

Influence of measurements on the statistics of work performed on a quantum system

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The recently demonstrated robustness of fluctuation theorems against measurements [M. Campisi, P. Talkner, and P. Hänggi, *Phys. Rev. Lett.* **105**, 140601 (2010).] does not imply that the probability distributions of nonequilibrium quantities, such as heat and work, remain unaffected. We determine the impact of measurements that are performed during a running force protocol on the characteristic function of work. The results are illustrated by means of the Landau-Zener(-Stückelberg-Majorana) model. In the limit of continuous measurements the quantum Zeno effect suppresses any unitary dynamics. It is demonstrated that the characteristic function of work is the same as for an adiabatic protocol when the continuously measured quantity coincides with the Hamiltonian governing the unitary dynamics of the system in the absence of measurements.

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I. INTRODUCTION

The Jarzynski equality and the fluctuation theorems are general and surprisingly robust exact relations of nonequilibrium thermodynamics. The validity of these relations was rigorously established irrespective of the speed of the externally applied forcing protocol for driven classical [1–3] and quantum systems [4,5], staying either in complete isolation or in weak [6] or even strong contact with an environment [7,8]. See also the reviews in Refs. [9] and [10] and references therein. In this work we will focus on the Tasaki-Crooks work fluctuation theorem, reading

$$\frac{p_F(w)}{p_B(-w)} = e^{\beta(w-\Delta F)}, \quad (1)$$

where $p_F(w)$ is the probability density function (pdf) of work performed by a forcing protocol denoted as λ_t . This forcing acts on the system between the times $t = 0$ and $t = \tau$. Accordingly, $p_B(w)$ is the pdf of work performed on the system when the backward (B) protocol $\lambda_{\tau-t}$ describes the forcing of the system. The forward and backward processes start in the Gibbs equilibrium states at the same inverse temperature β and at the initial parameter values λ_0 and λ_τ , respectively. The free-energy difference between these two states is denoted by ΔF .

In the derivations [4,5] of the quantum work fluctuation theorem, Eq. (1), the energy of the system is measured at times $t = 0$ and $t = \tau$, and the work w is determined by the difference of the obtained eigenvalues. Recently, we showed that the Tasaki-Crooks work fluctuation theorem, Eq. (1), as well as other quantum fluctuation theorems, remain unaffected for scenarios other than this two-measurement scheme: The ratio of forward and backward pdf's in Eq. (1) stays unaltered even if further projective quantum measurements of any sequence of arbitrary observables are performed while the protocol is in action [11]. In Sec. II we provide an alternative proof of this result, based on the calculation of the characteristic function of work of a driven quantum system whose dynamics is interrupted by projective quantum measurements.

Based on the fact that the process of measurement of a quantum system is rather invasive due to the collapse of the wave function, it has been argued [11,12] that, although the value of the ratio of backward and forward pdf's in

Eq. (1) remains unchanged, additional measurements affect the values of the individual work pdfs. One must expect that, in a driven-measured quantum system, the work done is not only determined by the interaction with the manipulated external field λ , but also by the measurements themselves, which are physically realized by a measurement apparatus. In Sec. III we calculate the statistics of work in a prototypical model of driven quantum system, namely, the Landau-Zener (-Stückelberg-Majorana) [13–16] model, and illustrate how it is influenced by projective quantum measurements. We will also draw the attention on interesting features related to the quantum Zeno effect appearing when the measurements become very frequent. As we will see, if the observable that is measured at time t is the Hamiltonian $H(t)$, then in the Zeno limit the work characteristic function approaches the same expression as for an adiabatic protocol with no intermediate measurements.

Section IV closes the paper with some concluding remarks.

II. FLUCTUATION THEOREMS FOR DRIVEN-MEASURED QUANTUM SYSTEMS

We consider a quantum system that, in the time span $[0, \tau]$, is thermally isolated, and interacts with the external world only through a mechanical coupling to a time-dependent external force field and a measurement apparatus. The information regarding this interaction is encoded in a protocol which we denote as

$$\sigma = \{H(\lambda_t), (t_i, \mathcal{A}_i)\}. \quad (2)$$

It specifies (a) the system Hamiltonian $H(\lambda_t)$ at each time t in terms of the external forces λ_t , and (b) the times $t_i \in (0, \tau)$, $i = 1, \dots, N$, at which measurements of the observables \mathcal{A}_i occur. For the sake of simplicity, in the following we will adopt the notation $H(t)$ for $H(\lambda_t)$. As in the two-measurement scheme [4,5] of the Tasaki-Crooks theorem, we assume that, besides the N intermediate measurements of $\mathcal{A}_1, \dots, \mathcal{A}_N$, the energies determined by $H(0)$ and $H(\tau)$ are measured at times $t_0 = 0$ and $t_f = \tau$. For simplicity we then set $f = N + 1$ and

$$\mathcal{A}_0 = H(0), \quad \mathcal{A}_f = H(\tau). \quad (3)$$

The remaining observables, $\mathcal{A}_1, \dots, \mathcal{A}_N$, represented by Hermitean operators are required to have discrete spectra, but otherwise can be chosen arbitrarily.

