

BUBBLES AND ONIS

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ABSTRACT. In this article we study the gradient flow equation of a variant of the Rabinowitz action functional on very negative line bundles and relate it to periodic orbits on the base of this bundle. On very negative line bundles there are generically no holomorphic spheres.

Dedicated to Paul Rabinowitz

1. INTRODUCTION

Rabinowitz pioneered the application of global methods in Hamiltonian dynamics, see [17, 18]. This motivated Weinstein to formulate his famous conjecture in [22] on the existence of periodic solutions of Hamiltonian systems with fixed energy and arbitrary period. For existence of periodic solutions of fixed period of time-dependent Hamiltonian systems Arnold formulated in the 60s his famous conjecture, see [15, Chapter 11] for a detailed account. Arnold's conjecture motivated Floer to introduce his semi-infinite dimensional Morse homology, nowadays referred to as Floer homology, more precisely as Hamiltonian Floer homology in this context. The analog of Floer homology for the fixed energy problem was introduced by Cieliebak-Frauenfelder in [3] and is referred to as **Rabinowitz Floer homology** in reference to the action functional used in Rabinowitz' fundamental article [17]. In the present article we study a variant of the Rabinowitz action functional for the fixed period problem.

Defining Hamiltonian Floer homology on general closed symplectic manifolds (M, ω) is notoriously difficult due to bubbling-off of holomorphic spheres. Formally bubbling-off of holomorphic spheres is a codimension two phenomenon, however this leads to severe transversality issues which forces the use of multi-valued perturbations and thus rational coefficients, see for example [10, 11]. On a sufficiently negative line bundle over the closed symplectic manifold holomorphic spheres generically do not exist. In this article we analyze what will happen to the bubbles.

We lift the Hamiltonian dynamics from M to the negative line bundle via a modified Rabinowitz action functional. If (M, ω) is semi-positive one gets a version of Rabinowitz Floer homology on the negative line bundle which is canonically isomorphic to the Floer homology on M . In this paper we don't want to assume semi-positivity.

We describe here a moduli space of a new PDE-type problem. We refer to solutions to this problem as onis.¹ As with holomorphic spheres onis are the obstructions to a well-defined boundary operator in Rabinowitz Floer homology. The advantage of the onis is that for generic almost complex structure one has transversality. In the second part of the paper we explain how this observation is potentially useful. To tame onis we marry them, i.e. with $\mathbb{Z}/2$ -coefficients the oni-obstruction vanishes and one obtains a well-defined boundary operator.

¹Oni (鬼) is Japanese for an imp.

So far, it seems that marrying onis is only possible if the Hamiltonian diffeomorphism ϕ we are studying admits a square root, i.e. $\phi = \psi^2$ for some other Hamiltonian ψ . Unfortunately, this is not always the case as we showed in [1]. Marrying onis apparently comes at a price: we explain that marrying onis in the worst case costs 80% of the sum of the Betti numbers of M . The undaunted reader is cordially invited to read this gloomy story.

2. FROM BUBBLES TO ONIS

In Floer homology one considers a closed symplectic manifold (M, ω) and a Hamiltonian $H \in C^\infty(M \times S^1)$ which is periodic in time. For $S^1 = \mathbb{R}/\mathbb{Z}$ the circle we denote by

$$\mathcal{L}_M \subset C^\infty(S^1, M)$$

the component of the free loop space $C^\infty(S^1, M)$ consisting of contractible loops. We consider the cover

$$\widetilde{\mathcal{L}}_M \rightarrow \mathcal{L}_M$$

consisting of equivalence classes $[v, \bar{v}]$ where $v \in \mathcal{L}_M$ and \bar{v} is a filling disk of v . Here two pairs are equivalent if the loops coincide and the integrals of ω as well as of one and hence every representative of the first Chern class $c_1(TM)$ agree on the filling disk. Therefore the group of decktransformations of the cover $\widetilde{\mathcal{L}}_M \rightarrow \mathcal{L}_M$ is the group

$$\Gamma = \frac{\pi_2(M)}{\ker \omega \cap \ker(c_1(TM))}.$$

The action functional of classical mechanics

$$\mathcal{A}_H: \widetilde{\mathcal{L}}_M \rightarrow \mathbb{R}$$

is given by

$$\mathcal{A}_H([v, \bar{v}]) = - \int \bar{v}^* \omega - \int_0^1 H(v(t), t) dt.$$

If $[v, \bar{v}]$ is a critical point of \mathcal{A}_H , then v is a contractible time-one periodic orbit of the Hamiltonian vector field of H , i.e. a solution of the ODE

$$\partial_t v(t) = X_{H_t}(v(t)), \quad t \in S^1,$$

where the Hamiltonian vector field is defined by $dH_t = \omega(X_{H_t}, \cdot)$, with $H_t = H(\cdot, t) \in C^\infty(M)$ for $t \in S^1$. The Hamiltonian H is called nondegenerate if for each contractible periodic orbit v the time-one map ϕ_H of the Hamiltonian vector field of H satisfies

$$\det(d\phi(v(0)) - \text{id}|_{T_{v(0)}M}) \neq 0.$$

If the Hamiltonian is nondegenerate there are only finitely many Γ -orbits of critical points of \mathcal{A}_H and the Floer chain space $CF_*(H)$ can be defined as the \mathbb{Z}_2 -vector space consisting of infinite sums

$$\xi = \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c c$$

with coefficients $\xi_c \in \mathbb{Z}_2$ which satisfy for any $r \in \mathbb{R}$ the finiteness condition

$$\#\{c \in \text{crit}(\mathcal{A}_H) : \xi_c \neq 0, \mathcal{A}_H(c) > r\} < \infty.$$

The grading on $CF_*(H)$ is given by the Conley-Zehnder index. Using the action of Γ on $\text{crit}(\mathcal{A}_H)$ the Floer chain space can be endowed with the structure of a module over the Novikov ring of Γ . As a matter of fact the Novikov ring over a field is itself a field.

To define a boundary operator on $CF_*(H)$ Floer considers the L^2 -gradient flow equation for the action functional of classical mechanics. We denote by \mathcal{J} circle families J_t for $t \in S^1$ of ω -compatible almost complex structures on M . For $J \in \mathcal{J}$ the metric \mathfrak{m}_J on $\widetilde{\mathcal{L}}_M$ at a point $[v, \bar{v}]$ is given for two tangent vectors $\hat{v}_1, \hat{v}_2 \in T_{[v, \bar{v}]} \widetilde{\mathcal{L}}_M = \Gamma(S^1, v^*TM)$ by

$$\mathfrak{m}_J(\hat{v}_1, \hat{v}_2) = \int_0^1 \omega(\hat{v}_1(t), J_t(v(t))\hat{v}_2(t)) dt.$$

