A Remark on Truncation Schemes of Cumulant Hierarchies

P. Hänggi[1,2] and P. Talkner[1]

We discuss some problems obtained by truncating a cumulant hierarchy at order $n > 2$. We show that such a truncation scheme is not consistent with a in its whole range of definition nonnegative probability function.

KEY WORDS: Cumulant hierarchy; nonlinear coupling; Gaussian approximation; characteristic function; quasiprobability.

In recent years it has become evident that the problem of fluctuation effects in nonlinear statistical theories is a very general one, and is presently attracting a great deal of interest. Usually the dynamical coupling between first moments and nonlinear fluctuations can be treated exactly in terms of a moment hierarchy. In this context, a widely used concept to obtain a closed system of equations is given by the Hartree–Fock or Gaussian approximation. Due to the useful mathematical properties, it is often more convenient to work in terms of cumulants.[(1)] Then, to calculate the first moments the strongest form of a truncation scheme for the resulting cumulant hierarchy is obtained by setting all cumulants c_n of higher order than one equal to zero. This amounts to the assumption that the influence of the fluctuations can be neglected entirely. The Gaussian approximation mentioned above is equivalently realized by setting all c_n with $n \geqslant 3$ equal to zero. But this approximation is known not to always give a satisfactory description of the properties as they are of importance for the study of critical dynamics or metastable states. The truncation scheme can obviously be generalized by neglecting cumulants up to the order $n > n_0 > 2$, yielding a closed set of

[1] Institut für Theoretische Physik, Universität Stuttgart, West Germany.
[2] Present address: Department of Chemistry, University of California, San Diego, La Jolla, California.

equations for the cumulants c_n, $n \leqslant n_0$. This technique has been applied to various statistical problems, such as, for example, to turbulence theory[2] and nonlinear stochastic models, which may display a phase transition.[3,4]

In this paper, we show that such a cutoff hypothesis cannot yield a consistent approximation scheme for classical systems. The reason for the inconsistency lies in the fact that for a corresponding classical probability function which is supposed to give the same cumulants up to order $n_0 > 2$ (or the corresponding moments to all orders), the semipositivity of this probability cannot hold in its whole range of definition.

Considering a one-dimensional classical process $x(t)$, we obtain with a cutoff hypothesis for $n > n_0$ for the characteristic function $\phi_t(\omega)$

$$\phi_t(\omega) = \langle \exp i\omega x(t) \rangle \tag{1}$$

$$= \exp\left\{ \sum_{j=1}^{n_0} \frac{(i\omega)^j}{j!} C_j(t) \right\} \tag{2}$$

$$= \exp P_t^{(n_0)}(\omega) \tag{3}$$

In Eq. (1) the angular brackets denote the statistical expectation over the probability function $p(xt)$. But in order for $\phi_t(\omega)$ to be a characteristic function, Marcinkiewicz[5] has proved that the polynomial $P_t^{(n_0)}(\omega)$ in Eq. (3) must be of first or second degree. Henceforth, we must conclude that if all cumulants $c_n(t)$ vanish for some $n_0 > 2$, then all cumulants vanish for $n > 2$ and hence that the probability is Gaussian.

Remark. For the set of moments generated from the truncated cumulant hierarchy at the order $n_0 \geqslant 3$, the problem of moments[6] has consequently no solution!

Nevertheless, this does not mean that a truncation scheme beyond $n_0 \geqslant 3$ represents an empty concept; it only means that the neglect of cumulants beyond a given order cannot be justified a priori. For example, this method has been applied successfully in a recent study of a nonlinear stochastic model[4] where the convergence for the first cumulants obtained by solution of the truncated hierarchy has been tested numerically. On the other hand, for the problem of the isoprobability of a pair of metastable states located at x_1 and x_2 which approaches in the thermodynamic limit the expression $\frac{1}{2}\{\delta(x - x_1) + \delta(x - x_2)\}$, a low-order truncation scheme for the calculation of the mean is expected to yield poor convergence.

Further, it is worth emphasizing that Marcinkiewicz's theorem *implies* the theorem of Pawula[7]: Considering the Kramers–Moyal expansion of a Markovian master equation with derivate moments $\{a_n(xt)\}$ and assuming that $a_n(xt)$ vanishes identically for some $n_0 > 2$, then it follows that the $\{a_n(xt)\}$ have to vanish identically for all $n > 2$.

Dealing with a quantum system, one may use a description in terms of a quasiprobability function[8] which allows for negative values, so that the above reasoning loses its validity.

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