For times $t \leq 0$ the system is assumed to stay in the Gibbs thermal state at inverse temperature β :

$$\rho_0^\beta = e^{-\beta H(0)} / Z_0, \quad (4)$$

where

$$Z_0 = \text{Tr} e^{-\beta H(0)} \quad (5)$$

is the canonical partition function.

We denote the orthogonal eigenprojectors of the observable \mathcal{A}_i by Π_k^i . These satisfy the eigenvalue equations

$$\mathcal{A}_i \Pi_k^i = \alpha_k^i \Pi_k^i, \quad (6)$$

where α_k^i are the eigenvalues of \mathcal{A}_i . Hence, with the above choice of the first and last measured observables, α_k^0 and α_k^f represent the instantaneous eigenenergies of the system at $t = 0$ and $t = \tau$, respectively.

In the following we compute the conditional probability $p_\sigma(m, \tau | n, 0)$ to find the eigenvalue α_m^f obtained in the last measurement at time τ , provided that the eigenvalue α_n^0 was the result of the first measurement, under those conditions that are specified by the protocol σ . We begin our discussion considering only one intermediate measurement ($N = 1$) of the observable \mathcal{A}_1 .

According to the von Neumann postulates, immediately after the eigenvalue α_n^0 is measured at $t = 0$, the system density matrix becomes

$$\rho_n(0^+) = \Pi_n^0 \rho_0^\beta \Pi_n^0 / p_n, \quad (7)$$

where

$$p_n = \text{Tr} \Pi_n^0 \rho_0^\beta \Pi_n^0 / Z_0 = e^{-\beta \alpha_n^0} / Z_0 \quad (8)$$

is the probability to find the system initially in the state with energy α_n^0 . Since we assumed that the system is thermally isolated, it subsequently evolves until time t_1 according to the unitary time evolution $U_{t_1,0}$ that is governed by the Schrödinger equation,

$$i\hbar \partial_t U_{t,0} = H(t) U_{t,0}, \quad U_{0,0} = \mathbb{1}. \quad (9)$$

Thus, immediately before the measurement of \mathcal{A}_1 , occurring at t_1 , the density matrix is

$$\rho_n(t_1^-) = U_{t_1,0} \rho_n(0^+) U_{t_1,0}^\dagger, \quad (10)$$

and the subsequent measurement projects it into

$$\rho_n(t_1^+) = \sum_r \Pi_r^1 U_{t_1,0} \rho_n(0^+) U_{t_1,0}^\dagger \Pi_r^1. \quad (11)$$

Likewise, just before the measurement of $\mathcal{A}_f = H(\tau)$ at time τ , the density matrix becomes

$$\rho_n(\tau^-) = U_{\tau,t_1} \rho_n(t_1^+) U_{\tau,t_1}^\dagger, \quad (12)$$

and the probability that the outcome of the measurement of $\mathcal{A}_f = H(\tau)$ at time τ is α_m^f is

$$p_\sigma(m, \tau | n, 0) = \text{Tr} \Pi_m^f U_{\tau,t_1} \rho_n(t_1^+) U_{\tau,t_1}^\dagger. \quad (13)$$

Finally, the pdf of work w , $P_\sigma(w)$, is obtained as the sum of the joint probabilities $p_\sigma(m, \tau | n, 0) p_n$ restricted to

$$w = \alpha_m^f - \alpha_n^0, \quad (14)$$

and hence becomes

$$P_\sigma(w) = \sum_{n,m} \delta[w - \alpha_m^f - \alpha_n^0] p_\sigma(m, \tau | n, 0) p_n. \quad (15)$$