The gradient $\nabla_J \mathcal{A}_H$ of the action functional \mathcal{A}_H with respect to the metric \mathfrak{m}_J at a point $[v, \bar{v}] \in \widetilde{\mathcal{L}}_M$ is given by

$$\nabla_J \mathcal{A}_H([v, \bar{v}]) = J(v)(\partial_t v - X_H(v)).$$

A gradient flow line $w = [v, \bar{v}] \in C^\infty(\mathbb{R}, \widetilde{\mathcal{L}}_M)$ is formally a solution of the ODE

$$\partial_s w(s) + \nabla_J \mathcal{A}_H(w(s)) = 0, \quad s \in \mathbb{R}$$

and therefore $v \in C^\infty(\mathbb{R} \times S^1, M)$ is a solution of the PDE

$$\partial_s v + J(v)(\partial_t v - X_H(v)) = 0$$

which is a perturbed holomorphic curve equation. For critical points $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ we denote by $\mathcal{M}(c_-, c_+; J)$ the moduli space of unparametrised gradient flow lines $[w]$ of \mathcal{A}_H which asymptotically satisfy $\lim_{s \rightarrow \pm\infty} w(s) = c_\pm$. By

$$\mathcal{J}_{\text{reg}}(H) \subset \mathcal{J}$$

we denote the subset of second category of $J \in \mathcal{J}$ such that the linearization of Floer's gradient flow equation with respect to J_t along any finite energy gradient flow line is surjective. If $J \in \mathcal{J}_{\text{reg}}$ then the moduli space of gradient flow lines is a smooth manifold of dimension

$$\dim \mathcal{M}(c_-, c_+; J) = \mu_{CZ}(c_-) - \mu_{CZ}(c_+) - 1.$$

In particular if $\mu_{CZ}(c_-) = \mu_{CZ}(c_+) + 1$ then the moduli space is zero dimensional. Unfortunately without additional assumptions on (M, ω) like monotonicity or more generally semipositivity there is little hope that this moduli space is also compact. This is due to bubbling of holomorphic spheres. If it is compact then it is a finite set of points and one sets

$$n(c_-, c_+, J) = \#_2 \mathcal{M}(c_-, c_+; J)$$

where $\#_2$ denotes cardinality modulo two. In this case the Floer boundary map

$$\partial: CF_*(H) \rightarrow CF_{*-1}(H)$$

is defined for $\xi = \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c c \in CF_*(H)$ by

$$\partial(\xi) = \sum_{c' \in \text{crit}(\mathcal{A}_H)} \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c n(c, c'; J) c'.$$

If the moduli spaces are not compact then one way to still define the Floer boundary operator is by using abstract perturbation theory. In this case one first compactifies the moduli space via bubbles and then abstractly perturbs it. Because of the possibility that the bubbles have nontrivial automorphism group this perturbation is multivalued and one has to restrict in this case to rational coefficients.

In this paper we study a different kind of perturbation of the moduli spaces of Floer's gradient flow equation which also leads to compact moduli spaces. We have to assume in

addition that (M, ω) satisfies the Bohr-Sommerfeld condition, i.e. $[\omega]$ lies in the image of $H^2(M; \mathbb{Z}) \rightarrow H_{dR}^2(M)$. Under this assumption there exists a hermitian line bundle

$$E_\omega \rightarrow M$$

whose first Chern class satisfies

$$c_1(E_\omega) = -[\omega].$$

For $\nu \in \mathbb{N}$ we consider the hermitian line bundle

$$E = E_\omega^\nu = E_\omega^{\otimes \nu}$$

whose first Chern class satisfies

$$c_1(E) = -\nu[\omega].$$

We fix a hermitian connection α on E whose curvature satisfies

$$F_\alpha = \nu\omega.$$

If $p: E \rightarrow M$ denotes the canonical projection, we endow E with the symplectic form

$$\omega_E = d(\pi|u|^2\alpha) + p^*\omega.$$

Thinking of M as the zero section in E the restriction of ω_E to M coincides with ω . However, the virtual dimension of the moduli space of holomorphic curves drops in E , because the line bundle is negative. If the line bundle is negative enough generically there are no holomorphic curves left in E . To have a quantitative statement to say what "negative enough" means we consider the Auroux constant. If $\beta \in \Omega^2(M)$ and J is an ω -compatible almost complex structure we set

$$\gamma_{\beta, J} = \beta(\cdot, J\cdot) \in \Gamma(T^*M \oplus T^*M)$$

and abbreviate

$$\kappa_\beta(J) = \|\gamma_{\beta, J}\|_J$$

where the notation $\|\cdot\|_J$ means that we take the norm with respect to the metric $\omega(\cdot, J\cdot)$. We then set

$$\kappa(J) = \inf \{ \kappa_\beta(J) : d\beta = 0, [\beta] = c_1(TM) \}$$

and finally

$$\kappa(\omega) = \inf \{ \kappa(J) : J \text{ } \omega\text{-compatible} \}.$$

We assume in the following that $\nu \in \mathbb{N}$ satisfies

$$\nu > \max \{ n + \kappa(\omega) - 2, \kappa(\omega) \}.$$

This condition turns out to be sufficient to make sure that there are ω -compatible almost complex structures for which no holomorphic spheres exist.

We denote by $\mu \in C^\infty(E)$ the function

$$\mu(u) = \pi(|u|^2 - 1), \quad u \in E.$$

Denote by \mathcal{L}_E the component of contractible loops in the free loop space $C^\infty(S^1, E)$ and by $\widetilde{\mathcal{L}}_E$ the cover of \mathcal{L}_E consisting of equivalence classes of pairs $[u, \bar{u}]$ where $u \in \mathcal{L}_E$, $\bar{u} \in C^\infty(D, E)$ with D the unit disk is a filling disk of u , and two filling disks are equivalent if ω_E and each representative of $p^*c_1(TM)$ agrees on them. For $H \in C^\infty(M \times S^1)$ the time-dependent Hamiltonian on the base M , we denote for $t \in S^1$

$$\widehat{H}_t(u) = \nu\pi|u|^2 H_t(p(u)), \quad u \in E$$

the fiberwise quadratic lift of H to E . We consider the following variant of Rabinowitz action functional

$$\mathcal{A}_H^\mu: \widetilde{\mathcal{L}}_E \times \mathbb{R} \rightarrow \mathbb{R}$$

given for $([u, \bar{u}], \eta) \in \widetilde{\mathcal{L}}_E \times \mathbb{R}$ by

$$\mathcal{A}_H^\mu([u, \bar{u}], \eta) = - \int_D \bar{u}^* \omega_E - \int_0^1 \widehat{H}_t(u) dt - \eta \int_0^1 \mu(u) dt.$$

If $P: \widetilde{\mathcal{L}}_E \times \mathbb{R} \rightarrow \widetilde{\mathcal{L}}_M$ denotes the projection induced from the projection $p: E \rightarrow M$, then

$$P(\text{crit}(\mathcal{A}_H^\mu)) = \text{crit}(\mathcal{A}_H).$$

However, the correspondence is not one to one, but for each $c \in \text{crit}(\mathcal{A}_H)$ there is a whole $\mathbb{Z} \times S^1$ -family of critical points of $\text{crit}(\mathcal{A}_H^\mu)$, i.e.

$$P|_{\text{crit}(\mathcal{A}_H^\mu)}^{-1}(c) \cong \mathbb{Z} \times S^1, \quad c \in \text{crit}(\mathcal{A}_H).$$

The geometric origin of this fact is the following. The circle acts on E by

$$r * u = e^{-2\pi i r} u, \quad r \in S^1 = \mathbb{R}/\mathbb{Z}, \quad u \in E.$$

In fact this action is Hamiltonian with respect to ω_E with moment map μ . This action gives rise to a $\mathbb{Z} \times S^1$ -action on \mathcal{L}_E which is given for $u \in \mathcal{L}_E$ by

$$((n, r)_* u)(t) = (nt + r)_* u(t), \quad t \in S^1, \quad (n, r) \in \mathbb{Z} \times S^1.$$

