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Radially symmetric critical points of nonconvex functionals

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Abstract

We investigate critical points of the functional

$$E(u) = \int_{B_R(0)} W(\nabla u) + G(u)\, dx$$

over a ball in $\mathbb{R}^n$*. Here,* W *is radially symmetric but not convex. We embed the functional into a family of functionals*

$$E_{\varepsilon,\lambda}(u) = \int_{B_R(0)} \frac{\varepsilon}{2}(\Delta u)^2 + W(\lambda, \nabla u) + G(u)\, dx$$

where $E_{0,0}(u) = E(u)$*. A global bifurcation analysis yields a branch of nontrivial critical points depending on* λ *and positive* ε *where we can set* $\lambda = 0$*. The geometric properties preserved on that branch, due to the maximum principle, prove compactness such that the singular limit as* $\varepsilon \searrow 0$ *exists. Under natural conditions on* W *and* G *the critical point obtained in this way is a minimizer of the original functional. That programme can be carried out only under the restriction of radial symmetry since the maximum principle applies only to special elliptic equations of fourth order. That restriction, however, is not essential since every minimizer of the functional is radially symmetric.*

Keywords: Nonconvex variational problem, radial symmetry, singular perturbation, global bifurcation
MSC: 35B25, 35B32, 35J40

1 Introduction

We investigate critical points of the functional

$$E(u) = \int_{B_R(0)} W(\nabla u) + G(u)\, dx \tag{1.1}$$

over the ball $B_R(0) \subset \mathbb{R}^n$ with radius R and center 0. The functions W and G are smooth satisfying certain growth conditions. Moreover, we assume that W depends only on the euclidean norm of ∇u and that W is not convex. Consequently the direct methods of the calculus of variations are not applicable and the Euler-Lagrange equation for critical points is not elliptic. Therefore the existence of critical points, in particular of minimizers, is not at all obvious even if the functional is coercive and bounded from below.
One possibility to overcome the difficulties is given by a singular perturbation

$$E_\varepsilon(u) = \int_{B_R(0)} \frac{\varepsilon}{2}(\Delta u)^2 + W(\nabla u) + G(u)\, dx \quad \text{for } \varepsilon > 0. \tag{1.2}$$

Since (1.2) has a uniformly elliptic Euler–Lagrange equation the chances to prove the existence of critical points are considerably increased. Moreover, if critical points of (1.2) converge as ε tends to zero, then one can hope for critical points of (1.1).
For the first step we embed (1.2) into a family of functionals

$$E_{\varepsilon,\lambda}(u) = \int_{B_R(0)} \frac{\varepsilon}{2}(\Delta u)^2 + W(\lambda, \nabla u) + G(u)\, dx \tag{1.3}$$

where $E_{\varepsilon,0}(u) = E_\varepsilon(u)$. Apart from its aforementioned technical benefits, the functional (1.2) is motivated by physical models where the function W is nonconvex and does not attain its minimum at a single point. A related model for example emerges in the theory of liquid crystals [8]. Moreover, (1.3) can be considered as a model for the energy of an elastic thin film attached to a planar substrate, where u is the vertical displacement of the film, neglecting in–plane displacements. In this model, ε is corresponds to the thickness of the film, and a possible physical meaning of λ is the temperature: If the substrate and the elastic film respond to heating with different rates of expansion, changes in temperature can cause a transition from convex W to nonconvex W, the latter corresponding to the case where the film on the substrate is compressed along the plane. If $G = 0$, one expects the formation of microstructure for small ε as investigated in [12]. An interesting open problem in this context is to obtain the Γ–limit of (1.2) (after scaling with the factor $\varepsilon^{-\frac{1}{2}}$), see [4] for partial results and further references. Our attention is focused on a different setting, however, because our assumptions on G (with prototype $G(u) = -u^2$) would correspond to an external force pulling the film away from the substrate.

A global bifurcation analysis gives us a branch of nontrivial critical points of (1.3) depending on λ and ε and where we can set $\lambda = 0$. The benefits of that continuation in λ are not only existence but also geometric properties of critical points: Due to the maximum principle the shape of solutions is essentially preserved along a continuum. That knowledge, in turn, yields the key for compactness of critical points as $\varepsilon \searrow 0$.
In order to carry out that programme we have to impose a restriction: The maximum

principle applies only to special elliptic equations of fourth order. The Euler-Lagrange equation of (1.3) is not admissible, in general, unless it is restricted to radially symmetric functions.

That restriction, however, is not essential: If the infimum of (1.1) is attained and if W is radially symmetric and coercive then at least one minimizer is radially symmetric as shown in [10]. Furthermore, if W is radially symmetric, if G is decreasing on $[0, \infty)$ and $G(-\mu) \geq G(\mu)$ for every $\mu > 0$, and if the functional (1.1) is coercive, then a positive minimizer of (1.1) does exist. The first result in this direction was obtained in [3], assuming additionally that G is convex. This assumption can be dropped if G is of class C^2, see [9] (or [10]). In both cases, strict monotonicity of G on $[0, \infty)$ implies radial symmetry of every minimizer. (For convex G, the minimizer is unique, cf. [3].) Also note that the general result of [2] can be applied to obtain existence (but not symmetry) of a minimizer of (1.1) under our assumptions on W below and fairly general conditions on G.

By another result of [9], we know that every positive minimizer has the geometric properties of a critical point obtained by our method described above. As a consequence, our uniqueness result, Theorem 8.2 below, guarantees that we really end up with a minimizer of (1.1), provided a positive minimizer exists. The benefits of our approach compared to a mere existence result are the following: the analysis can be mimicked by a numerical simulation, which already has been employed successfully for the Cahn–Hilliard equation [11]: a pathfollowing device along the branch of critical points of (1.3) starting at a specific bifurcation point with linear positive eigenfunction gives for $\lambda = 0$ a good approximation of the minimizer of (1.1) provided $\varepsilon > 0$ is small enough. Moreover, our approach yields a critical point of (1.1) also in cases where existence of a global minimizer of (1.1) is unknown, i.e., if no assumptions on the shape of G are made on $(-\infty, 0)$.

The method presented here is a generalization of the one–dimensional case studied in [6] and it has been worked out in [9].

2 Preliminaries

For every $\xi \in \mathbb{R}^n, n \geq 2$, and $\lambda \in \mathbb{R}$ we assume the following:

(W_0)	$W(\lambda, \xi) = \frac{1}{2}\lambda^2 \lvert\xi\rvert^2 + W_0(\xi)$	(Dependence on λ)
(W_1)	$W_0(\xi) = \tilde{W}_0(\lvert\xi\rvert)$	(Symmetry)
(W_2)	$\tilde{W}_0 : \mathbb{R} \to \mathbb{R}$ is of class C^3 and even	(Regularity)
(W_3)	$\nabla W_0(\xi) \cdot \xi = \tilde{W}_0'(\lvert\xi\rvert)\lvert\xi\rvert \geq \nu_1 \lvert\xi\rvert^p - C$	(Coercivity)
(W_4)	$\lvert\nabla W_0(\xi)\rvert = \lvert\tilde{W}_0'(\lvert\xi\rvert)\rvert \leq \nu_2 \lvert\xi\rvert^{p-1} + C$	(Growth)
(W_5)	$\frac{1}{t}\tilde{W}_0'(t) \leq \tilde{W}_0''(t)$ for all $t \in \mathbb{R}\backslash\{0\}$, $\tilde{W}_0''(0) < 0$	(Shape)

Here, $p > 1, \nu_2 \geq \nu_1 > 0$, and $C > 0$ are real constants. The assumptions on G are:

(G_1)	$G : \mathbb{R} \to \mathbb{R}$ is of class C^3	(Regularity)
(G_2)	$\lvert G'(\mu)\rvert \leq \nu_3 \lvert\mu\rvert^{\tilde{p}-1} + C \quad$ for $\mu \in \mathbb{R}$	(Growth)
(G_3)	$G'(\mu) \leq 0$ for $\mu \geq 0 \quad$ and $G'(0) = 0$, $G''(0) < 0$	(Shape)

where $1 \leq \tilde{p} < p$ and $\nu_3 \geq 0$. Typical examples are

$$\tilde{W}_0(t) = (t^2 - 1)^2, \; G(\mu) = -\mu^2, \tag{2.1}$$

with $p = 4$ and $\tilde{p} = 2$. Note, however, that G is not necessarily concave.