A. The characteristic function of work

Next we focus on the characteristic function of work $G_\sigma(u)$, given by the Fourier transform of the work pdf $P_\sigma(w)$:

$$G_\sigma(u) = \int dw P_\sigma(w) e^{iuw}. \quad (16)$$

Substituting (15) into (16) we obtain for the characteristic function

$$\begin{aligned} G_\sigma(u) &= \sum_{m,n} e^{iu[\alpha_m^f - \alpha_n^0]} p_\sigma(m, \tau | n, 0) e^{-\beta \alpha_n^0} / Z_0 \\ &= \sum_{m,n,r} e^{iu[\alpha_m^f - \alpha_n^0]} \text{Tr} \Pi_m^f U_{\tau,t_1} \Pi_r^1 U_{t_1,0} \Pi_n^0 \\ &\quad \times \rho_0^\beta \Pi_n^0 U_{t_1,0}^\dagger \Pi_r^1 U_{\tau,t_1}^\dagger \\ &= \sum_r \text{Tr} e^{iuH(\tau)} U_{\tau,t_1} \Pi_r^1 U_{t_1,0} e^{-iuH(0)} \rho_0^\beta U_{t_1,0}^\dagger \Pi_r^1 U_{\tau,t_1}^\dagger \\ &= \text{Tr}[e^{iuH(\tau)}]_\sigma e^{-iuH(0)} \rho_0^\beta, \end{aligned} \quad (17)$$

where $[X]_\sigma$ denotes the time evolution of an operator X from $t = 0$ to $t = \tau$ in the presence of the protocol σ , which implies a unitary evolution governed by $H(t)$ interrupted by a measurement of an observable \mathcal{A}_1 at time $t = t_1$. It takes the form

$$[X]_\sigma = \sum_r U_{t_1,0}^\dagger \Pi_r^1 U_{\tau,t_1}^\dagger X U_{\tau,t_1} \Pi_r^1 U_{t_1,0}, \quad N = 1. \quad (18)$$

In the case of a protocol σ containing N interrupting measurements, the formal expression of the characteristic function is the same as for one interrupting measurement, Eq. (17), with the time evolution $[X]_\sigma$ given by

$$\begin{aligned} [X]_\sigma &= \sum_{r_1 \dots r_N} U_{t_1,0}^\dagger \Pi_{r_1}^1 U_{t_2,t_1}^\dagger \Pi_{r_2}^2 \dots \\ &\quad \dots U_{t_N,t_{N-1}}^\dagger \Pi_{r_N}^N U_{\tau,t_N}^\dagger X U_{\tau,t_N} \Pi_{r_N}^N U_{t_N,t_{N-1}} \dots \\ &\quad \dots \Pi_{r_2}^2 U_{t_2,t_1} \Pi_{r_1}^1 U_{t_1,0}. \end{aligned} \quad (19)$$

B. The Jarzynski equality

Putting $u = i\beta$, one recovers the Jarzynski equality

$$\begin{aligned} G_\sigma(i\beta) &= \langle e^{-\beta w} \rangle_\sigma = \text{Tr}[e^{-\beta H(\tau)}]_\sigma e^{\beta H(0)} \rho_0^\beta \\ &= Z_0^{-1} \text{Tr}[e^{-\beta H(\tau)}]_\sigma \\ &= Z_0^{-1} \text{Tr} e^{-\beta H(\tau)} = Z_f / Z_0 = e^{-\beta \Delta F}, \end{aligned} \quad (20)$$

because $\text{Tr}[X]_\sigma = \text{Tr} X$ for any trace class operator X [17].

This follows from the cyclic invariance of the trace, the unitarity relation $U_{t,s}^\dagger U_{t,s} = \mathbb{1}$, and the completeness of the projection operators $\sum_r \Pi_r^\dagger = \mathbb{1}$. Here $Z_f = \text{Tr} e^{-\beta H(\tau)}$ denotes the partition function of the Gibbs state at the initial temperature and final parameter values,

$$\rho_f^\beta = e^{-\beta H(\tau)} / Z_f, \quad (21)$$

and $\Delta F = \beta^{-1} \ln Z_0 / Z_f$ is the difference of the free energies of the thermal equilibrium states ρ_f^β and ρ_0^β .

The symbol $\langle \cdot \rangle_\sigma$ denotes an average with respect to the work pdf $P_\sigma(w)$. Equation (20) says that the Jarzynski equality holds irrespective of the details of the interaction protocol σ . Independent of the number and nature of the measured observables as well as strength, speed, and functional form of the driving force, the average exponentiated work is solely determined by the free-energy difference [8].