We lift this action to the cover $\widetilde{\mathcal{L}}_E \rightarrow \mathcal{L}_E$ and extend it trivially to $\widetilde{\mathcal{L}}_E \times \mathbb{R}$. The differential $d\mathcal{A}_H^\mu$ is invariant under this action and hence we get a $\mathbb{Z} \times S^1$ -action on $\text{crit}(\mathcal{A}_H^\mu)$. The projection P induces a bijection

$$\frac{\text{crit}(\mathcal{A}_H^\mu)}{\mathbb{Z} \times S^1} \cong \text{crit}(\mathcal{A}_H).$$

A section for the \mathbb{Z} -action on $\text{crit}(\mathcal{A}_H^\mu)$ is given by the *winding number*

$$\mathfrak{w}: \text{crit}(\mathcal{A}_H^\mu) \rightarrow \mathbb{Z}$$

which for a critical point $([u, \bar{u}], \eta)$ is given by

$$\mathfrak{w}([u, \bar{u}], \eta) = \int u^* \alpha - \nu \int \bar{u}^* p^* \omega.$$

The connection α induces a splitting

$$(1) \quad TE = \mathcal{V} \oplus \mathcal{H}$$

into vertical and horizontal subbundles. Recalling that $p: E \rightarrow M$ denotes the canonical projection, then for each $e \in E$ we have canonical identifications

$$\mathcal{H}_e = T_{p(e)}M, \quad \mathcal{V}_e = E_{p(e)}.$$

We denote by I the complex structure in \mathcal{V} coming from the complex structure of the hermitian vector bundle $E \rightarrow M$. For $J \in \mathcal{J}$ we denote by abuse of notation its lift to a family of complex structure on \mathcal{H} also by J . We extend the vector bundle $\text{End}(\mathcal{H}, \mathcal{V})$ over E trivially to $E \times S^1$. For a section $B \in \Gamma_0(E \times S^1, \text{End}(\mathcal{H}, \mathcal{V}))$ we abbreviate

$$B_t = B(\cdot, t) \in \Gamma_0(E, \text{End}(\mathcal{H}, \mathcal{V})), \quad t \in S^1.$$

Since H is non-degenerate X_H has only finitely many periodic orbits. From now on we fix around each periodic orbit a neighborhood which contracts onto the orbit. Furthermore, we

assume that all these neighborhoods are disjoint. Let \mathcal{U} be the union of these neighborhoods. If Γ_0 stands for sections with compact support we introduce the vector space

$$(2) \quad \mathfrak{B}(J) = \left\{ B \in \Gamma_0(E \times S^1, \text{End}(\mathcal{H}, \mathcal{V})) : B_t J_t = -I B_t, t \in S^1 \text{ and } B_t(e) = 0 \forall e \text{ with } p(e) \in \mathcal{U} \right\}.$$

Even though $\mathfrak{B}(J)$ depends on \mathcal{U} we will suppress this in the notation.

With respect to the splitting (1) we introduce for $B \in \mathfrak{B}(J)$ the following circle family of almost complex structures on E

$$J_t^B = \begin{pmatrix} I & B_t \\ 0 & J_t \end{pmatrix}.$$

If B is different from zero then J^B is not ω_E -compatible, however if B is a small enough perturbation then J^B is still ω_E -tame. We therefore introduce the nonempty open convex subset

$$\mathfrak{B}^T(J) \subset \mathfrak{B}(J)$$

consisting of all $B \in \mathfrak{B}(J)$ such that J_t^B is ω_E -tame for any $t \in S^1$. If J_t is a smooth family of ω -compatible almost complex structures and B_t is a smooth family of compactly supported sections from E to $\text{End}(\mathcal{H}, \mathcal{V})$ such that $B_t \in \mathfrak{B}(J_t)$ for every $t \in S^1$ we denote by \mathfrak{m}_B the bilinear form on $T(\widetilde{\mathcal{L}}_E \times \mathbb{R})$ which is given for $([u, \bar{u}], \eta) \in \widetilde{\mathcal{L}}_E \times \mathbb{R}$ and $(\hat{u}_1, \hat{\eta}_1), (\hat{u}_2, \hat{\eta}_2) \in T_{([u, \bar{u}], \eta)}(\widetilde{\mathcal{L}}_E \times \mathbb{R}) = \Gamma(S^1, u^*TE) \times \mathbb{R}$ by the formula

$$\mathfrak{m}_B((\hat{u}_1, \hat{\eta}_1), (\hat{u}_2, \hat{\eta}_2)) = - \int_0^1 \omega_E(J_t^{B_t}(u(t)) \hat{u}_1(t), \hat{u}_2(t)) dt + \hat{\eta}_1 \cdot \hat{\eta}_2.$$

If B is different from zero, the bilinear form \mathfrak{m}_B is not symmetric. However, it is nondegenerate, and if B is small it is positive. Denote by

$$R = X_\mu \in \Gamma(TE)$$

the infinitesimal generator of the S^1 -action on E . The gradient of Rabinowitz action functional with respect to the bilinear form \mathfrak{m}_B at $w = ([u, \bar{u}], \eta) \in \widetilde{\mathcal{L}}_E \times \mathbb{R}$ defined implicitly by the condition

$$d\mathcal{A}_H^\mu(w) = \mathfrak{m}_B(\nabla_B \mathcal{A}_H^\mu(w), \hat{w}), \quad \forall \hat{w} \in T_w(\widetilde{\mathcal{L}}_E \times \mathbb{R})$$

is given by

$$\nabla_B \mathcal{A}_H^\mu([u, \bar{u}], \eta) = \begin{pmatrix} J^B(\partial_t u - X_{\widehat{H}}(u) - \eta R(u)) \\ - \int_0^1 \mu(u) dt \end{pmatrix}.$$

We point out that $\nabla_B \mathcal{A}_H^\mu$ is not an honest gradient since \mathfrak{m}_B is not symmetric. But

$$d\mathcal{A}_H^\mu(w) \nabla_B \mathcal{A}_H^\mu(w) = \mathfrak{m}_B(\nabla_B \mathcal{A}_H^\mu(w), \nabla_B \mathcal{A}_H^\mu(w)) > 0$$

away from the critical points. Moreover \mathfrak{m}_B is a honest metric on loops contained in $p^{-1}(\mathcal{U})$. Thus, the vector field $\nabla_B \mathcal{A}_H^\mu(w)$ is a pseudo-gradient for \mathcal{A}_H^μ .