The Euler-Lagrange equation of (1.3) in its weak form is

$$\int_{B_R(0)} \varepsilon \Delta u \Delta \varphi + \nabla W(\lambda, \nabla u) \cdot \nabla \varphi + G'(u)\varphi \, dx = 0, \tag{2.2}$$

for all test functions $\varphi \in C^\infty(\overline{B}_R(0))$. Here ∇W denotes the gradient of W with respect to its second variable. Equation (2.2) makes sense for $u \in W^{2,1}(B_R(0)) \cap W^{1,q}(B_R(0))$ with $q = \max\{p-1, 1\}$, and elliptic regularity theory gives more for any solution of (2.2) in $W^{2,2}(B_R(0)) \cup W^{1,p}(B_R(0))$. We do not go into the details since we confine ourselves to radially symmetric functions for which stronger regularity results are valid.

Denoting $\tilde{W}(\lambda, |\xi|) = \frac{1}{2}\lambda^2|\xi|^2 + \tilde{W}_0(|\xi|)$ the Euler-Lagrange equation of (1.3) for radially symmetric functions in its weak form is

(2.3)
$$\int_0^R [\varepsilon \Delta u \Delta \varphi + \tilde{W}'(\lambda, u')\varphi' + G'(u)\varphi] r^{n-1} dr = 0$$
for all radially symmetric test functions $\varphi \in C^\infty(\overline{B_R(0)})$, where
$$\Delta u = u'' + \frac{n-1}{r} u',$$

and where $\tilde{W}'$ denotes the derivative of $\tilde{W}$ with respect to its second variable.
Radially symmetric functions can be considered as functions over $B_R(0)$ or over $(0, R)$, the transition is simply given by $u(x) = \tilde{u}(|x|) = \tilde{u}(r)$. As mentioned before, the integral (2.2) makes sense only if the functions u and φ have a certain regularity. For radially symmetric functions the integral over the ball $B_R(0)$ reduces to an integral over the interval $(0, R)$, differential operators of order greater than one have a singularity at $r = 0$, and the integrand is multiplied by the weight r^{n-1}. Therefore it is not convenient to give the regularity of the functions u and φ for the existence of (2.3) in terms of the variable r (the usual Sobolev spaces are not adequate), but it seems to be more natural to give the conditions when they are considered as functions over the ball $B_R(0)$. In our notation here and in the sequel we identify u and $\tilde{u}$.

For radially symmetric functions $u \in W^{2,1}(B_R(0)) \cap W^{1,q}(B_R(0))$ we obtain $|\nabla u| = |u'| \in L^q_r(0, R)$ and $|\Delta u| = |u'' + \frac{n-1}{r} u'| \in L^1_r(0, R)$, where the subscript r denotes the weight r^{n-1}.

Thus $u' \in L^q(r_0, R)$ and $u \in C[r_0, R]$ for any $0 < r_0 < R$. By $G'(u) \in C[r_0, R]$, $\tilde{W}'(\lambda, u') \in L^1(r_0, R)$, $u'', u', \Delta u \in L^1(r_0, R)$, the fundamental lemma of the calculus of variations in one dimension applies to (2.3) and gives the regularity $u \in C^3(0, R]$. Thus $\tilde{W}'(\lambda, u') \in C^1(0, R)$ which implies finally $u \in C^4(0, R]$ such that u satisfies the strong Euler-Lagrange equation in the classical sense:

$$\varepsilon \Delta^2 u - \frac{1}{r^{n-1}} \frac{d}{dr}[r^{n-1}\tilde{W}'(\lambda, u')] + G'(u) = 0 \quad \text{for } r \in (0, R]. \tag{2.4}$$

In the sequel we impose the geometric boundary condition $u(R) = 0$. Choosing a test function φ with support in $(0, R], \varphi(R) = 0$, and $\varphi'(R) = 1$, then (2.3) and (2.4) give the natural boundary condition. We summarize:

$$\begin{aligned} &u(R) = 0 \text{ (geometric boundary condition)},\\ &\Delta u(R) = u''(R) + \frac{n-1}{R} u'(R) = 0 \ \text{ (natural boundary condition)}. \end{aligned} \tag{2.5}$$

For radially symmetric solutions in $W^{1,q}(B_R(0))$ with $q = \max\{1, \tilde{p} - 1\}$ we also obtain a natural boundary condition at $r = 0$: Choose a test function φ that is identically equal to 1 on $[0, r_0]$ for some $r_0 \in (0, R)$ and satisfies $\varphi(R) = 0$. Then (2.3) yields via integration by parts for any $s \in (0, r_0)$

$$\begin{aligned} &\int_s^R [\varepsilon \Delta u \Delta\varphi + \tilde{W}'(\lambda, u')\varphi' + G'(u)\varphi] r^{n-1} dr + \int_0^s G'(u) r^{n-1} dr \\ &= r^{n-1}[\varepsilon \tfrac{d}{dr}\Delta u(r) - \tilde{W}'(\lambda, u')]|_{r=s} + \int_0^s G'(u) r^{n-1} dr = 0. \end{aligned} \tag{2.6}$$

By (G_2) $G'(u) \in L^1(B_R(0))$ and the limit $s \searrow 0$ gives a boundary condition at $r = 0$, namely,

$$\lim_{r \to 0} r^{n-1}[\varepsilon \frac{d}{dr} \Delta u(r) - \tilde{W}'(\lambda, u'(r))] = 0. \tag{2.7}$$

By integration of (2.4) over $[s, x] \subset (0, R]$,

$$\begin{aligned} &\int_s^x \tfrac{d}{dr}\left(r^{n-1}[\varepsilon \tfrac{d}{dr}\Delta u - \tilde{W}'(\lambda, u')]\right) + G'(u) r^{n-1} dr \\ &= r^{n-1}[\varepsilon \tfrac{d}{dr}\Delta u - \tilde{W}'(\lambda, u')]|_{r=s}^{r=x} + \int_s^x G'(u) r^{n-1} dr = 0, \end{aligned} \tag{2.8}$$

we obtain in the limit $s \searrow 0$ by (2.7) an integrated version of (2.4).

$$r^{n-1}[\varepsilon \frac{d}{dr} \Delta u - \tilde{W}'(\lambda, u')] + \int_0^r G'(u(s)) s^{n-1} ds = 0, \tag{2.9}$$

for $r \in (0, R]$.

3 A priori estimates

A first a priori estimate is valid for any solution of the Euler-Lagrange equation in its weak form, irrespective of its symmetries.

Proposition 3.1 *Any solution* $u \in W^{2,2}(B_R(0)) \cap W_0^{1,p}(B_R(0))$ *of*

$$\int_{B_R(0)} \varepsilon \Delta u \Delta \varphi + \nabla W(\lambda, \nabla u) \cdot \nabla \varphi + G'(u)\varphi \, dx = 0 \tag{3.1}$$

for all $\varphi \in W^{2,2}(B_R(0)) \cap W^{1,p}(B_R(0))$ *(where* ∇W *is the gradient of* W *with respect to its second variable) satisfies*

$$\varepsilon \int_{B_R(0)} (\Delta u)^2 dx + \lambda^2 \int_{B_R(0)} |\nabla u|^2 dx + \int_{B_R(0)} |\nabla u|^p dx \leq C \tag{3.2}$$

with a constant $C > 0$ *not depending on* u, ε*, and* λ*. In particular*

$$\|u\|_{W^{2,2}(B_R(0))} \leq \frac{\tilde{C}}{\sqrt{\varepsilon}} \quad \text{and} \quad \|u\|_{W^{1,p}(B_R(0))} \leq \tilde{C} \tag{3.3}$$

for some uniform constant $\tilde{C} > 0$*.*

Proof. Using $\varphi = u$ as a test function in (3.1) we infer from (W_3) and (G_2)

$$\varepsilon \textstyle\int (\Delta u)^2 dx + \lambda^2 \int |\nabla u|^2 dx + \nu_1 \int |\nabla u|^p dx \leq \nu_3 \int |u|^{\tilde{p}} dx + C_1, \tag{3.4}$$

where the integration is over $B_R(0)$. Using Poincaré's and Hölder's inequality the right hand side of (3.4) is estimated as

$$\nu_3 \textstyle\int |u|^{\tilde{p}} dx + C_1 \leq C_2 \left(\int |\nabla u|^p \right)^{\tilde{p}/p} + C_1 \tag{3.5}$$

with an exponent $\tilde{p}/p < 1$. Moreover, there is a constant C_3 such that

$$-\frac{\nu_1}{2} t + C_2 t^{\tilde{p}/p} + C_1 \leq C_3 \quad \text{for all} \quad t \geq 0. \tag{3.6}$$

Inserting $t = \int |\nabla u|^p dx$ gives (3.2). The assertions (3.3) follow by an elliptic a priori estimate for the Laplacian and Poincaré's inequality. □