C. The work fluctuation theorem

Besides the Jarzynski equality, also the Tasaki-Crooks theorem is robust under repeated quantum measurements. In the presence of many intermediate measurements it reads

$$P_\sigma(w) = P_{\tilde{\sigma}}(-w) e^{\beta(w - \Delta F)}, \quad (22)$$

where the tilde ($\tilde{\sigma}$) indicates the temporal inversion of the protocol σ , that is,

$$\tilde{\sigma} = \{H(\tau - t), (\tau - t_i, \mathcal{A}_i)\}. \quad (23)$$

Hence, $\tilde{\sigma}$ specifies the succession of force values and measurements in the reversed order, specifically with the measurement of the observables \mathcal{A}_i at times $\tau - t_i$. In particular, it implies that at time $t = 0$, the observable $\mathcal{A}_f = H(\tau)$, and at time $t = \tau$, the observable $\mathcal{A}_0 = H(0)$ are measured. Accordingly, the initial state of the backward process is given by the Gibbs state ρ_f^β , Eq. (21), i.e., the system is at equilibrium with inverse temperature β and force value λ_τ .

Equation (22) holds under the assumptions that both $H(t)$ and \mathcal{A}_i commute with the quantum-mechanical antiunitary time-reversal operator Θ [18]. That is, for all $t \in [0, \tau]$, and $i = 0, \dots, f$, we assume

$$H(t)\Theta = \Theta H(t), \quad (24)$$

$$\mathcal{A}_i\Theta = \Theta \mathcal{A}_i. \quad (25)$$

We prove Eq. (22) for the simplest case of a single intermediate measurement, $N = 1$. The generalization to many measurements is straightforward. From Eq. (17) we have

$$G_\sigma(u) = \text{Tr} \sum_r U_{t_1,0}^\dagger \Pi_r^\dagger U_{\tau,t_1}^\dagger e^{iuH(\tau)} \times U_{\tau,t_1} \Pi_r U_{t_1,0} e^{i(\beta-u)H(0)} / Z_0. \quad (26)$$

Introducing the notation $\tilde{U}_{t',t}$ for the time evolution governed by $\tilde{H}(t) \equiv H(\tau - t)$, the backward characteristic function of

work can be written as

$$\begin{aligned} G_{\tilde{\sigma}}(u) &= \text{Tr} \sum_r \tilde{U}_{\tau-t_1,0}^\dagger \Pi_r^\dagger \tilde{U}_{\tau,t_1}^\dagger e^{iu\tilde{H}(\tau)} \\ &\quad \times \tilde{U}_{\tau,t_1} \Pi_r \tilde{U}_{\tau-t_1,0} e^{-iu\tilde{H}(0)} e^{-\beta H(\tau)} / Z_f \\ &= \text{Tr} \sum_r \Theta [\Theta^\dagger \tilde{U}_{\tau-t_1,0}^\dagger \Theta \Theta^\dagger \Pi_r^\dagger \Theta \Theta^\dagger \tilde{U}_{\tau,t_1}^\dagger \Theta \\ &\quad \times \Theta^\dagger e^{iu\tilde{H}(\tau)} \Theta \Theta^\dagger \tilde{U}_{\tau,t_1} \Theta \Theta^\dagger \Pi_r \Theta \\ &\quad \times \Theta^\dagger \tilde{U}_{\tau-t_1,0} \Theta \Theta^\dagger e^{-i(u-i\beta)\tilde{H}(0)} \Theta] \Theta^\dagger / Z_f, \end{aligned} \quad (27)$$

where we used the antiunitarity $\Theta \Theta^\dagger = \mathbb{1}$. From Eq. (25) it follows that all eigenprojection operators commute with the time-reversal operator, i.e., $\Pi_r^\dagger \Theta = \Theta \Pi_r^\dagger$. The time-reversal invariance, expressed by Eq. (24), implies $\Theta^\dagger e^{sH(t)} \Theta = e^{s^*H(t)}$ for any C -number s . Further, microreversibility of driven systems [10,19] implies

$$\begin{aligned} \Theta^\dagger \tilde{U}_{\tau-t,0} \Theta &= U_{t,\tau}, \\ \Theta^\dagger \tilde{U}_{\tau,\tau-t} \Theta &= U_{0,t}. \end{aligned} \quad (28)$$

Using these relations and recalling that for any trace class operator X , $\text{Tr} \Theta X \Theta^\dagger = \text{Tr} X^\dagger$, one ends up with

$$\begin{aligned} G_{\tilde{\sigma}}(u) &= \text{Tr} \sum_r e^{i(\beta-u)H(\tau)} U_{\tau,t_1} \Pi_r^\dagger U_{t_1,0} \\ &\quad \times e^{iuH(0)} U_{t_1,0}^\dagger \Pi_r^\dagger U_{\tau,t_1}^\dagger / Z_f. \end{aligned} \quad (29)$$

By comparison with Eq. (26), we finally find

$$Z_f G_{\tilde{\sigma}}(i\beta - u) = Z_0 G_\sigma(u), \quad (30)$$

hence, by means of an inverse Fourier transform the searched fluctuation theorem (22).

