus \mathbb{R}^{2n-k}$ is identified with $S^{k-1} \times \mathbb{C}^n$ by viewing $TS^{k-1} \oplus \underline{\mathbb{R}}$ as the tangent bundle to the ball $B^k \times 0 \times 0 \subset \mathbb{R}^k \times i\mathbb{R}^k \times \mathbb{C}^{n-k}$.

An *isotopy* of HATs is an isotopy of embeddings f_t covered by homotopies β_t, γ_t. Attaching a handle with isotopic HATs yields diffemorphic smooth manifolds with homotopic almost complex structures.

Note that the homotopy group $\pi_{k-1}SO(2n-k)$ acts transitively on the framings β (considered up to homotopy) by composition. For a HAT (f, β, γ) and g in the kernel of the map $\pi_{k-1}SO(2n-k) \to \pi_{k-1}SO(2n)$, $(f, \beta \cdot g, \gamma)$ is again a HAT (which leads to a different smooth manifold when attaching the handle).

The maximal complex subspaces of ∂W define a contact structure ξ on ∂W. An embedding into ∂W is called *isotropic* if it is tangent to ξ. An isotropic embedding of the maximal possible dimension, $n-1$, is called *Legendrian*. A HAT (f, β, γ) is called *special* if f is an isotropic embedding, $\beta = JDf \oplus \theta$ for an injective complex bundle homomorphism $\theta : S^{k-1} \times \mathbb{C}^{n-k} \to f^*TW$, and $\gamma = Df \oplus JDf \oplus \theta$. It is called *stably special* if there exists a $g \in \ker[\pi_{k-1}SO(2n-k) \to \pi_{k-1}SO(2n)]$ such that $(f, \beta \cdot g, \gamma)$ is special. The main inductional lemma in [3] states that if (W, J) is a Stein domain and (f, β, γ) a special HAT, then a handle can be attached in such a way that the Stein structure extends over the handle.

Now let (V, J_0) be a Stein domain of complex dimension $n-1$ with plurisubharmonic Morse function ϕ_0, $\phi_0|_{\partial V} \equiv c$. Let $W \subset V \times \mathbb{C}$ be the Stein domain $(\phi_0 + |z|^2)^{-1}(-\infty, c]$, where z is the coordinate on $\mathbb{C}$. W is equipped with the complex structure $J = J_0 \times i$ and the plurisubharmonic function $\phi = \phi_0 + |z|^2$. Note that ∂W has a natural open book structure with trivial monodromy,

$$\partial W \cong \partial V \times B^2 \cup V \times S^1.$$

Let (f, β, γ) be a HAT for W of index $k < n$.

A HAT (f_0, β_0, γ_0) for V naturally induces a HAT $(\hat{f}_0, \hat{\beta}_0, \hat{\gamma}_0)$ for W with $\hat{f}_0 = f_0 \times 0 : S^{k-1} \hookrightarrow \partial V \times 0$, $\hat{\beta}_0 = \beta_0 \times \mathbb{1}_{\mathbb{C}}$ and $\hat{\gamma}_0 = \gamma_0 \times \mathbb{1}_{\mathbb{C}}$.

Lemma 2.1. *There exists an embedding $f_0 : S^{k-1} \hookrightarrow \partial V$ such that $\hat{f}_0$ is isotopic (through embeddings into ∂W) to f.*

Proof. Let $\Delta \subset V$ be the skeleton, i.e. the union of all descending manifolds of critical points of ϕ_0 (with respect to some Riemannian metric). Note that the negative gradient flow of ϕ_0 retracts V onto Δ. Since $\Delta \times S^1$ has codimension at least $n-1$ and S^{k-1} has dimension $< n-1$, a small perturbation of $f : S^{k-1} \hookrightarrow \partial W$ makes it avoid $\Delta \times S^1$. Then we use the gradient flow of ϕ_0 to isotop f to an embedding into $\partial V \times B^2$. Since $2 \dim S^{k-1} \leq \dim \partial V - 1$, the latter embedding is isotopic to an embedding into $\partial V \times 0$. □

Lemma 2.2. *There exist a HAT (f_0, β_0, γ_0) on V such that the HAT $(\hat{f}_0, \hat{\beta}_0, \hat{\gamma}_0)$ is isotopic to (f, β, γ).*

Proof. The previous lemma shows that after an isotopy of HATs, we may assume that $f = \hat{f}_0$. Complete f_0 to any HAT $(f_0, \bar{\beta}_0, \bar{\gamma}_0)$ on V. This allows us to identify homotopy classes of framings β_0 for f_0 with $\pi_{k-1}SO(2n-k-2)$, and similarly for γ_0, β, γ. Now consider the following commutative diagram:

$$\begin{array}{ccc}
\beta_0 \in \pi_{k-1}SO(2n-k-2) & \xrightarrow{\sigma} & \pi_{k-1}SO(2n-k) \ni \beta \\
\downarrow & & \downarrow \cong \\
\pi_{k-1}SO(2n-2) & \xrightarrow{\cong} & \pi_{k-1}SO(2n) \\
\uparrow & & \uparrow \\
\gamma_0 \in \pi_{k-1}U(n-1) & \xrightarrow{\cong} & \pi_{k-1}U(n) \ni \gamma.
\end{array}$$

Here the map σ is surjective by Bott periodicity (see [7], p. 230): $\pi_i O(l-1) \to \pi_i O(l)$ is an isomorphism for $i < l-2$ and surjective for $i \leq l-2$. With $i = k-1$ and $l-1 = 2n-k-2$, the condition for surjectivity becomes $k-1 \leq 2n-k-3$, or $k \leq n-1$, which is fulfilled by hypothesis. The same argument yields the isomorphisms in the middle row and at the vertical arrow. The isomorphism in the bottom row follows similarly (see [8]): $\pi_i U(n-1) \to \pi_i U(n)$ is surjective for $i \leq 2n-2$ and an isomorphism for $i < 2n-2$.