By $(W_0) - (W_2)$ and (G_3) we have $\nabla W(\lambda, 0) = 0$ and $G'(0) = 0$ such that

$$u \equiv 0 \quad \text{is the trivial solution of (3.1).} \tag{3.7}$$

Proposition 3.2 *There are two constants* $\zeta_i > 0$ *such that* $u \equiv 0$ *is the unique solution of (3.1) in* $W^{2,2}(B_R(0)) \cap W_0^{1,p}(B_R(0))$ *whenever* $\lambda^2 \geq \zeta_1$ *or* $\varepsilon \geq \zeta_2$*.*

Proof. By $\nabla W_0(0) = 0$ and $G'(0) = 0$ we can replace (W_3) and (G_2) by

$$
\begin{aligned}
&\nabla W_0(\xi)\cdot\xi \geq \nu_1|\xi|^p - \tilde{C}|\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^n,\\
&|G'(\mu)\mu| \leq \nu_3|\mu|^{\tilde{p}} + \tilde{C}\mu^2 \quad \text{for} \quad \mu \in \mathbb{R},
\end{aligned} \tag{3.8}
$$

with some constant $\tilde{C}$. Then (3.1) with $\varphi = u$ implies

$$
\begin{aligned}
&\varepsilon \textstyle\int (\Delta u)^2 dx + \lambda^2 \int |\nabla u|^2 dx + \nu_1 \int |\nabla u|^p dx\\
&\leq \tilde{C} \textstyle\int (|\nabla u|^2 + u^2) dx + \nu_3 \int |u|^{\tilde{p}} dx.
\end{aligned} \tag{3.9}
$$

By the elliptic a priori estimate

$$
\textstyle\int (\Delta u)^2 dx \geq \eta \|u\|^2_{W^{2,2}} \geq \eta \int |\nabla u|^2 dx \tag{3.10}
$$

for some $\eta > 0$ (3.9) implies

$$
\begin{aligned}
&(\lambda^2 + \varepsilon\eta) \textstyle\int |\nabla u|^2 dx + \nu_1 \int |\nabla u|^p dx\\
&\leq \tilde{C}_1 \textstyle\int (|\nabla u|^2 dx + \tilde{C}_2 \int |\nabla u|^{\tilde{p}} dx,
\end{aligned} \tag{3.11}
$$

where Poincaré's inequality is used to estimate the right hand side.
Since $\tilde{p} < p$ we can find a constant $\zeta > 0$ such that

$$
\tilde{C}_1 t^2 + \tilde{C}_2 |t|^{\tilde{p}} \leq \zeta t^2 + \frac{\nu_1}{2}|t|^p \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.12}
$$

Then (3.11) implies $u \equiv 0$ whenever $\zeta \leq \lambda^2 + \varepsilon\eta$. □

The next proposition sharpens $(3.3)_2$ considerably for radially symmetric solutions of (3.1) that are slightly more regular.

Proposition 3.3 *Any radially symmetric solution* $u \in W^{2,2}(B_R(0)) \cap W_0^{1,p}(B_R(0)) \cap W^{1,\infty}(B_R(0))$ *of (3.1) (or of (2.4)) satisfies*

$$
|u'(r)| \leq C \quad \text{for all} \quad r \in (0, R] \tag{3.13}
$$

with a constant $C > 0$ *not depending on* $u, \varepsilon,$ *and* λ.

Proof. By the boundedness of $|u'|$ we can choose a function

$$
h_\beta(r) = \alpha_0 r^{-\beta} + |u'(R)|\ , 0 < \beta \leq \frac{1}{2}, \tag{3.14}
$$

where $\alpha_0 = \alpha_0(\beta) \geq 0$ is minimal such that h_β touches $|u'|$:

$$
\begin{aligned}
|u'(r)| &\leq h_\beta(r) \quad \text{for all} \quad r \in (0, R],\\
|u'(r_0)| &= h_\beta(r_0) \quad \text{for some} \quad r_0 \in (0, R].
\end{aligned} \tag{3.15}
$$

We show that $r_0 \in (0, R)$. By

$$
h'_\beta(R) + \tfrac{n-1}{R} h_\beta(R) = \alpha_0(n - 1 - \beta)R^{-\beta-1} + \tfrac{n-1}{R}|u'(R)| \tag{3.16}
$$

we see that $(\alpha_0, u'(R)) \neq (0,0)$ (otherwise $u' \equiv 0$) and therefore, in view of $n-1-\beta > 0$ and $\Delta u(R) = 0$, cf. (2.5),

$$h'_\beta(R) + \tfrac{n-1}{R} h_\beta(R) \pm (u''(R) + \tfrac{n-1}{R} u'(R)) > 0. \tag{3.17}$$

Assume first $u'(r_0) \le 0$. If $r_0 = R$ then $(h_\beta + u')(R) = 0$ by $(3.15)_2$, $(h_\beta + u')'(R) > 0$ by (3.17), contradicting $h_\beta + u' \ge 0$ on $(0, R]$ by $(3.15)_1$.
Therefore $r_0 \in (0, R)$ whence

$$\begin{aligned} &(h_\beta + u')(r_0) = (h_\beta + u')'(r_0) = 0 \quad \text{and} \\ &(h_\beta + u')''(r_0) \ge 0. \end{aligned} \tag{3.18}$$

This, in turn, means

$$\begin{aligned} &\tfrac{d}{dr}\Delta u(r_0) \ge -h''_\beta(r_0) - \tfrac{n-1}{r_0} h'_\beta(r_0) + \tfrac{n-1}{r_0^2} h_\beta(r_0) \\ &= \alpha_0(\beta+1)(n-1-\beta) r_0^{-\beta-2} + \tfrac{n-1}{r_0^2}|u'(R)| > 0, \end{aligned} \tag{3.19}$$

and in view of the integrated Euler-Lagrange equation (2.9),

$$-r_0^{n-1}\tilde{W}'(\lambda, u'(r_0)) \le \int_0^{r_0} |G'(u(s))| s^{n-1} ds. \tag{3.20}$$

If $u'(r_0) \ge 0$ replace u by $-u$ and obtain

$$r_0^{n-1}\tilde{W}'(\lambda, u'(r_0)) \le \int_0^{r_0} G'(u(s)) s^{n-1} ds. \tag{3.21}$$

By the oddness $\tilde{W}'(\lambda, -\xi) = -\tilde{W}'(\lambda, \xi)$ and the coercivity (W_3), we end up in both cases with

$$\begin{aligned} &r_0^{n-1}(\nu_1 |u'(r_0)|^{p-1} - \tilde{C}) \le \int_0^{r_0} |G'(u(s))| s^{n-1} ds \\ &\text{where } |u'(r_0)| = h_\beta(r_0). \end{aligned} \tag{3.22}$$

Next we derive an estimate for the right hand side of (3.22):

$$\begin{aligned} &|u(s)| \le \int_0^R |u'(r)| dr \le \int_0^R h_\beta(r) dr \\ &= \tfrac{1}{1-\beta}\alpha_0 R^{1-\beta} + R|u'(R)| \le 2\alpha_0(R+1) + R|u'(R)|, \end{aligned} \tag{3.23}$$

whence

$$|G'(u(s))| \le C_1(\alpha_0 + |u'(R)|)^{\tilde{p}-1} + C \quad \text{by} \quad (G_2). \tag{3.24}$$

The constant C_1 depends only on R that is fixed. Using $r_0 < R$ and $r_0^{-\beta} > R^{-\beta} \ge \min\{1, R^{-\frac{1}{2}}\}$ for $0 < \beta \le \frac{1}{2}$ the estimates (3.22) and (3.24) imply

$$C_2(\alpha_0 + |u'(R)|)^{p-1} \le C_3(\alpha_0 + |u'(R)|)^{\tilde{p}-1} + C_4. \tag{3.25}$$

Since $\tilde{p} < p$ this last inequality can only be true if

$$0 < \alpha_0(\beta) + |u'(R)| \le C_5 \quad \text{for all} \quad 0 < \beta \le \frac{1}{2} \tag{3.26}$$

and for a uniform constant C_5. In view of (3.15) this means

$$
\begin{array}{ll} |u'(r)| \le \alpha_0(\beta) r^{-\beta} + |u'(R)| & \text{for all } 0 < \beta \le \frac{1}{2} \text{ or} \\ |u'(r)| \le C_5 \quad \text{for} & r \in (0, R]. \end{array} \tag{3.27}
$$

$\square$

Corollary 3.4 *Any radially symmetric solution $u \in W^{2,2}(B_R(0)) \cap W_0^{1,p}(B_R(0)) \cap W^{1,\infty}(B_R(0))$ of (3.1) is in $W^{4,q}(B_R(0))$ for every $q \in (1,\infty)$, and moreover*

$$
\|u\|_{W^{4,q}(B_R(0))} \le C, \tag{3.28}
$$

where C depends only on ε, λ, and q, and is bounded for bounded λ and ε^{-1}. If $u \in C^{2,\alpha}(\overline{B_R(0)})$ for an $\alpha \in (0,1)$, then we know that $u \in C^{4,\alpha}(\overline{B_R(0)})$.