We thus have proved that the fluctuation theorem of Tasaki-Crooks remains unchanged if, additionally to the measurements of energy at time $t = 0$ and $t = \tau$, intermediate measurements of time-reversal invariant observables \mathcal{A}_i are performed at times t_i , provided the order of measurements in the backward protocol is properly changed in accordance with the corresponding times $\tau - t_i$.

III. EXAMPLE

As mentioned in the Introduction, one expects measurements to strongly influence the pdf of work, although the Jarzynski equality and the Tasaki-Crooks relation are insensitive to intermediate measurements. To illustrate this point in more detail, we consider the example of the Landau-Zener-(Stückelberg-Majorana) [13–16] model described by the Hamiltonian

$$H(t) = \frac{vt}{2} \sigma_z + \Delta \sigma_x. \quad (31)$$

It governs the dynamics of a two-level quantum system whose energy separation, vt , varies linearly in time, and whose states are coupled via the interaction energy Δ . Here, σ_x and σ_z denote Pauli matrices.

The Landau-Zener model is one of the few time-dependent quantum-mechanical problems that have an analytic solution. The elements of the 2×2 unitary time evolution matrix $U_{t,s}$

can be expressed in terms of parabolic cylinder functions [20]. The instantaneous eigenvalues of $H(t)$ are

$$E_n^t = (n - 1/2)\sqrt{v^2 t^2 + 4\Delta^2}, \quad n = 0, 1. \quad (32)$$

Since these energies are symmetric with respect to an inversion about $t = 0$ we choose the initial and final times t_0 and t_f as $-\tau/2$ and $\tau/2$, respectively, instead of 0 and τ , as in the previous discussion.

Figure 1(a) depicts the survival probability $p^1(0, \tau/2|0, -\tau/2)$ as a function of the instant t_1 of a single intermediate measurement of $H(t_1)$ for positive times t_1 and fixed length τ of the protocol. As an even function of t_1 , this also specifies $p^1(0, \tau/2|0, -\tau/2)$ for negative t_1 . The straight horizontal line shows the value of the survival probability $p(0, \tau/2|0, -\tau/2)$ without intermediate measurement. It is obvious that the intermediate measurement in general alters the survival

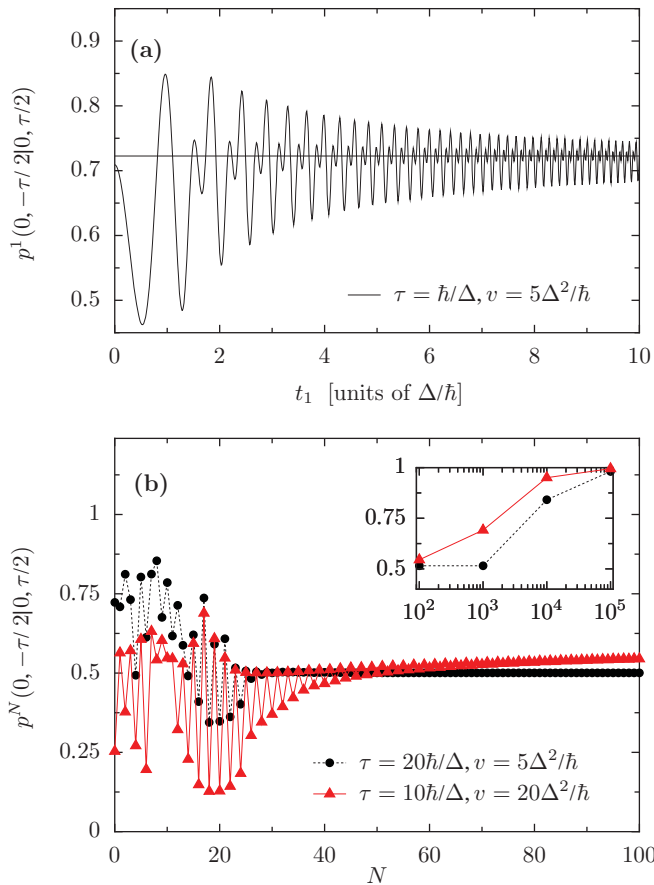


FIG. 1. (Color online) Survival probability of the Landau-Zener system, Eq. (31), interrupted by measurements. (a) Survival probability as a function of the time t_1 of a single energy measurement of $H(t_1)$. The parameter values are $\tau = 20\hbar/\Delta$, $v = 5\Delta^2/\hbar$. The horizontal line is the survival probability $p(0, -\tau/2|0, \tau/2)$ with no intermediate measurements. (b) Survival probability as a function of the number N of equally spaced intermediate measurements of $H(t)$. The parameter values are $\tau = 20\hbar/\Delta$, $v = 5\Delta^2/\hbar$ (black circles), and $\tau = 10\hbar/\Delta$, $v = 20\Delta^2/\hbar$ (red triangles). Inset: The survival probabilities approach the value 1 in the Zeno limit.