We see that given β, γ with the same image in $\pi_{k-1}SO(2n)$ we find preimages β_0, γ_0 under the vertical maps. Since β_0, γ_0 have the same image in $\pi_{k-1}SO(2n-2)$, (f_0, β_0, γ_0) is a HAT with the desired properties.

Note that here we make a choice if σ is not bijective: We may change β_0 by any element in the kernel of σ and still get a HAT with the desired properties. This freedom will be important later on. □

Lemma 2.3. *There exist a special HAT (f_0, β_0, γ_0) on V such that the HAT $(\hat{f}_0, \hat{\beta}_0, \hat{\gamma}_0)$ is isotopic to (f, β, γ).*

Proof. Let (f_0, β_0, γ_0) be a HAT as provided by Lemma 2.2. It is shown in [3] that (f_0, β_0, γ_0) is isotopic (on V) to a stably special HAT (f_1, β_1, γ_1). Moreover,

for $n > 3$ or $k < n-1$, the stably special HAT (f_1, β_1, γ_1) is isotopic to a special one. The same holds for $n = 2$ by elementary reasons.

It remains to treat the case $n = 3$ and $k = 2$. Then the diagram in the proof of Lemma 2.2 becomes

$$\begin{array}{ccc} \beta_1 \in \pi_1 SO(2) & \xrightarrow{\sigma} & \pi_1 SO(4) \ni \beta \\ \downarrow & & \downarrow \cong \\ \pi_1 SO(4) & \xrightarrow{\cong} & \pi_1 SO(6) \\ \uparrow & & \uparrow \\ \gamma_1 \in \pi_1 U(2) & \xrightarrow{\cong} & \pi_1 U(3) \ni \gamma. \end{array}$$

The embedding f_1 is a Legendrian knot in the contact 3-manifold ∂V. It determines a normal framing JDf in ∂V which corresponds to an element in $\pi_1 SO(2) \cong \mathbb{Z}$. The HAT is special iff $\beta_1 = JDf \in \pi_1 SO(2) \cong \mathbb{Z}$. Adding "kinks" to f_1 allows us to *decrease* JDf by an arbitrary integer, but by Bennequin's inequality [2], we cannot increase it. However, recall that in the proof of Lemma 2.2 we had the freedom of changing β_0 (and hence β_1) by an arbitrary element in $\ker[\sigma : \pi_1 SO(2) \to \pi_1 SO(4)] \cong 2\mathbb{Z}$. So we can make β_1 smaller than JDf by subtracting an even integer and then decrease JDf by adding kinks to f_1 until $JDf = \beta_1$. The result is a special HAT as desired.

Note that the modification of β_1 changes the diffeomorphism type of the Stein surface we get after attaching the handle. Since we have the freedom of making β_1 more negative and adding more kinks to f_1, the diffeomorphism type is not uniquely determined (see Example 1.2). □

Proof of Theorem 1.1. The main extension lemma in [3] states that given a special HAT (f_0, β_0, γ_0), a handle can be attached in such a way that the Stein structure extends over the handle. So using Lemma 2.3 and induction, we obtain a Stein manifold (V, J_0) and a diffeomorphism $F : V \times \mathbb{C} \to W$ such that F^*J is homotopic (through almost complex structures) to $J_0 \times i$. Moreover, by construction, there are exhausting plurisubharmonic functions ϕ on W and ϕ_0 on V such that $F^*\phi$ and $\phi_0 + |z|^2$ have the same critical points. Now another theorem of Eliashberg [4] implies that $J_0 \times i$ and f^*J are Stein homotopic.

The same arguments work for Stein domains. This proves Theorem 1.1. □

References

[1] P. Biran and K. Cieliebak, *Symplectic topology on subcritical manifolds*, Comment. Math. Helv. **76**, no. 4, 712-753 (2001).

[2] D. Bennequin, *Entrelacements et équations de Pfaff*, Asterisque **107-108**, 87-161 (1983).

[3] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Int. J. Math. **1**, 29-46 (1990).

[4] Y. Eliashberg, *Symplectic geometry of plurisubharmonic functions*, Notes by M. Abreu, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **488**, Gauge Theory and Symplectic Geometry (Montreal 1995), Kluwer Acad. Publ., Dordrecht, 49-67 (1997).

[5] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, Geom. and Funct. Anal., Special Volume (2000), 560-673.

[6] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Math. **20**, Am. Math. Soc., Providence (1999).

[7] A. Kosinski, *Differential Manifolds*, Pure and Appl. Math. **138**, Acad. Press, Boston (1993).

[8] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton (1963).

[9] K. Mohnke, *Holomorphic disks and the Chord Conjecture*, Ann. Math. **154**, no. 1, 219-222 (2001).

[10] C. Viterbo, *Functors and computations in Floer homology with applications I*, Geom. Funct. Anal. **9**, no. 5, 985-1033 (1999).

[11] M.-L. Yau, *Contact homology of subcritical Stein manifolds*, PhD thesis, Stanford University (1999).