A gradient flow line $w \in C^\infty(\mathbb{R}, \widetilde{\mathcal{L}}_E \times \mathbb{R})$ of Rabinowitz action functional with respect to \mathfrak{m}_B formally satisfies

$$\partial_s w(s) + \nabla_B \mathcal{A}_H^\mu(w(s)) = 0, \quad s \in \mathbb{R}.$$

Hence $(u, \eta) \in C^\infty(\mathbb{R} \times S^1, E) \times C^\infty(\mathbb{R}, \mathbb{R})$ is a solution of the problem

$$(3) \quad \left. \begin{aligned} \partial_s u + J^B(u)(\partial_t u - X_{\widehat{H}}(u) - \eta R(u)) &= 0 \\ \partial_s \eta - \int_0^1 \mu(u) dt &= 0. \end{aligned} \right\}$$

Writing

$$\partial_s u = \partial_s^v u + \partial_s^h u, \quad \partial_t u = \partial_t^v u + \partial_t^h u$$

with respect to the splitting (1) we can rewrite (3) to

$$(4) \quad \left. \begin{aligned} \partial_s^v u + I \left(\partial_t^v u - (\nu H(p(u)) + \eta) R(u) \right) + B \left(\partial_t^h(u) - X_H(p(u)) \right) &= 0 \\ \partial_s^h u + J(p(u)) \left(\partial_t^h(u) - X_H(p(u)) \right) &= 0 \\ \partial_s \eta - \int_0^1 \mu(u) dt &= 0. \end{aligned} \right\}$$

We abbreviate by $\mathcal{R}(B)$ the moduli space of all unparametrised flow lines $[w]$ of $\nabla_B \mathcal{A}_H^\mu$ such that $\lim_{s \rightarrow \pm\infty} w(s) \in \text{crit}(\mathcal{A}_H^\mu)$ exists. We further introduce the evaluation maps

$$\text{ev}_\pm: \mathcal{R}(B) \rightarrow \text{crit}(\mathcal{A}_H^\mu)$$

defined by

$$\text{ev}_\pm([w]) = \lim_{s \rightarrow \pm\infty} w(s).$$

We denote by $\mathcal{J}^\nu \subset \mathcal{J}$ the nonempty open subset of $J \in \mathcal{J}$ such that

$$\nu > \max \{n + \kappa(J_t) - 2, \kappa(J_t)\}, \quad t \in S^1.$$

and set

$$\mathcal{J}_{\text{reg}}^\nu(H) = \mathcal{J}^\nu \cap \mathcal{J}_{\text{reg}}(H).$$

For $J \in \mathcal{J}^\nu$ we abbreviate by

$$\mathfrak{B}_{\text{reg}}(J) \subset \mathfrak{B}(J)$$

the subset of perturbations $B \in \mathfrak{B}(J)$ which satisfy the following three conditions.

- (i): The linearization of the flow equation for $\nabla_B \mathcal{A}_H^\mu$ on $\mathcal{R}(B)$ is surjective.
- (ii): The evaluation maps ev_+ and ev_- are transverse to each other.
- (iii): For each $t \in S^1$ there are no nonconstant J_t^B -holomorphic spheres on E .

Proposition 2.1. *For $J \in \mathcal{J}_{\text{reg}}^\nu$ the subset $\mathfrak{B}_{\text{reg}}(J) \subset \mathfrak{B}(J)$ is of second category.*

Recall that $\mathfrak{B}^T(J)$ denotes the set of perturbations $B \in \mathfrak{B}(J)$ such that J^B is ω_E -tame. We set

$$\mathfrak{B}_{\text{reg}}^T(J) = \mathfrak{B}^T(J) \cap \mathfrak{B}_{\text{reg}}(J).$$

We make the following definition. A *lift*

$$\ell: \text{crit}(\mathcal{A}_H) \rightarrow \text{crit}(\mathcal{A}_H^\mu)$$

is a section for the projection $P: \text{crit}(\mathcal{A}_H^\mu) \rightarrow \text{crit}(\mathcal{A}_H)$, i.e. $P \circ \ell = \text{id}|_{\text{crit}(\mathcal{A}_H)}$, satisfying

$$\mathfrak{w}(\ell(c)) = 0, \quad c \in \text{crit}(\mathcal{A}_H).$$

Note that for each $c \in \text{crit}(\mathcal{A}_H)$ there is an S^1 -ambiguity for the choice of $\ell(c)$. For a lift ℓ we introduce the following subset of $\mathcal{R}(B)$

$$\mathcal{R}(B, \ell) = \{[w] \in \mathcal{R}(B) : \text{ev}_-([w]) \in \ell(\text{crit}(\mathcal{A}_H))\}.$$

We say that a lift is *B-admissible* if the following two conditions hold.

- (i): The restriction of the linearization of the flow equation of $\nabla_B \mathcal{A}_H^\mu$ to $\mathcal{R}(B, \ell)$ is surjective.
- (ii): The evaluation map $\text{ev}_+|_{\mathcal{R}(B, \ell)}$ is transverse to ev_- .

If $B \in \mathfrak{B}_{\text{reg}}(J)$, then the evaluation maps ev_- and ev_+ are transverse to each other by assumption and hence a generic lift ℓ is B -admissible. For $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$, $J \in \mathcal{J}$, $B \in \mathfrak{B}(J)$ and a lift ℓ we abbreviate

$$\mathcal{R}(c_-, c_+; B, \ell) = \{[w] \in \mathcal{R}(B) : \text{ev}_-([w]) = \ell(c_-), \text{ev}_+([w]) \in S^1 \cdot \ell(c_+)\}.$$

The condition for the positive asymptotic can be rephrased by saying that $\text{ev}_+([w])$ is a critical point of \mathcal{A}_H^μ of winding number zero. Hence the condition for the positive asymptotic is actually independent of the choice of the lift. If $B \in \mathfrak{B}_{\text{reg}}(J)$, ℓ is a B -admissible lift, and $c_- \neq c_+ \in \text{crit}(\mathcal{A}_H)$, then the moduli space $\mathcal{R}(c_-, c_+; B, \ell)$ is a smooth manifold of dimension

$$\dim \mathcal{R}(c_-, c_+; B, \ell) = \mu_{CZ}(c_-) - \mu_{CZ}(c_+) - 1$$

and therefore coincides with the expected dimension of the moduli space of unparametrized Floer gradient flow lines from c_- to c_+ .

Theorem 2.2. *Assume that the Conley-Zehnder indices of $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ satisfy $\mu_{CZ}(c_-) = \mu_{CZ}(c_+) + 1$, that $J \in \mathcal{J}^\nu$, $B \in \mathfrak{B}_{\text{reg}}^T(J)$, and ℓ is a B -admissible lift, then the moduli space $\mathcal{R}(c_-, c_+; B)$ is a finite set.*

Under the assumptions of the theorem, we can now set

$$\varrho(c_-, c_+; B, \ell) = \#_2 \mathcal{R}(c_-, c_+; B, \ell)$$

the cardinality modulo two of the moduli spaces. If the Conley-Zehnder indices do not satisfy $\mu_{CZ}(c_-) - \mu_{CZ}(c_+) = 1$, then we set $\varrho(c_-, c_+; B, \ell) = 0$.

Proposition 2.3. *If $J \in \mathcal{J}_{\text{reg}}^\nu$, $B \in \mathfrak{B}_{\text{reg}}^T(J)$ and ℓ is a B -admissible lift, then the numbers $\varrho(c_-, c_+; B, \ell)$ are independent of the choice of ℓ .*

Hence under the assumptions of the proposition we are allowed to set

$$\varrho(c_-, c_+; B) = \varrho(c_-, c_+; B, \ell)$$

for any B -admissible lift ℓ . The following Lemma is an immediate consequence of Gromov compactness applied to the shadows $P(w)$ of flow lines of $\nabla_B \mathcal{A}_H^\mu$ by noting that these are flow lines of $\nabla_J \mathcal{A}_H$.