Proof. We apply elliptic regularity theory to (2.2) or (3.1). By Proposition 3.3 all solutions u are uniformly bounded in $W^{1,\infty}(B_R(0))$ and thus in $W^{1,q}(B_R(0))$ for all $q \in (1,\infty)$. By (W_2) and (G_2) the functions $\nabla W(\lambda, \nabla u)$ and $G'(u)$ are in $L^\infty(B_R(0))$. Therefore $u \in W^{3,q}(B_R(0))$, and thus in $C^{2,\alpha}(\overline{B_R(0)})$ provided q is large enough. Rewriting (2.2) as

$$
\int_{B_R(0)} \varepsilon \Delta u \Delta \varphi = \int_{B_R(0)} [div(\nabla W(\lambda, \nabla u)) - G'(u)]\varphi \, dx, \tag{3.29}
$$

the inhomogeneity of the right hand side is in $L^q(B_R(0))$ for all $q \in (1,\infty)$. Furthermore, it is in $C^\alpha(\overline{B_R(0)})$ provided $u \in C^{2,\alpha}(\overline{B_R(0)})$. Hence the claims of Corollary 3.4 follow. $\square$

4 Bifurcation analysis

We formulate (2.2) as

$$
F_\varepsilon(\lambda, u) = 0 \qquad \text{for} \qquad (\lambda, u) \in \mathbb{R} \times X \tag{4.1}
$$

such that local and global bifurcation theory applies to (4.1). We define a Hölder space

$$
X = C^{3,\alpha}(\overline{B_R(0)}) \cap \left\{ \begin{array}{l} u \text{ is radially symmetric,} \\ u(R) = \Delta u(R) = 0 \end{array} \right\} \tag{4.2}
$$

endowed with the usual Hölder norm of $C^{3,\alpha}(\overline{B_R(0)})$, and the mapping

$$
\begin{array}{l} F_\varepsilon : \mathbb{R} \times X \to X \quad \text{by} \\ F_\varepsilon(\lambda, u) = u + \frac{1}{\varepsilon} K(\lambda, u), \text{where} \\ K(\lambda, u) = \Delta^{-2}(-div[\nabla W(\lambda, \nabla u)] + G'(u)). \end{array} \tag{4.3}
$$

Note that $\Delta^2 : W^{4,q}(B_R(0)) \cap \{u(R) = \Delta u(R) = 0\} \to L^q(B_R(0))$ is an isomorphism, where we choose $q \in (1,\infty)$ large enough such that $W^{4,q}(B_R(0))$ is compactly embedded into $C^{3,\alpha}(\overline{B_R(0)})$. For this reason and by the assumptions on W and G the mapping $K : \mathbb{R} \times X \to X$ is completely continuous. Observe that any solution of (4.1) solves the

Euler-Lagrange equation (2.3) with the boundary conditions (2.5), as well as the strong form of (2.2), i.e.,

$$\varepsilon\Delta^2 u - div[\nabla W(\lambda, \nabla u)] + G'(u) = 0 \text{ in } B_R(0), \tag{4.4}$$

since $u \in C^4(\overline{B_R(0)})$ by Corollary 3.4. As already mentioned in (3.7) we have for all $\lambda \in \mathbb{R},\ \varepsilon > 0$

$$F_\varepsilon(\lambda, 0) = 0, \text{ i.e. } \quad u \equiv 0 \text{ is the trivial solution.} \tag{4.5}$$

In the following analysis we fix $\varepsilon > 0$. For bifurcation from the trivial solution line $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ the linearization

$$D_u F_\varepsilon(\lambda, 0)v = v + \frac{1}{\varepsilon}\Delta^{-2}(-\tilde{W}''(\lambda, 0)\Delta v + G''(0)v) = 0 \tag{4.6}$$

has to have nontrivial solutions $v \in X$, or equivalently,

$$\begin{aligned} &\varepsilon\Delta^2 v - \tilde{W}''(\lambda, 0)\Delta v + G''(0)v = 0 \quad \text{in} \quad B_R(0), \\ &v(R) = \Delta v(R) = 0 \end{aligned} \tag{4.7}$$

has to have nontrivial classical radially symmetric solutions v. Let $0 < \mu_0 < \mu_1 < \cdots < \mu_k < \cdots$ be the simple eigenvalues of the negative Laplacian $-\Delta$ over $B_R(0)$ with radially symmetric eigenfunctions $v_0, v_1, ..., v_k, ...$ satisfying homogeneous Dirichlet boundary conditions $v_k(R) = 0$. Then (4.7) has a nontrivial solution v if and only if $\lambda \in \mathbb{R}$ fulfills the characteristic equation

$$\varepsilon\mu_k^2 + \tilde{W}''(\lambda, 0)\mu_k + G''(0) = 0 \quad \text{for some} \quad k \in \mathbb{N}_0. \tag{4.8}$$

By (W_0) we have $\tilde{W}''(\lambda, 0) = \lambda^2 + \tilde{W}_0''(0)$ and $\tilde{W}_0''(0) < 0$, $G''(0) < 0$ is assumed in $(W_5), (G_3)$. Therefore, for any fixed μ_k, equation (4.8) has precisely two solutions $\pm\lambda_k$ provided $\varepsilon > 0$ is small enough. Moreover, the total number of solutions $\pm\lambda_0, \pm\lambda_1, ..., \pm\lambda_k, ..., \pm\lambda_N$ is finite and the number $N = N(\varepsilon)$ tends to infinity as $\varepsilon \searrow 0$. Finally,

$$\begin{aligned} &\dim Ker(D_u F_\varepsilon(\pm\lambda_k, 0)) = 1,\ Ker(D_u F_\varepsilon(\pm\lambda_k, 0)) = span[v_k], \\ &D^2_{u\lambda} F_\varepsilon(\pm\lambda_k, 0)v_k = \pm\tfrac{2\lambda_k}{\varepsilon\mu_k} v_k \notin R(D_u F_\varepsilon(\pm\lambda_k, 0)) \end{aligned} \tag{4.9}$$

by the symmetry of $D_u F_\varepsilon(\pm\lambda_k, 0)$, such that the local bifurcation theorem with one-dimensional kernel of Crandall and Rabinowitz is applicable at each $(\pm\lambda_k, 0)$. For details we refer to [7].

For a global continuation of the local bifurcating curves we apply the global bifurcation theorem of Rabinowitz. To this purpose it has to be shown that the local Leray-Schauder degree (= index) of $F_\varepsilon(\lambda, \cdot)$ at the trivial solution $u = 0$ jumps at $\pm\lambda_k$. This, in turn, follows from an odd crossing number of its linearization $D_u F_\varepsilon(\lambda, 0)$ at $\pm\lambda_k$. In our case the local Morse index changes by one: For λ near $\pm\lambda_k$, the simple eigenvalue 0 of $D_u F_\varepsilon(\pm\lambda_k, 0)$ is perturbed to $\nu = \nu(\lambda)$ that satisfies

$$\varepsilon(1 - \nu)\mu_k^2 + \tilde{W}''(\lambda, 0)\mu_k + G''(0) = 0. \tag{4.10}$$

By the monotonicity of $\tilde{W}''(\lambda, 0)$ near $\pm\lambda_k$ the eigenvalue ν changes sign when λ passes through $\pm\lambda_k$.