probability thus affecting the work pdf. For example, the average work,

$$\langle w \rangle = 2E_1^{\tau/2} \tanh[\beta E_1^{\tau/2}][1 - p(0, \tau/2|0, -\tau/2)], \quad (33)$$

evidently changes when $p(0, \tau/2|0, -\tau/2)$ is replaced by $p^1(0, \tau/2|0, -\tau/2)$. The same can be said for the standard deviation of work that reads

$$\langle \Delta w^2 \rangle = (2E_1^{\tau/2})^2[1 - p(0, \tau/2|0, -\tau/2)] - \langle w \rangle^2. \quad (34)$$

Notably, the introduction of an intermediate measurement may lower the average work.

Figure 1(b) shows the N dependence of the survival probability $p^N(0, \tau/2|0, -\tau/2)$ for N equally spaced intermediate measurements of $H(t)$, and two sets of model parameters. Oscillatory behavior is observed for small values of N , while as N increases the asymptotic value 1 is approached (see the inset). When the measurement frequency is high enough, the unitary dynamics between subsequent measurements becomes increasingly suppressed until the dynamics is completely frozen, and consequently the survival probability reaches the asymptotic value 1 for $N \rightarrow \infty$. This phenomenon is known as the quantum Zeno effect [21–24]. We investigate it further in the following section.

A. Quantum Zeno effect

To formally elucidate the quantum Zeno effect observed in this particular example, and under more general conditions as well, we analyze the form of the characteristic function given by Eq. (17) in the limit of infinitely many measurements of energy. We approach this limit by considering a finite number of N intermediate measurements of energy that take place at equally spaced instants,

$$t_k = t_0 + k\varepsilon, \quad k = 1, \dots, N, \quad (35)$$

where $\varepsilon = (t_f - t_0)/(N + 1)$ denotes the time elapsing between two subsequent measurements. We denote the corresponding protocol with the symbol ν , that is,

$$\nu = \{H(t), [t_k, H(t_k)]\}. \quad (36)$$

Since we are interested in the limiting case of infinitely many measurements, we may choose N sufficiently large such that the Hamiltonian between two subsequent measurements can safely be approximated by its value at the later measurement, i.e., $H(t) \approx H(t_k)$ for $t \in (t_{k-1}, t_k)$ with $t_k = t_0 + k\varepsilon$ being the instant of the k th measurement. The time evolution within such a short period then becomes

$$U_{t_k, t_{k-1}} \simeq e^{-iH(t_k)\varepsilon/\hbar}. \quad (37)$$

According to Eq. (19) the time evolution $U_{t_k, t_{k-1}}^\dagger$ acts on the projection operator $\Pi_{r_k}(t_k)$, resulting in the phase factor $e^{i\alpha_{r_k}(t_k)\varepsilon/\hbar}$ while the complex-conjugate phase factor is obtained from the product $\Pi_{r_k}(t_k)U_{t_k, t_{k-1}}$, which appears to the right of the operator X . In slight deviation from our previous notation, Eq. (6), we denote the eigenvalues of $\mathcal{A}_k = H(t_k)$ by $\alpha_r(t_k)$ and the respective projection operators by $\Pi_r(t_k)$. Hence,

these phase factors cancel each other and one obtains for the v -propagated exponential operator $X = e^{iuH(\tau)}$ the expression

$$[e^{iuH(\tau)}]_v = \sum_{r_1, r_2, \dots, r_N} \Pi_{r_1}(t_1) \Pi_{r_2}(t_2) \cdots \times \Pi_{r_N}(t_N) e^{iuH(\tau)} \Pi_{r_N}(t_N) \cdots \times \Pi_{r_2}(t_2) \Pi_{r_1}(t_1). \quad (38)$$

Assuming first that $H(t)$ is nondegenerate at any time $t \in (t_0, t_f)$, all projection operators are of the one-dimensional form, $\Pi_{r_k}(t_k) = |\psi_{r_k}(t_k)\rangle\langle\psi_{r_k}(t_k)|$, with instantaneous eigenfunctions $|\psi_{r_k}(t_k)\rangle$ of the Hamiltonian $H(t_k)$. The labeling of the instantaneous eigenstates can be arranged in such a way that the eigenvalues $\alpha_k(t)$ are continuous functions of time. In other words, each adiabatic energy branch is labeled by an index r . The scalar products of eigenfunctions at neighboring times then deviate from a Kronecker delta by terms of the order ε^2 , which consequently can be neglected, i.e., states on different adiabatic energy branches at neighboring times are almost orthogonal:

$$\langle\psi_{r_k}(t_k)|\psi_{r_{k+1}}(t_{k+1})\rangle = \delta_{r_k, r_{k+1}} + O(\varepsilon^2). \quad (39)$$