Lemma 2.4. *If $J \in \mathcal{J}_{\text{reg}}^\nu$, and $B \in \mathfrak{B}_{\text{reg}}^T(J)$, then for each real number $r \in \mathbb{R}$ and $c_- \in \text{crit}(\mathcal{A}_H)$ the set $\{c_+ \in \text{crit}(\mathcal{A}_H) : \mathcal{A}_H(c_+) > r, \varrho(c_-, c_+; B) \neq 0\}$ is finite.*

In view of the Lemma we can define a \mathbb{Z}_2 -linear map

$$\mathfrak{R} : CF_*(H) \rightarrow CF_{*-1}(H)$$

which is given for $\xi = \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c c \in CF_*(H)$ by

$$\mathfrak{R}(\xi) = \sum_{c' \in \text{crit}(\mathcal{A}_H)} \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c \varrho(c, c'; B) c'.$$

We refer to \mathfrak{R} as the Rabinowitz Floer map. Since the winding number is unchanged under the action of $\Gamma_0 = \pi_2(M)/\ker(\omega) = \pi_2(E)/\ker(\omega_E)$ on $\widetilde{\mathcal{L}}_E$ we conclude that the Rabinowitz Floer map is linear with respect to the action of the group ring $\mathbb{Z}_2[\Gamma_0]$ on $CF_*(H)$.

We next describe the square of the Rabinowitz Floer map. For this we need the following Definition.

Definition 2.5. Assume that $J \in \mathcal{J}$ and $B \in \mathfrak{B}(J)$. An Oni is an element $[w] \in \mathcal{R}(B)$ whose asymptotics satisfy $\mathfrak{w}(\text{ev}_-(w)) = 1$ and $\mathfrak{w}(\text{ev}_+(w)) = 0$.

We denote by $\mathcal{O}(B)$ the moduli space of Onis. If $B = 0 \in \mathfrak{B}(J)$ then M interpreted as zero section in E is a complex submanifold of J^0 . Therefore by positivity of intersections Onis cannot exist and we have

$$\mathcal{O}(0) = \emptyset.$$

But without perturbation all bubbles in M survive. In view of this the following statement makes sense: *Onis are born out of bubbles.*

If $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ we abbreviate

$$\mathcal{O}(c_-, c_+; B) = \{[w] \in \mathcal{O}(B) : P(\text{ev}_\pm(w)) = c_\pm\}.$$

If $B \in \mathfrak{B}_{\text{reg}}(J)$ the moduli space of Onis is a smooth manifold and its dimension is given by

$$\dim \mathcal{O}(c_-, c_+; B) = \mu_{CZ}(c_-) - \mu_{CZ}(c_+) - 2.$$

If $J \in \mathcal{J}^\nu$, $B \in \mathfrak{B}_{\text{reg}}^T(J)$, and $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ satisfy $\mu(c_-) - \mu(c_+) = 2$, then the same compactness arguments which lead to Theorem 2.2 also show that the moduli space of Onis $\mathcal{O}(c_-, c_+; B)$ is a finite set. Hence in this situation we set

$$\varpi(c_-, c_+; B) = \#_2 \mathcal{O}(c_-, c_+; B)$$

and define a map

$$\mathfrak{D}: CF_*(H) \rightarrow CF_{*-2}(H)$$

which is given for $\xi = \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c c \in CF_*(H)$ by

$$\mathfrak{D}(\xi) = \sum_{c' \in \text{crit}(\mathcal{A}_H)} \sum_{c \in \text{crit}(\mathcal{A}_H)} \xi_c \varpi(c, c'; B) c'.$$

We refer to the map \mathfrak{D} as the *Oni-map*. The Oni-map is the obstruction for the Rabinowitz Floer map to be a boundary operator as the following theorem shows.

Theorem 2.6. Assume that $J \in \mathcal{J}_{\text{reg}}^\nu$, $B \in \mathfrak{B}_{\text{reg}}^T(J)$, then $\mathfrak{R}^2 = \mathfrak{D}$.

3. PROOFS

3.1. Proof of Proposition 2.3. Given two B -admissible lifts $\ell_0, \ell_1: \text{crit}(\mathcal{A}_H) \rightarrow \text{crit}(\mathcal{A}_H^\mu)$ and two critical points $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ satisfying $\mu_{CZ}(c_-) = \mu_{CZ}(c_+) + 1$ we have to show that

$$(5) \quad \varrho(c_-, c_+; B, \ell_0) = \varrho(c_-, c_+; B, \ell_1).$$

Choose a smooth family $L = \{\ell_r\}_{r \in [0,1]}$ of lifts $\ell_r: \text{crit}(\mathcal{A}_H) \rightarrow \text{crit}(\mathcal{A}_H^\mu)$ interpolating between ℓ_0 and ℓ_1 . We consider the moduli space

$$\mathcal{R}(B, L) = \{(r, [w]) : r \in [0, 1], [w] \in \mathcal{R}(B, \ell_r)\}.$$

The boundary of this moduli space is given by

$$\partial \mathcal{R}(B, L) = \mathcal{R}(B, \ell_0) \sqcup \mathcal{R}(B, \ell_1).$$

Therefore to prove (5) it suffices to show that $\mathcal{R}(B, L)$ is compact. In view of compactness for the gradient flow equation of Rabinowitz action functional, see Theorem 3.4, the only obstruction to compactness is breaking of gradient flow lines. Since B is regular, and we

consider a 1-dimensional moduli problem it remains to rule out two times broken flow lines $[w^1]\#[w^2]$ of $\nabla_B\mathcal{A}_H^\mu$ for which there exists $r \in [0, 1]$ such that

$$(6) \quad \left. \begin{aligned} \text{ev}_-([w^1]) &= \ell_r(c_-), \\ \text{ev}_+([w^1]) &= \text{ev}_-([w^2]), \\ \mathfrak{w}(\text{ev}_+[w^2]) &= 0, \\ P\text{ev}_+[w^2] &= c_+. \end{aligned} \right\}.$$

Its shadow $[Pw^1]\#[Pw^2]$ is then a broken flow line of $\nabla_J\mathcal{A}_H$ satisfying

$$(7) \quad \left. \begin{aligned} \text{ev}_-([Pw^1]) &= c_-, \\ \text{ev}_+([Pw^1]) &= \text{ev}_-([Pw^2]), \\ \text{ev}_+[Pw^2] &= c_+. \end{aligned} \right\}.$$

Since J is regular, we conclude that either Pw^1 or Pw^2 has to be constant. We first rule out the case that Pw^1 is constant. If this case occurred, the positive asymptotic of the first flow line would satisfy

$$P\text{ev}_+([w^1]) = c_-, \quad \mathfrak{w}(\text{ev}_+[w^1]) > 0.$$

But then w^2 belongs to a moduli space of negative virtual dimension which contradicts the assumption that B is regular. Hence the case that Pw^1 is constant does not occur. If Pw^2 is constant, the negative asymptotic of the second flow line meets the condition

$$P\text{ev}_-([w^2]) = c_+, \quad \mathfrak{w}(\text{ev}_-[w^2]) < 0$$

implying that w^1 lies in a moduli space of negative virtual dimension. Again this contradicts the regularity of the perturbation B . Hence no breaking occurs and the Proposition is proved. \square

3.2. Proof of Theorem 2.6. We pick $c_-, c_+ \in \text{crit}(\mathcal{A}_H)$ satisfying $\mu_{CZ}(c_-) = \mu_{CZ}(c_+) + 2$. We have to prove

$$(8) \quad \sum_{c \in \text{crit}(\mathcal{A}_H)} \varrho(c_-, c; B) \varrho(c, c_+; B) = \varpi(c_-, c_+; B).$$