The global alternative is refined for bifurcation with one-dimensional kernel. By (4.9) the bifurcating branch is locally a smooth curve of the form

$$\begin{aligned}&\{(\lambda(s), u(s)) \mid |s| < \delta\} \subset \mathbb{R} \times X,\ \lambda(0) = \pm\lambda_k,\\ &u(0) = 0,\ u'(0) = v_k,\end{aligned} \tag{4.11}$$

and each part $\{(\lambda(s), u(s)) \mid -\delta < s < 0\}$ and $\{(\lambda(s), u(s)) \mid 0 < s < \delta\}$ has a global extension $\mathcal{C}^-_{\pm\lambda_k}$ and $\mathcal{C}^+_{\pm\lambda_k}$ in the set of nontrivial solutions of $F_\varepsilon(\lambda, u) = 0$. We summarize:

Theorem 4.1 *Let $\pm\lambda_k$ be the two solutions of the characteristic equation (4.8) for fixed $\varepsilon > 0$. Then $(\pm\lambda_k, 0)$ are bifurcation points of two global continua $\mathcal{C}^\pm_{\pm\lambda_k}$ of nontrivial solutions of $F_\varepsilon(\lambda, u) = 0$ in $\mathbb{R} \times X$, respectively, and each of $\mathcal{C}^-_{\pm\lambda_k}$ and $\mathcal{C}^+_{\pm\lambda_k}$ satisfies*

(i) $\mathcal{C}^\pm_{\pm\lambda_k}$ is unbounded, or

(ii) $\mathcal{C}^\pm_{\pm\lambda_k}$ meets the trivial solution line $\{(\lambda, 0)\}$ at a different bifurcation point, or

(iii) $\mathcal{C}^\pm_{\pm\lambda_k}$ contains a pair of points $(u, \lambda), (-u, \lambda)$, where $u \neq 0$.

All details about Theorem 4.1 can be found in [7].

As a matter of fact, $\mathcal{C}^+_{\lambda_0} = \mathcal{C}^+_{-\lambda_0}$ connects $(-\lambda_0, 0)$ and $(\lambda_0, 0)$ (see Theorem 5.4), but for $k \geq 1$ the continua $\mathcal{C}^+_{+\lambda_k}$ and $\mathcal{C}^+_{-\lambda_k}$ might differ.

Remark 4.2 *A similar bifurcation analysis can be carried out for fixed λ and with $\varepsilon > 0$ as bifurcation parameter. If ε_k is a solution of the characteristic equation (4.8) for fixed λ then $(\varepsilon_k, 0)$ is a bifurcation point for nontrivial solutions of $F_\varepsilon(\lambda, u) = 0$ subject to analogous alternatives given in Theorem 4.1.*

Remark 4.3 *If $\tilde{W}_0$ and G are regular enough, the bifurcation formulas given in [7] can be evaluated to determine the local behaviour of the bifurcation curves near the bifurcation point. For the example (2.1) the following turns out: all bifurcations are pitchfork bifurcations which are supercritical at $-\lambda_k < 0$, subcritical at $\lambda_k > 0$, and subcritical at $\varepsilon_k > 0$.*

5 A global branch of positive solutions

In this section we have a closer look at $\mathcal{C}^+_0 := \mathcal{C}^+_{\lambda_0}$ emanating at $(\lambda_0, 0)$ where $\lambda_0 > 0$ solves (4.8) with $\mu_0 > 0$ which is the first eigenvalue of $-\Delta$ with positive radially symmetric eigenfunction v_0. The case of $\mathcal{C}^+_{-\lambda_0}$ can be treated analogously.

Theorem 5.1 *Let $(\lambda, u) \in \mathcal{C}_0^+$. Then*

(i) $u > 0$ *and* $\Delta u = u'' + \dfrac{n-1}{r} u' < 0$ *in* $B_R(0)$,

(ii) $u' < 0$ *in* $\overline{B}_R(0) \backslash \{0\}$, *and* $\dfrac{d}{dr} \Delta u(R) > 0$.

Proof. We define a cone in X as follows:

$$P = \{u \in X \big| u \text{ satisfies (i) and (ii) of Theorem 5.1}\} \tag{5.1}$$

which is open in X. By definition, $\mathcal{C}_0^+$ is connected in $\mathbb{R} \times X$ and contains the bifurcating curve $\{(\lambda(s), u(s)) | 0 < s < \delta\}$ which satisfies (4.11). Since $v_0 \in P$ the vector $u(s)/s \in P$ which is close to v_0 for small positive s. This proves that $\{(\lambda(s), u(s)) | 0 < s < \delta\} \subset \mathbb{R} \times P$ for small $\delta > 0$ or $\mathcal{C}_0^+ \cap (\mathbb{R} \times P) \neq \emptyset$.
By definition $\mathcal{C}_0^+ \cap (\mathbb{R} \times P)$ is relatively open in $\mathcal{C}_0^+$. We show that it is also relatively closed in $\mathcal{C}_0^+$. Let $(\lambda_k, u_k) \in \mathcal{C}_0^+ \cap (\mathbb{R} \times P)$ converging to some $(\lambda, u) \in \mathcal{C}_0^+$ which means $u \neq 0$ and $u \in \overline{P}$. Hopf's maximum principle and boundary lemma applied to $\Delta u \leq 0$ in $B_R(0)$ and $u(R) = 0$ give $u > 0$ in $B_R(0)$ and $u'(R) < 0$. Next we use the Euler-Lagrange equation in its integrated version (2.9):

$$\begin{aligned} &\varepsilon[(u')'' + \tfrac{n-1}{r}(u')' - \tfrac{n-1}{r^2} u'] - \tfrac{\tilde{W}'(\lambda, u')}{u'} u' \\ &= -\int_0^r G'(u(s)) \left(\tfrac{s}{r}\right)^{n-1} ds \geq 0 \text{ by } (G_3) \text{ for } r \in (0, R]. \end{aligned} \tag{5.2}$$

Here $\tilde{W}'(\lambda, u')/u' \to \tilde{W}''(\lambda, 0)$ if $u' \to 0$, and by $u' \leq 0$ the sign of the coefficient of u' plays no role for the application of Hopf's maximum principle. Therefore $u' < 0$ in $\overline{B_R(0)} \backslash \{0\}$. Since $div[\nabla W(\lambda, u)] = \tilde{W}''(\lambda, u'))u'' - \frac{n-1}{|x|} \tilde{W}'(\lambda, u')$, (4.4) yields

$$\begin{aligned} &\varepsilon \Delta^2 u - \tilde{W}''(\lambda, u') u'' - \tfrac{n-1}{r} \tilde{W}'(\lambda, u') + G'(u) = 0, \quad \text{or} \\ &\varepsilon \Delta(\Delta u) - \tilde{W}''(\lambda, u') \Delta u = -G'(u) + V(x) \text{ in } B_R(0), \quad \text{where} \\ &\Delta u = u'' + \tfrac{n-1}{|x|} u', \ V(x) = \left[\tfrac{\tilde{W}'(\lambda, u')}{u'} - \tilde{W}''(\lambda, u)\right] \tfrac{n-1}{|x|} u'. \end{aligned} \tag{5.3}$$

By (G_3) we have $-G'(u) \geq 0$ and by $(W_2), (W_5)$ the function $V \geq 0$ for $u' \leq 0$. Therefore, again by Hopf's maximum principle, $\Delta u < 0$ in $B_R(0)$ (recall that $\Delta u \in C^2(\overline{B_R(0)})$; due to $\Delta u \leq 0$ the sign of $\tilde{W}''(\lambda, u)$ is irrelevant) and Hopf's boundary lemma implies $\frac{d}{dr} \Delta u(R) > 0$. This completes the proof that the limit $(\lambda, u) \in \mathcal{C}_0^+ \cap (\mathbb{R} \times P)$ or that $\mathcal{C}_0^+ \cap (\mathbb{R} \times P)$ is relatively closed in $\mathcal{C}_0^+$. By the connectedness of $\mathcal{C}_0^+$ we end up with $\mathcal{C}_0^+ = \mathcal{C}_0^+ \cap (\mathbb{R} \times P)$ proving (i) and (ii). □

Remark 5.2 *Assuming* (G_3^-) $G'(\mu) \geq 0$ *for* $\mu \leq 0$, *an analogous theorem can be proved for* $(\lambda, u) \in \mathcal{C}_0^-$, *namely* $u < 0$, $\Delta u > 0$ *in* $B_R(0)$, $u' > 0$ *in* $\overline{B}_R(0) \backslash \{0\}$, *and* $\frac{d}{dr} \Delta u(R) < 0$.