Herewith the left-hand side of Eq. (38) simplifies to read in the limit of infinitely many dense measurements

$$[e^{iuH(\tau)}]_v = \sum_r |\psi_r(t_0)\rangle\langle\psi_r(t_f)| e^{iuH(t_f)} |\psi_r(t_f)\rangle\langle\psi_r(t_0)|. \quad (40)$$

For the generating function this yields

$$G_v^\infty(u) = \sum_r e^{iu[\alpha_r(t_f) - \alpha_r(t_0)]} e^{-\beta\alpha_r(t_0)} / Z_0. \quad (41)$$

We therefore find that in the Zeno limit the characteristic function of work and accordingly the pdf of work coincide with the respective expressions obtained for an adiabatic protocol in the absence of any intermediate measurement.

The same line of reasoning also applies for the case that an observable that does not change in time is repeatedly measured, i.e., for a protocol

$$\mu = \{H(t), (t_k, \mathcal{A})\}. \quad (42)$$

For such an observable \mathcal{A} with eigenprojection operators Π_r , one then obtains in the Zeno limit for the characteristic function

$$G_\mu^\infty(u) = \text{Tr} \sum_r (\Pi_r e^{iuH(t_f)} \Pi_r) e^{-iuH(t_0)} e^{-\beta H(t_0)} / Z_0. \quad (43)$$

1. Landau-Zener

In the case of the Landau-Zener problem studied in the previous section, Eq. (31), the eigenvalues on each adiabatic branch coincide at the beginning and the end of the protocol, and therefore the coefficients of u in the exponential terms on the right-hand side of Eq. (41) vanish, leading to

$$G_v^\infty(u) = 1. \quad (44)$$

This leads to the expected result that under permanent observation of the Hamiltonian no work is done, i.e.,

$$P_v^\infty(w) = \delta(w). \quad (45)$$

The Zeno effect sets in when the measurement frequency is so large that any unitary evolution within the time leaps between two subsequent measurements can be neglected, hence for $\varepsilon\sqrt{v^2\tau^2/4 + 4\Delta/\hbar} \ll 1$, or equivalently if $N \gg \tau\sqrt{v^2\tau^2/4 + 4\Delta/\hbar}$. With the parameters used in Fig. 1(b), the Zeno effect sets in for $N \gg 10^3$, as shown in the inset of Fig. 1(b).

In the case of degeneracy, Eq. (41) continues to hold if the instantaneous energy branches are labeled such that the corresponding instantaneous eigenvalues smoothly vary along each branch. For the Landau-Zener problem a degeneracy happens at $t = 0$ if the coupling strength Δ vanishes. One then finds the characteristic function

$$G_v^\infty(u) = \frac{\cosh[(\beta + 2iu)v\tau/2]}{\cosh(\beta v\tau/2)} \quad [\Delta = 0], \quad (46)$$

yielding, for the pdf of work,

$$P_v^\infty(w) = p_- \delta(w - v\tau) + p_+ \delta(w + v\tau) \quad [\Delta = 0], \quad (47)$$

where $p_\pm = e^{\mp\beta v\tau/2} / [2 \cosh(\beta\tau/2)]$ denote the thermal populations of the ground and the excited states at the initial time $t = -\tau/2$, respectively.

In the case of continuous measurement of the Pauli operators σ_i , $i = x, y, z$, the driving-measurement protocols are

$$\mu_i = \{H(t), (t_k, \sigma_i)\}, \quad i = x, y, z. \quad (48)$$

The characteristic functions of work then become

$$G_{\mu_x}^\infty(u) = \frac{1}{2} \left(1 - \frac{\Delta^2}{q^2}\right) (p_- e^{2iqu} + p_+ e^{-2iqu}) + \frac{1}{2} \left(1 + \frac{\Delta^2}{q^2}\right),$$

$$G_{\mu_y}^\infty(u) = \frac{1}{2} (p_- e^{2iqu} + p_+ e^{-2iqu} + 1), \quad (49)$$

$$G_{\mu_z}^\infty(u) = \frac{1}{2} \left(1 + \frac{\Delta^2}{q^2}\right) (p_- e^{2iqu} + p_+ e^{-2iqu}) + \frac{1}{2} \left(1 - \frac{\Delta^2}{q^2}\right),$$