By Proposition 2.3 the number $\varrho(c, c_+; B) = \varrho(c, c_+; B, \ell)$ is independent of the choice of the B -admissible lift ℓ . Therefore for a B -admissible lift ℓ the left hand side of (8) can be interpreted as the modulo two number of unparametrized nonconstant broken flow lines $[w^1]\#[w^2]$ of $\nabla_B\mathcal{A}_H^\mu$ subject to the following asymptotic conditions

$$(9) \quad \left. \begin{aligned} \text{ev}_-([w^1]) &= \ell(c_-), \\ \text{ev}_+([w^1]) &= \text{ev}_-([w^2]), \\ \mathfrak{w}(\text{ev}_+[w^1]) &= 0, \\ \mathfrak{w}(\text{ev}_+[w^2]) &= 0, \\ P\text{ev}_+[w^2] &= c_+. \end{aligned} \right\}$$

Let us consider the one dimensional moduli space $\mathcal{R}(\ell_{c_-}, c_+; B)$ of all unparametrized flow lines from $\nabla_B\mathcal{A}_H^\mu$ from ℓ_{c_-} to a point in $S^1\ell_{c_+}$. By compactness for gradient flow lines of $\nabla_B\mathcal{A}_H^\mu$ we conclude that $\mathcal{R}(\ell_{c_-}, c_+; B)$ can be compactified to a one dimensional manifold with boundary whose boundary points are unparametrized nonconstant broken flow lines

$[w^1] \# [w^2]$ of $\nabla_B \mathcal{A}_H^\mu$ which meet the asymptotic conditions

$$\left. \begin{aligned} \text{ev}_-([w^1]) &= \ell(c_-), \\ \text{ev}_+([w^1]) &= \text{ev}_-([w^2]), \\ \mathfrak{w}(\text{ev}_+[w^2]) &= 0, \\ P\text{ev}_+[w^2] &= c_+. \end{aligned} \right\}$$

Since the number of boundary points of a compact one dimensional manifold is even, we conclude that the modulo two number of broken flow lines $[w^1] \# [w^2]$ subject to the asymptotic condition (9) coincides with the modulo two number of unparametrized nonconstant broken flow lines satisfying

$$(10) \quad \left. \begin{aligned} \text{ev}_-([w^1]) &= \ell(c_-), \\ \text{ev}_+([w^1]) &= \text{ev}_-([w^2]), \\ \mathfrak{w}(\text{ev}_+[w^1]) &\neq 0, \\ \mathfrak{w}(\text{ev}_+[w^2]) &= 0, \\ P\text{ev}_+[w^2] &= c_+. \end{aligned} \right\}$$

In order to ease notation we abbreviate

$$\gamma = \text{ev}_+([w^1]) \in \text{crit}(\mathcal{A}_H^\mu).$$

We first note that there exists

$$\varepsilon \in \{0, 1\}$$

such that

$$\mu_{CZ}(\ell(c_-)) - \mu_{CZ}(\gamma) = 1 + \varepsilon, \quad \mu_{CZ}(\gamma) - \mu_{CZ}(\ell(c_+)) = 1 - \varepsilon.$$

The reason for the ambiguity in the index computation lies in the fact that γ is not an isolated critical point of \mathcal{A}_H^μ but lies in a circle family of critical points. If $P: \widetilde{\mathcal{L}}_E \times \mathbb{R} \rightarrow \widetilde{\mathcal{L}}_M$ is the projection we have

$$\mu_{CZ}(\gamma) = \mu_{CZ}(P\gamma) - 2\mathfrak{w}(\gamma).$$

Since the winding numbers of $\ell(c_-)$ and $\ell(c_+)$ vanish, we get

$$\mu_{CZ}(\ell(c_-)) = \mu_{CZ}(c_-), \quad \mu_{CZ}(\ell(c_+)) = \mu_{CZ}(c_+).$$

Using these facts we compute

$$\mu_{CZ}(c_-) - \mu_{CZ}(P\gamma) = \mu_{CZ}(\ell(c_-)) - \mu_{CZ}(\gamma) - 2\mathfrak{w}(\gamma) = 1 + \varepsilon - 2\mathfrak{w}(\gamma)$$

and

$$\mu_{CZ}(P\gamma) - \mu_{CZ}(c_+) = \mu_{CZ}(\gamma) + 2\mathfrak{w}(\gamma) - \mu_{CZ}(\ell(c_+)) = 1 - \varepsilon + 2\mathfrak{w}(\gamma).$$

We now consider the broken flow line $[Pw^1] \# [Pw^2]$ of $\nabla_J \mathcal{A}_H$. Since $J \in \mathcal{J}_{\text{reg}}^\nu$ there are three cases to distinguish.

Case 1: The two flow lines Pw^1 and Pw^2 are not constant. In this case

$$\mu_{CZ}(c_-) - \mu_{CZ}(P\gamma) = 1, \quad \mu_{CZ}(P\gamma) - \mu_{CZ}(c_+) = 1.$$

Hence

$$\varepsilon = 0, \quad \mathfrak{w}(\gamma) = 0.$$

Therefore the broken flow line $[w^1] \# [w^2]$ does not satisfy the asymptotic condition (10).

Case 2: The flow line Pw^1 is constant. In this case $P\gamma = c_-$ and

$$\mu_{CZ}(c_-) - \mu_{CZ}(P\gamma) = 0, \quad \mu_{CZ}(P\gamma) - \mu_{CZ}(c_+) = 2.$$

Hence

$$\varepsilon = 1, \quad \mathfrak{w}(\gamma) = 1.$$

Therefore the broken flow line $[w^1]\#[w^2]$ satisfies the asymptotic condition (10).

Case 3: The flow line Pw^2 is constant. In this case $P\gamma = c_+$ and

$$\mu_{CZ}(c_-) - \mu_{CZ}(P\gamma) = 2, \quad \mu_{CZ}(P\gamma) - \mu_{CZ}(c_+) = 0.$$

Hence

$$\varepsilon = 1, \quad \mathfrak{w}(\gamma) = 0.$$

Since $P\gamma = c_+$ and $\mathfrak{w}(\gamma) = 0$ we conclude that $\gamma \in S^1\ell(c_+)$. Since the action of a nonconstant gradient flow line is strictly decreasing we conclude that w^2 itself is constant, which contradicts our assumption. Hence Case 3 never occurs.

Summarizing we have shown that the only broken flow lines $[w^1]\#[w^2]$ meeting the asymptotic condition (10) are the one's from Case 2, i.e. Pw^1 is constant and $\mathfrak{w}(\gamma) = 1$. We conclude that w^1 is a vortex and w^2 is an Oni. Since the vortex number is one by Theorem A.2 we deduce that the modulo two number of broken gradient flow lines subject to condition (10) coincides with the number of Onis. This finishes the proof of the Theorem. \square

3.3. Proof of Proposition 2.1. To establish that for generic $B \in \mathfrak{B}(J)$ regularity conditions (i) and (ii) are met, we first observe that since $J \in \mathcal{J}_{\text{reg}}^\nu$ the linearization of the gradient flow equation (4) along a finite energy gradient flow line w is already surjective in the horizontal directions. To show that it is also surjective in the vertical directions we distinguish two cases. In the first Pw is nonconstant, i.e. Pw is a finite energy solution of Floer's gradient flow equation. We claim that Pw necessarily leaves the neighborhood \mathcal{U} . We recall that \mathcal{U} is the union of disjoint neighborhoods of the periodic orbits of X_H where each such neighborhood contracts onto the periodic orbit, see the discussion before equation (2). If Pw is contained in \mathcal{U} then it has to be a gradient trajectory connecting the same periodic orbit with cappings d and $d\#Pw$. Since Pw is contained in \mathcal{U} the two cappings are homotopic to each other and thus Pw is constant.