Remark 5.3 *Without radial symmetry of* u *the Euler-Lagrange equation* $\varepsilon \Delta^2 u - div[\nabla W(\lambda, \nabla u)] + G'(u) = 0$ *(cf. (2.2)) does not give an equation for* Δu *with a definite sign on the right hand side like in (5.3)*$_2$. *Recall that a maximum principle holds only for second order equations. Therefore the possibility to separate a positive branch from other branches as in Theorem 5.4 below fails in the general case.*

Theorem 5.4 *For fixed $\varepsilon > 0$, sufficiently small such that (4.8) has a solution $\lambda_0 > 0$ for $k = 0$, the continuum $\mathcal{C}_0^+$ connects $(-\lambda_0, 0)$ and $(\lambda_0, 0)$. In particular, there is a radially symmetric solution $u \in P$ of the Euler-Lagrange equation (2.2) with boundary conditions (2.5) for any $\lambda \in (-\lambda_0, \lambda_0)$.*

Proof. By Proposition 3.2 and Corollary 3.4 alternative (i) of Theorem 4.1 rules out. Since $\mathcal{C}_0^+ \subset \mathbb{R} \times P$, cf. Theorem 5.1, alternative (iii) is not possible, either. Finally, by the positivity of u for $(\lambda, u) \in \mathcal{C}_0^+$, the only different bifurcation point where $\mathcal{C}_0^+$ meets the trivial solution line is a point $(\tilde{\lambda}, 0)$ where $D_u F_\varepsilon \tilde{\lambda}, 0)$ has a positive kernel vector $\tilde{v}_0$, cf. [7]. □

Remark 5.5 *The global continuum bifurcating at $(\varepsilon_0, 0)$ according to Remark 4.2 is bounded in $[a, \infty) \times X$ for any $a > 0$, cf. Proposition 3.2 and Corollary 3.4. Since there is only one bifurcation point for positive solutions the only possibility left is that the branch in $(0, \infty) \times P$ tends to the hyperplane $\{0\} \times X$ with a blow up as $\varepsilon \searrow 0$. We have not investigated the precise rate of blowing up of the norm of X. Nonetheless there is a limit in $W^{1,q}(B_R(0))$ as shown in the next section.*

6 The singular limit

The continuum $\mathcal{C}_0^+$ connecting the two bifurction points $(\pm\lambda_0, 0)$ depends on $0 < \varepsilon < \varepsilon_0$. Since ε was fixed in the preceding analysis that dependence was suppressed. In this section we investigate its singular limit as $\varepsilon \searrow 0$. A look at the characteristic equation (4.8) reveals the convergence of the bifurcation points $\pm\lambda_0(\varepsilon) \to \pm\lambda_0(0) \neq 0$. Therefore we can study the singular limit of $(\lambda, u) = (\lambda, u_\varepsilon) \in \mathcal{C}_0^+ = \mathcal{C}_0^+(\varepsilon)$ for any $\lambda \in (-\lambda_0(0), \lambda_0(0))$, in particular for $\lambda = 0$.
Let $u_k = u_{\varepsilon_k}$ be such that $(\lambda, u_k) \in \mathcal{C}_0^+(\varepsilon_k)$ and $\varepsilon_k \searrow 0$. Then

$$\frac{d}{dr}(r^{n-1}u_k'(r)) = r^{n-1}\Delta u_k(r) < 0, \tag{6.1}$$

which means that all $z_k(r) = r^{n-1}u_k'(r)$ are monotonically decreasing on $(0, R)$. By Proposition 3.2 the sequence $(z_k)_{k\in\mathbb{N}}$ is uniformly bounded on $[0, R]$ such that Helly's theorem [5] implies the existence of a subsequence, again denoted by (z_k), having the following properties:

$$\begin{array}{l} (z_k) \text{ and therefore } (u_k') \text{ converge pointwise in } (0,R) \text{ whence} \\ u_k \to u \quad \text{in } W^{1,q}(0,R) \text{ for some } u \in W^{1,q}(0,R) \text{ and for any } q \in [1,\infty) \end{array} \tag{6.2}$$

by Lebesgue's dominated converge theorem and $u_k(R) = 0$. Clearly $u(R) = 0$, and by one-dimensional embedding

$$u_k \to u \text{ in } C^\alpha[0, R] \text{ and in } C^\alpha(\overline{B}_R(0)) \text{ for any } \alpha \in (0, 1). \tag{6.3}$$

From the sequence (u_k) the limit inherits the properties

$$u \geq 0 \text{ and } u \text{ is monotonically decreasing in } (0, R), \tag{6.4}$$

and finally (6.2) and the bound of $(|u_k'|) = (|\nabla u_k|)$ imply

$$u_k \to u \ \text{ in } W^{1,q}(B_R(0)) \text{ for any } \ q \in [1, \infty). \tag{6.5}$$

The convergence of the sequence (u_k) together with the uniform boundedness of $(|u_k'|)$ is strong enough to pass to the limit in the Euler-Lagrange equation (2.2):

$$
\begin{aligned}
&\int_{B_R(0)} \varepsilon_k u_k \Delta^2\varphi + \nabla W(\lambda, \nabla u_k)\cdot\nabla\varphi + G'(u_k)\varphi\, dx = 0 \\
&\text{converges to} \\
&\int_{B_R(0)} \nabla W(\lambda, \nabla u)\cdot\nabla\varphi + G'(u)\varphi\, dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(B_R(0)).
\end{aligned} \tag{6.6}
$$

Setting $\lambda = 0$ we end up with a radially symmetry critical point $u \in W^{1,p}(B_R(0))$ of the functional (1.1). However, that result is of interest only if we have shown that u is nontrivial, i.e. $u \neq 0$.

We prove this by contradiction. Assume $u = 0$.

By the monotonicity of z_k and the pointwise convergence to $z = 0$ in $(0, R)$ the convergence of (z_k), and therefore also of (u_k'), is uniform on any compact subinterval of $(0, R)$. In view of (6.3) our assumption $u = 0$ means

$$
\begin{aligned}
u_k &\to 0 \quad \text{in} \quad C[0, R], \\
u_k' &\to 0 \quad \text{in} \quad C[a, b] \text{ for any } [a, b] \subset (0, R).
\end{aligned} \tag{6.7}
$$

We denote the norm in $u \in W^{1,1}(B_R(0))$ by $\|\ \ \|_{1,1}$ and define

$$
v_k = \frac{u_k}{\|u_k\|_{1,1}} \quad \text{for all} \quad k \in \mathbb{N}. \tag{6.8}
$$

Using $\Delta v_k < 0$ we see as in (6.1) that the functions $w_k(r) = r^{n-1} v_k'(r)$ are monotonically decreasing on $(0, R)$, whence

$$
\begin{aligned}
w_k(r) &\geq \tfrac{1}{R-r} \int_r^R w_k(s) ds \geq -\tfrac{1}{(R-r)|S^{n-1}|} \|\nabla v_k\|_{L^1(B_R(0))}, \\
w_k(r) &\leq \tfrac{1}{r} \int_0^r w_k(s) dx \leq \tfrac{1}{r|S^{n-1}|} \|\nabla v_k\|_{L^1(B_R(0))}.
\end{aligned} \tag{6.9}
$$

Since $\|v_k\|_{1,1} = 1$, the sequence $(|w_k|)$ is uniformly bounded on compact subintervals of $(0, R)$. Helly's theorem allows then to select a subsequence, again denoted by (w_k), that converges pointwise to a limit function. Thus (v_k') and also (v_k) converge pointwise in $(0, R)$. The boundedness of $(|w_k|)$ on compact intervals of $(0, R)$ implies also the boundedness of $(|v_k'|)$ and of $(|v_k|)$ such that Lebesgue's dominated convergence theorem yields

$$
v_k \to v \ \text{ in } W^{1,1}_{loc}(B_R(0)\backslash\{0\}) \text{ for some } \ v. \tag{6.10}
$$

Finally, by the compact embedding of $W^{1,1}(B_R(0))$ into $L^1(B_R(0))$ we can also assume that

$$
v_k \to v \quad \text{in} \quad L^1(B_R(0)). \tag{6.11}
$$

As in (6.4) the limit function v satisfies

$$
v \geq 0 \quad \text{and } v'(r) \leq 0 \text{ for almost all} \quad r \in (0, R). \tag{6.12}
$$

Next we show that $v \neq 0$.
To this purpose we claim

$$\|u_k'\|_{L^1(B_R(0))} \leq \frac{2n}{R}\|u_k\|_{L^1(B_R(0))} \tag{6.13}$$

which entails

$$\|v_k\|_{L^1(B_R(0))} \geq \frac{R}{2n+R} \quad \text{for all } k \in \mathbb{N} \text{ and } v \neq 0 \tag{6.14}$$

by (6.11). For a proof of (6.13), observe that integration by parts yields

$$0 \leq -\int_r^R u_k'(s)s^{n-1}ds = u_k(r)r^{n-1} + (n-1)\int_r^R u_k(s)s^{n-2}ds, \tag{6.15}$$

and by integration over $(0, R)$,

$$\begin{aligned} & -\int_0^R u_k(r)r^{n-1}dr + (n-1)\int_0^R\int_r^R u_k(s)s^{n-2}dsdr \\ & = \int_0^R u_k(r)r^{n-1}dr + (n-1)\int_0^R\int_0^s u_k(s)s^{n-2}drds \\ & = \tfrac{n}{|S^{n-1}|}\|u_k\|_{L^1(B_R(0))}. \end{aligned} \tag{6.16}$$