where p_\pm denotes the populations of the excited and the ground states of the initial Hamiltonian, respectively, given by

$$p_\pm = e^{\mp\beta q} / (e^{\beta q} + e^{-\beta q}), \quad (50)$$

with

$$q = \sqrt{(v\tau/4)^2 + \Delta^2}. \quad (51)$$

The work pdf follows then, by means of an inverse Fourier transform, to read

$$P_{\mu_x}^\infty = \frac{1}{2} \left(1 - \frac{\Delta^2}{q^2}\right) [p_- \delta(w - 2q) + p_+ \delta(w + 2q)] + \frac{1}{2} \left(1 + \frac{\Delta^2}{q^2}\right) \delta(w),$$

$$P_{\mu_y}^\infty = \frac{1}{2} [p_- \delta(w - 2q) + p_+ \delta(w + 2q) + \delta(w)], \quad (52)$$

$$P_{\mu_z}^\infty = \frac{1}{2} \left(1 + \frac{\Delta^2}{q^2}\right) [p_- \delta(w - 2q) + p_+ \delta(w + 2q)] + \frac{1}{2} \left(1 - \frac{\Delta^2}{q^2}\right) \delta(w).$$

IV. DISCUSSION

We provided an alternative proof of the robustness of the Tasaki-Crooks fluctuation theorem against repeated measurements [11]. The present proof, however, is more general in regard to the fact that it allows for possibly degenerate eigenvalues of the measured observables. Further, it was found that the characteristic function of work assumes the form of a quantum two-time correlation function of the exponentiated initial Hamiltonian, and the exponentiated final Hamiltonian in the Heisenberg picture generated by the unitary evolution interrupted by quantum collapses.

Our proof keeps holding also if quantum measurements are done with respect to positive operator-valued measures (POVMs) [25] which are less invasive than those described by von Neumann projection-valued measures. For the case of measurements with respect to a POVM,

$$\sum_r M_r M_r^\dagger = \mathbb{1}, \quad (53)$$

the interrupted evolution $[X]_\sigma$ is obtained from Eq. (19), with the projection operators Π_r on the left-hand side of X being replaced by the weak measurement operators M_r , and those to the right of X by the adjoint operators M_r^\dagger . Due to Eq. (53), $\text{Tr}[X]_\sigma = \text{Tr} X$ continues to hold, implying the validity of the Tasaki-Crooks relation, Eq. (22), for weak measurement of time-reversal invariant observables with $M_k \Theta = \Theta M_k$.

In proving Eq. (22) we assumed that the Hamiltonian and the measured observables commute with the time-reversal operator. This assumption may be relaxed and the Tasaki-Crooks fluctuation theorem continues to hold if the backward protocol is defined as

$$\tilde{\sigma} = \{\Theta H(\tau - t) \Theta^\dagger, (\tau - t_i, \Theta A_i \Theta^\dagger)\}. \quad (54)$$

Notwithstanding the robustness of the Tasaki-Crooks theorem, we found, in accordance with the intuitive expectation, that the interruption of the dynamics of a driven quantum system by means of projective measurements alters the statistics of work performed on the system. We illustrated the influence

of measurements on the work distribution by the example of the Landau-Zener(-Stückelberg-Majorana) model. We noticed that, depending on the positions in time or frequency of one or more interrupting measurements, the average work $\langle w \rangle$ may be lowered (Fig. 1).

Schmiedl *et al.* [26] studied the problem of designing optimal protocols that minimize the average work spent during the forcing, without considering intermediate measurements. In order to further minimize the average work, one might expand the optimization parameter space and include the possibility of performing intermediate measurements. This opens for the possibility of a much greater control over the energy flow into the system.

In the limit of very frequent measurements the unitary dynamics becomes completely suppressed due to the quantum Zeno effect. If at each measurement time the instantaneous Hamiltonian is measured, then in the Zeno limit the work characteristic function approaches the same form that it would assume if the protocol was adiabatic.

The result of an intermediate measurement may be used to alter the subsequent force protocol. In this way a feedback control can be implemented for classical [27] as well as for quantum systems [12]. In both cases the Jarzynski equality only holds in a modified form.

Landau-Zener dynamics and frequent quantum measurements of a two-level system coupled to a thermal bath were recently shown to provide efficient means for quantum-state preparation [28] and purification [29], respectively. These are crucial prerequisites for the implementation of working quantum computers. The combined study of dissipative Landau-Zener dynamics with frequent observations could unveil unique practical methods for quantum-state control and manipulation.

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