By [7, Theorem 4.3] the set of regular points for Pw is open and dense. In view of Lemma 3.1 below a standard argument, see for instance [7, Section 5] or [16, Chapter 3] establishes that for generic $B \in \mathfrak{B}(J)$ the linearization of the gradient flow equation is also vertically surjective. In the second case Pw is constant. But then w is a vortex and vortices are by Proposition A.1 always transverse independent of the perturbation $B \in \mathfrak{B}(J)$. This shows that generically (i) and (ii) hold true.

The rest of the proof is devoted to show that generically the regularity condition (iii), i.e. the vanishing of J_t^B -holomorphic spheres, is satisfied as well. Assume that $u: S^2 \rightarrow E$ is a J_t^B -holomorphic sphere for some $t \in S^1$. Its shadow $v = p \circ u: S^2 \rightarrow M$ is a J_t -holomorphic sphere. Hence if $\beta \in \Omega^2(M)$ is a closed two-form representing the first Chern class of TM we obtain in local holomorphic coordinates of S^2 the inequality

$$|\beta(\partial_x v, \partial_y v)| = |\beta(\partial_x v, J_t(v)\partial_x v)| \leq \kappa_\beta(J_t) \|\partial_x v\|_{J_t}^2 = \kappa_\beta(J_t) \omega(\partial_x v, \partial_y v)$$

and therefore after integration

$$|\langle c_1(v^*TM), [S^2] \rangle| \leq \kappa_\beta(J_t) \omega([v]).$$

Since β was an arbitrary representative of the first Chern class we get by definition of the Auroux constant

$$|\langle c_1(v^*TM), [S^2] \rangle| \leq \kappa(J_t)\omega([v]).$$

Therefore we conclude

$$\begin{aligned} (11) \quad \langle c_1(u^*TE), [S^2] \rangle &= \langle c_1(v^*TE), [S^2] \rangle \\ &= \langle c_1(v^*TM), [S^2] \rangle + \langle c_1(v^*E), [S^2] \rangle \\ &\leq (\kappa(J_t) - \nu)\omega([v]) \\ &< \min\{2 - n, 0\}\omega([v]) \end{aligned}$$

where for the last inequality we used that $\nu > \max\{n + \kappa(J_t) - 2, \kappa(J_t)\}$ since $J \in \mathcal{J}^\nu$. The virtual dimension $\text{vir}dim_u(\mathfrak{H}(J_t^B))$ of the moduli space $\mathfrak{H}(J_t^B)$ of J_t^B -holomorphic spheres at u is given by the Riemann Roch formula

$$\text{vir}dim_u(\mathfrak{H}(J_t^B)) = 2\dim(E) + 2\langle c_1(u^*TE), [S^2] \rangle - 6$$

and hence can be estimated using (11)

$$(12) \quad \text{vir}dim_u(\mathfrak{H}(J_t^B)) < 2n - 4 - 2 \max\{n - 2, 0\}\omega([v]).$$

Since u is a nonconstant J_t^B -holomorphic curve it holds that

$$\omega([v]) = \omega_E([u]) > 0.$$

Since ω is integral, it follows that

$$(13) \quad \omega([v]) \geq 1.$$

Inequalities (12) and (13) imply that

$$(14) \quad \text{vir}dim_u(\mathfrak{H}(J_t^B)) < 2n - 4 - 2 \max\{n - 2, 0\} \leq 0.$$

Since the virtual dimension is an even integer we immediately obtain from (14) the stronger estimate

$$(15) \quad \text{vir}dim_u(\mathfrak{H}(J_t^B)) \leq -2.$$

Now assume that u is simple in the sense of [16, Chapter 2.5]. It follows from Lemma 3.3 below, that u is horizontally injective on a dense set, in particular, there are horizontally injective points on $E \setminus p^{-1}(\mathcal{U})$. Therefore the usual transversality arguments as explained in [16, Chapter 3.2] imply that for generic choice of the perturbation B the moduli space of simple J_t^B is a manifold whose dimension equals its virtual dimension. Therefore by (15) generically there are no simple J_t^B -holomorphic curves. Hence there are no nonconstant J_t^B -holomorphic curves at all, since every nonconstant J_t^B -holomorphic curve has an underlying simple curve. This finishes the proof of the Proposition. \square

It remains to show two lemmas which were used in the proof above.

Lemma 3.1. *Assume $e \in E$, $h_0 \in \mathcal{H}_e$, $v_0 \in \mathcal{V}_e$ such that $h_0 \neq 0$, and $t_0 \in S^1$. Then there exists $B \in \mathfrak{B}(J)$ such that $B_{t_0}h_0 = v_0$.*

Proof: By local triviality there exists a neighborhood U of e and sections $h \in \Gamma(U, \mathcal{H}|_U)$ and $v \in \Gamma(U, \mathcal{V}|_U)$ with the property that h is not vanishing and

$$h(e) = h_0, \quad v(e) = v_0.$$

Choose further a compactly supported function $\beta \in C^\infty(U, [0, 1])$ satisfying $\beta(e) = 1$. Since J_t is ω -compatible the orthogonal complement $\langle h, J_t h \rangle^{\perp_t} \subset \mathcal{H}$ with respect to the metric $\omega(\cdot, J_t \cdot)$ is invariant under J_t . We define $B \in \Gamma_0(E, \text{End}(\mathcal{H}, \mathcal{V}))$ as the section which vanishes outside U and on U is determined by

$$B_t h = \beta v, \quad B_t J_t h = -\beta I v, \quad B|_{\langle h, J_t h \rangle^{\perp_t}} = 0.$$

By construction we have $B_t J_t = -I B_t$ and $B_{t_0} h_0 = v_0$. This finishes the proof of the Lemma. \square

To state the second lemma we first need a definition. For $u \in C^\infty(S^2, E)$ we denote for $z \in S^2$ by

$$d^h u(z): T_z S^2 \rightarrow \mathcal{H}_{u(z)}$$

the composition of $du(z)$ with the projection from $T_{u(z)} E$ to $\mathcal{H}_{u(z)}$ along $\mathcal{V}_{u(z)}$.

Definition 3.2. *A J^B -holomorphic sphere $u: S^2 \rightarrow E$ is called somewhere horizontally injective if there exists $z \in S^2$ such that*

$$d^h u(z) \neq 0, \quad u^{-1}(u(z)) = \{z\}.$$

Lemma 3.3. *Assume that $u: S^2 \rightarrow E$ is a simple J^B -holomorphic curve. Then u is horizontally injective on a dense set.*

Proof: Denote by $I \subset S^2$ the subset of injective points of S^2 , by $S \subset S^2$ the subset of horizontally injective points and by $R(p(u)) \subset S^2$ the subset of nonsingular points of $p(u): S^2 \rightarrow M$. Then

$$(16) \quad S = I \cap R(p(u)).$$

We first observe that $p(u)$ is not constant, since otherwise u would lie in one fibre and hence itself must be constant, contradicting the assumption that it is simple. Hence it follows from [16, Lemma 2.4.1] that the complement of $R(p(u))$ is finite. Moreover, it follows from [16, Proposition 2.5.1] that the complement of I is countable. Hence by (16) the complement of S is countable. In particular, S is dense. \square

3.4. Proof of Theorem 2.2. If w is a flow line of $\nabla_B \mathcal{A}_H^\mu$, then Pw is a flow line of $\nabla_J \mathcal{A}^H$. Hence both functional \mathcal{A}_H^μ and $\mathcal{A}_H \circ P$ are decreasing along w . Therefore using that the perturbation is regular Theorem 2.2 follows from the usual breaking arguments in Morse homology, see [21], as soon as the following Theorem is established.