On the other hand,

$$\begin{aligned} & g_k(r) := -\int_r^R u_k'(s)s^{n-1}ds \geq 0, \ g_k(R) = 0, \\ & g_k'(r) = u_k'(r)r^{n-1} \leq 0, \ g_k'(0) = 0, \\ & g_k''(r) = r^{n-1}\Delta u_k(r) < 0, \end{aligned} \tag{6.17}$$

which shows that g_k is concave and maximal at $r = 0$. Therefore its integral over $(0, R)$ is larger than the area of the triangle with vertices $(0, 0)$, $(0, g_k(0))$, $(R, 0)$:

$$\int_0^R g_k(r)dr \geq \tfrac{R}{2}g_k(0) = -\tfrac{R}{2}\int_0^R u_k'(r)r^{n-1}dr = \tfrac{R}{2|S^{n-1}|}\|u_k'\|_{L^1(B_R(0))}, \tag{6.18}$$

and (6.15) shows

$$\int_0^R g_k(r)dr \leq \tfrac{n}{|S^{n-1}|}\|u_k\|_{L^1(B_R(0))} \tag{6.19}$$

which proves (6.13).
>From $(6.6)_1$ and (6.8) we obtain for radially symmetric test functions

$$\int_{B_R(0)} \varepsilon_k v_k \Delta^2\varphi + \frac{\tilde{W}'(\lambda, u_k')}{u_k'}v_k'\varphi' + \frac{G'(u_k)}{u_k}v_k\varphi \, dx = 0, \tag{6.20}$$

and in view of (6.7), (6.10) we can pass to the limit $k \to \infty$ for all radially symmetric test functions satisfying $\varphi' = 0$ in $B_a(0)$, and the limit is given by

$$\int_{B_R(0)} \tilde{W}''(\lambda, 0)v'\varphi' + G''(0)v\varphi \, dx = 0, \tag{6.21}$$

cf. (6.7). Choosing test functions such that $\varphi = 1$ on $[0, a]$, $\varphi > 0$, $\varphi' < 0$ on (a, b), and $\varphi = 0$ on $[b, R]$ then by (6.12), (W_5), and (G_3) equation (6.21) shows the following:

$$\text{If } \tilde{W}''(\lambda, 0) = \lambda^2 + \tilde{W}_0''(0) \le 0, \text{ then} \quad G''(0)v\varphi = 0 \text{ almost everywhere in}(0, R). \tag{6.22}$$

Since $G''(0) < 0$ and $b < R$ is arbitrary this implies $v = 0$, a contradiction to (6.14). This proves the following theorem:

Theorem 6.1 *The singular limit set*

$$\mathcal{C}_0^+(0) := \left\{ (\lambda, u) \in \mathbb{R} \times W_0^{1,p}(B_R(0)) \;\middle|\; \begin{array}{l} \text{There exists a sequence } (\varepsilon_k, \lambda_k, u_k) \\ \text{with } (\lambda_k, u_k) \in \mathcal{C}_0^+(\varepsilon_k) \text{ such that} \\ \varepsilon_k \searrow 0, \lambda_k \to \lambda, u_k \to u \text{ in } W_0^{1,p}(B_R(0)) \end{array} \right\} \tag{6.23}$$

is nonempty for $\lambda \in (-\lambda_0(0), \lambda_0(0))$ *where* $\pm\lambda_0(0)$ *are solutions of* $(\lambda^2 + \tilde{W}_0''(0))\mu_0 + G''(0) = 0$*, cf. (4.8),* $\mathcal{C}_0^+(0)$ *consists of nonnegative radially symmetric critical points of the functional*

$$E_{0,\lambda}(u) = \int_{B_R(0)} W(\lambda, \nabla u) + G(u)dx \tag{6.24}$$

which are nontrivial for $\lambda \in [-[-\tilde{W}_0''(0)]^{\frac{1}{2}}, +[-\tilde{W}_0''(0)]^{\frac{1}{2}}]$*. Moreover, the set* $\mathcal{C}_0^+(0)$ *is connected in* $\mathbb{R} \times W_0^{1,p}(B_R(0))$ *and contains the two points* $(\pm\lambda_0(0), 0)$*. In this sense it is a continuum of critical points bifurcating from the trivial line* $\{(\lambda, 0)\} \subset \mathbb{R} \times W_0^{1,p}(B_R(0))$*.*

The connectedness of $\mathcal{C}_0^+(0)$ follows from the results of [1]. In the next section we give more properties of the nontrivial critical points on $\mathcal{C}_0^+(0)$.

7 Geometric properties of radially symmetric critical points

By construction as a singular limit the elements of $\mathcal{C}_0^+(0)$ have more regularity than stated in Theorem 6.1: If $(\lambda, u) \in \mathcal{C}_0^+(0)$ then $u \in C^\alpha(\overline{B}_R(0))$ for any $\alpha \in (0, 1)$ and $u(R) = 0$ in the classical sense, cf. (6.3). We show more regularity and geometric properties for $(0, u) \in \mathcal{C}_0^+(0)$. Note that u is a critical point of the original functional (1.1):

$$E(u) = E_{0,0}(u) = \int_{B_R(0)} W_0(\nabla u) + G(u)dx. \tag{7.1}$$

In addition to $(W_1) - (W_5)$ we assume for $W_0(\nabla u) = \tilde{W}_0(|\nabla u|)$ that $\tilde{W}_0$ is a typical two-well potential of type (2.1), more precisely,

(W_6) $\quad \tilde{W}_0'' > 0$ on $\mathbb{R}\backslash[-\tilde{M}, \tilde{M}]$ and $\tilde{W}_0'' < 0$ on $(-\tilde{M}, \tilde{M})$ with an $\tilde{M} > 0$.

Hence $\tilde{W}_0'$ has exactly 3 zeros, namely $+M, -M$ and 0, where $\pm M$ are the Maxwell points where $\tilde{W}_0$ attains its global minimum.

The Euler-Lagrange equation of (7.1) for radially symmetric functions in its weak form is

$$\int_0^R [\tilde{W}_0'(u')\varphi' + G'(u)\varphi]r^{n-1}dr = 0 \tag{7.2}$$

for radially symmetric test functions $\varphi \in C_0^\infty(B_R(0))$, cf. (2.3). Let $u \in W_0^{1,p}(B_R(0))$ be a solution of (7.2). By one-dimensional embedding u is continuous on $(0, R]$, and the fundamental lemma of the calculus of variations in one dimension yields in view of $G'(u) \in C(0, R]$ that

$$\begin{aligned} &\tilde{W}_0'(u') \in C^1(0, R] \quad \text{and that} \\ &\frac{d}{dr}[r^{n-1}\tilde{W}_0'(u')] = r^{n-1}G'(u) \text{ holds on } (0, R] \end{aligned} \tag{7.3}$$

in the classical sense, cf. (2.4). By (2.7) and (2.9) the integrated version of (7.3) holds as well, namely,

$$r^{n-1}\tilde{W}_0'(u') = \int_0^r G'(u)s^{n-1}ds. \tag{7.4}$$

For $(0, u) \in \mathcal{C}_0^+(0)$ we know that $u \in C[0, R], u \geq 0$, and $u'(r) \leq 0$ for almost all $r \in (0, R)$, cf. (6.4). By (G_3), (7.3), and (7.4) we see that

$$\begin{aligned} &r^{n-1}\tilde{W}_0'(u'(r)) \text{ is monotonically decreasing for } r \in (0, R], \\ &\tilde{W}_0'(u') \leq 0 \text{ on } (0, R], \text{ and} \\ &\lim_{r\to 0} \tilde{W}_0'(u'(r)) = 0. \end{aligned} \tag{7.5}$$

Next we sharpen (G_3) to

(G_3') $\qquad G'(\mu) < 0 \qquad$ for $\mu > 0, \quad G'(0) = 0.$

This has two consequences:

$$\begin{aligned} u &> 0 \quad \text{on} \quad [0, R), \\ u' &< 0 \quad \text{on} \quad (0, R]. \end{aligned} \tag{7.6}$$

Assume $u'(r_0) = 0$ for some $r_0 \in (0, R]$. By (7.5) and $r_0^{n-1}\tilde{W}_0'(u'(r_0)) = 0$ we see that $r^{n-1}\tilde{W}_0'(u'(r)) = 0$ for all $r \in (0, r_0]$ which implies by (7.3) that $G'(u(r)) = 0$ for all $r \in (0, r_0]$. Since $u \geq 0$ assumption (G_3') implies $u(r) = 0$ for $r \in [0, r_0]$. If $u(r_0) = 0$ then

$$0 = u(R) - u(r_0) = \int_{r_0}^R u'(r)dr, \tag{7.7}$$

and since $u' \leq 0$ for almost all $r \in (r_0, R)$ it follows that $u'(r) = 0$ for almost all $r \in (r_0, R)$. The same preceding argument proves that $u(r) = 0$ for all $r \in [0, R]$ which contradicts $u \neq 0$ for $(0, u) \in \mathcal{C}_0^+(0)$.

Let $-M$ be the negative value where $\tilde{W}_0$ attains its minimum. By our assumption (W_6), $(7.5)_2$ and (7.6) imply that the range of u' is in $(-\infty, -M]$. The assumption (W_6) includes that $\tilde{W}_0''(t) > 0$ on $(-\infty, -M]$. This means that $\tilde{W}_0'$ has a continuously differentiable inverse $(\tilde{W}_0')^{-1} : (-\infty, 0] \to (-\infty, -M]$. Thus (7.4) can be rewritten as

$$u'(r) = (\tilde{W}_0')^{-1}\left[\frac{1}{r^{n-1}}\int_0^r G'(u)s^{n-1}ds\right] \quad \text{for} \quad r \in (0, R]. \tag{7.8}$$

Together with (7.5) the representation (7.8) shows that $u \in C^1[0, R]$, $u'(r) \le -M$ for $r \in [0, R]$, $u'(0) = -M$, $u \in C^2(0, R]$, and from (7.3) we obtain

$$\begin{aligned} &\tilde{W}_0''(u')(u'' + \frac{n-1}{r}u') - \frac{n-1}{r}H(u')u' = G'(u) \quad \text{for} \quad r \in (0, R], \\ &\text{where } H(t) := \tilde{W}_0''(t) - \frac{1}{t}\tilde{W}_0'(t) \text{ for } t \neq 0,\ H(0) := 0. \end{aligned} \tag{7.9}$$

Our assumptions on the shapes of $\tilde{W}_0$ and G, in particular (W_5), (W_6), and (G_3'), and the differential equation $(7.9)_1$ prove $\Delta u < 0$ in $(0, R)$. We summarize:

Theorem 7.1 *The nontrivial singular limit $(0, u) \in \mathcal{C}_0^+(0)$ provides a radially symmetric critical point u of (7.1) with the following properties:*

$$\begin{aligned} &u \in C^\alpha(\overline{B}_R(0)) \cap C^2(\overline{B}_R(0)\backslash\{0\}), \\ &u > 0 \ \text{ in } B_R(0),\ u(R) = 0, \\ &\Delta u < 0 \ \text{ in } B_R(0)\backslash\{0\}, \\ &u'(r) \le -M \ \text{ for } r \in [0, R],\ u'(0) = -M. \end{aligned} \tag{7.10}$$

The property of the derivative at $r = 0$ shows that the radially symmetric function u has a peak in the center of $B_R(0)$.

8 Uniqueness and minimizing property of singular limits

Under additional technical assumptions on $\tilde{W}_0$ and G, which are all fulfilled by the typical examples (2.1), we prove uniqueness of the critical point of (7.1) with the properties (7.10). These assumptions are:

$$\begin{aligned} &\beta(\rho, t) := \frac{H(\rho t)}{H(t)} \ge 1, \\ &0 < \beta(\rho, t)\tilde{W}_0''(t) < \tilde{W}_0''(\rho t) \quad \text{for} \quad \rho > 1,\ t \le -M, \\ &0 > \frac{1}{\mu}G'(\mu) \quad \text{is nondecreasing for} \quad \mu > 0. \end{aligned} \tag{8.1}$$

The function H is defined in (7.9). It is of class C^1 if we assume that $\tilde{W}_0$ is of class C^3. Note also that $\frac{1}{\mu}G'(\mu)$ extends continuously to $G''(0)$.

Theorem 8.1 *Under all assumptions on $\tilde{W}_0$ and G, including (8.1), the critical point u of (7.1) with the properties (7.10) is unique.*

Proof. Assume the existence of two such critical points u and v satisfying $u(R) = v(R) = 0$ and $u'(R) < v'(R)$. Since $(7.9)_1$ is a second order ODE the latter is necessary for two different solutions. Setting

$$\rho := \inf\{s > 0 \,|\, sv(r) \geq u(r) \ \text{ for all } \ r \in [0, R]\}, \tag{8.2}$$

we have $\rho > 1,\ \rho v \geq u$ on $[0, R]$, and there exists some $r_0 \in (0, R]$ such that

$$\rho v(r_0) = u(r_0) \quad \text{and} \quad \rho v'(r_0) = u'(r_0); \tag{8.3}$$

note that $r_0 = 0$ is excluded by $u'(0) = v'(0) = -M$. Since both u and v satisfy equation $(7.9)_1$ we obtain:

$$\begin{aligned} &\tilde{W}_0''(u')\Delta u - \beta(\rho, v')\tilde{W}_0''(v')\Delta(\rho v) - \frac{n-1}{r}[H(u')u' - \beta(\rho, v')H(v')\rho v'] \\ &= G'(u) - \beta(\rho, v')\frac{1}{v}G'(v)\rho v. \end{aligned} \tag{8.4}$$

In a neighborhood of r_0 in $(0, R]$ this reduces to a linear differential inequality for $u - \rho v$ as follows:

$$\tilde{W}_0''(u')\Delta(u - \rho v) - \frac{n-1}{r}C(r)(u' - \rho v') \geq \frac{1}{u}G'(u)(u - \rho v) \tag{8.5}$$

since $\beta(\rho, v')\tilde{W}_0''(v') \leq \tilde{W}_0''(u')$ by $(8.1)_2$ and $\Delta v < 0$,

$$H(u')u' - H(\rho v')\rho v' = C(r)(u' - \rho v') \tag{8.6}$$

with a continuous coefficient $C(r)$ since H is of class C^1, and finally since

$$\begin{aligned} &\beta(\rho, v')\frac{1}{v}G'(v) \leq \frac{1}{v}G'(v) \ \text{ by } (8.1)_1 \text{ and } v \geq 0, \\ &\frac{1}{v}G'(v) \leq \frac{1}{u}G'(u) \ \text{ by } (8.1)_3 \text{ and } v < u \text{ near } r_0. \end{aligned} \tag{8.7}$$

Since $u - \rho v \leq 0$ and $(u - \rho v)(r_0) = 0, (u - \rho v)'(r_0) = 0$, Hopf's maximum principle (or boundary lemma for $r_0 = R$) implies $(u - \rho v)(r) = 0$ for all r in a neighborhood of r_0 in $(0, R]$. Therefore the set $\{r \in (0, R] | (u - \rho v)(r) = 0\}$ is nonempty, open and closed in $(0, R]$, whence $u = \rho v$ on $(0, R]$. This contradicts $-M = u'(0) = \rho v'(0) = -\rho M$ and $\rho > 1$. □

In [9] it is shown that the functional (7.1) possesses a positive minimizer in $W_0^{1,p}(B_R(0))$ provided the potentials W_0 and G fulfill assumptions that are partially weaker than those assumed for Theorem 8.1 but also determine the shape on G on $(-\infty, 0]$, namely,

$$G(-\mu) \geq G(\mu) \quad \text{for every } \mu \geq 0. \tag{8.8}$$

Moreover, it is shown that a minimizer is radially symmetric and that under the assumptions for Theorem 8.1 it fulfills (7.10). Consequently the minimizer and the singular limit coincide.

Theorem 8.2 *Under all our assumptions on $\tilde{W}_0$ and G including (8.1) and (8.8), the singular limit $(0, u) \in \mathcal{C}_0^+(0)$ provides the unique global minimizer of the functional (7.1) on $W_0^{1,p}(B_R(0))$.*

A byproduct of uniqueness is that the limit in the definition (6.23) of $\mathcal{C}_0^+(0)$ does not depend on the choice of the sequence $(\lambda_k, u_k) \in \mathcal{C}_0^+(\varepsilon_k)$ for which $\varepsilon_k \searrow 0, \lambda_k \to 0$, and $u_k \to u$ in $W_0^{1,p}(B_R(0))$. This observation might be useful for a numerical approximation of u via a pathfollowing device along the branch $\mathcal{C}_0^+(\varepsilon_k)$ of the elliptic problem (2.2) or (2.3) with boundary conditions (2.5).

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