Theorem 3.4. *Assume that $J \in \mathcal{J}^\nu$ and $B \in \mathfrak{B}_{\text{reg}}^T(J)$. Suppose further that $w_\nu = ([u_\nu, \bar{u}_\nu], \eta_\nu)$ for $\nu \in \mathbb{N}$ is a sequence of flow lines of $\nabla_B \mathcal{A}_H^\mu$ for which there exists $a < b$ with the property that*

$$a \leq \mathcal{A}_H^\mu(w_\nu)(s) \leq b, \quad a \leq \mathcal{A}_H(Pw_\nu)(s) \leq b, \quad \nu \in \mathbb{N}, \quad s \in \mathbb{R}.$$

Then there exists a subsequence ν_j and a flow line w of $\nabla_B \mathcal{A}_H^\mu$ such that w_{ν_j} converges in the C_{loc}^∞ -topology to w .

Proof: The proof of Theorem 3.4 follows along standard lines, see [16, Chapter 4] if the following three conditions for w_ν can be established.

- (i): A uniform C^0 -bound on the Lagrange multipliers η_ν .
- (ii): A uniform C^0 -bound for the loops u_ν .
- (iii): A uniform bound on the derivatives of the loops u_ν .

Condition (i) is the content of Proposition 3.5, condition (ii) is the content of Proposition 3.6, and condition (iii) follows because there is no bubbling, since the perturbation B is regular. This proves the theorem. \square

Proposition 3.5. *Under the assumptions of Theorem 3.4 there exists a constant $c = c(a, b)$ such that $|\eta_\nu(s)| \leq c$ for every $\nu \in \mathbb{N}$ and every $s \in \mathbb{R}$.*

Proof: If $w = ([u, \bar{u}], \eta)$ is a critical point of \mathcal{A}_H^μ , then a computation shows that

$$\eta = \mathcal{A}_H^\mu(w) - \mathcal{A}_H(Pw).$$

By assumption $\mathcal{A}_H^\mu - \mathcal{A}_H \circ P$ is uniformly bounded along the gradient flow lines w_ν . By applying the arguments from [3] to $\mathcal{A}_H^\mu - \mathcal{A}_H \circ P$ instead of \mathcal{A}_H^μ the Lagrange multipliers η_ν can be bounded in terms of $\mathcal{A}_H^\mu - \mathcal{A}_H \circ P$ which gives a uniform bound for them. A similar argument was used in [5]. For complete details we refer to [9]. \square

Proposition 3.6. *Under the assumptions of Theorem 3.4 there exists a compact subset $K = K(a, b) \subset E$ such that $u_\nu(s, t) \in K$ for every $\nu \in \mathbb{N}$ and every $(s, t) \in \mathbb{R} \times S^1$.*

Proof: Since the perturbation B is compactly supported (E, J_t^B) is convex at infinity. A Laplace estimate for the gradient flow equation of Rabinowitz action functional then implies that u_ν stay in a bounded set of E . This Laplace estimate was also used in [4]. We refer to [4] or [9] for complete details. \square

4. HOMOTOPIES OF HOMOTOPIES

In the following we assume that $\nu > \max\{n + \kappa(\omega) - 1, \kappa(\omega)\}$ which allows us to conclude that for generic homotopies of homotopies there are still no holomorphic spheres. Instead of studying a fixed Hamiltonian H , we consider now the continuation from a C^2 -small Morse function H_0 to the Hamiltonian H . We can use the gradient flow equation of Rabinowitz action functional to define continuation homomorphisms

$$\Phi: CF_*(H_0) \rightarrow CF_*(H), \quad \Psi: CF_*(H) \rightarrow CF_*(H_0).$$

Namely choose a smooth monotone cutoff function $\beta \in C^\infty(\mathbb{R}, [0, 1])$ satisfying

$$\beta(s) = 0, \quad s < -1, \quad \beta(s) = 1, \quad s > 1$$

and consider the s -dependent Hamiltonians

$$H_s^+ = \beta(s)H + (1 - \beta(s))H_0, \quad H_s^- = \beta(s)H_0 + (1 - \beta(s))H$$

Then Φ is defined by counting gradient flow lines of the s -dependent Rabinowitz action functional $\mathcal{A}_{H_s^+}^\mu$ between a lift of a critical point of \mathcal{A}_{H_0} and a lift of a critical point of \mathcal{A}_H and similarly for Ψ one counts gradient flow lines of $\mathcal{A}_{H_s^-}^\mu$. To study their composition $\Psi \circ \Phi$ we consider as usual in Floer homology a homotopy of homotopies. Namely for $r \in [0, 1]$ choose a smooth family of functions $\beta_r \in C^\infty(\mathbb{R}, [0, 1])$ which satisfy the following conditions

- $\beta_0 = 0, \beta_1 = 1,$
- β_r is compactly supported for $r < 1,$
- For $r \in [0, 1]$ the functions β_r are monotone increasing for $s < 0$ and monotone decreasing for $s > 0,$
- The time-shifted functions $(\frac{1}{r-1})_*\beta_r$ defined by $(\frac{1}{r-1})_*\beta_r(s) = \beta_r(s + \frac{1}{r-1})$ for $s \in \mathbb{R}$ converge in the C_{loc}^∞ -topology to β as r goes to 1.

- The time-shifted functions $(\frac{1}{1-r})_* \beta_r$ converge in the C_{loc}^∞ -topology to $1 - \beta$ as r goes to 1.

For $r \in [0, 1)$ we consider the family of s -dependent Hamiltonians

$$H_r = H_0 + \beta_r(H - H_0).$$

A *Homotopy-Oni* is a pair (r, w) where $r \in [0, 1)$ and w is a finite energy gradient flow line of the time dependent Rabinowitz action function $\mathcal{A}_{H_r}^\mu$ whose asymptotic winding numbers satisfy $\mathfrak{w}(\text{ev}_-(w)) = 1$ and $\mathfrak{w}(\text{ev}_+(w)) = 0$. Counting Homotopy-Onis gives rise to a linear map

$$\mathfrak{D}: CF_*(H_0) \rightarrow CF_*(H_0).$$

We denote by

$$\partial: CF_*(H_0) \rightarrow CF_{*-1}(H_0)$$

the boundary operator obtained by counting Morse gradient flow lines of H_0 on M with respect to a Morse-Smale metric on M coming from a regular ω -compatible almost complex structure. We refer to [14, 20] for an existence proof of such metrics.

Conjecture 4.1. *There exist linear maps $T: CF_*(H_0) \rightarrow CF_{*+1}(H_0)$ such that*

$$\Psi \circ \Phi = \text{id}|_{CF_*(H_0)} + T\partial + \partial T + \mathfrak{D}.$$

The maps T in the conjecture arise in the same way as in the usual homotopy of homotopies argument in Floer homology, see [19, Section 3.4]. The proof of the conjecture should follow by basically the same argument as in the proof of Theorem 2.6, up to one point which involves abstract perturbation theory. This concerns the identification of the map $\mathfrak{R}(H_0): CF_*(H_0) \rightarrow CF_*(H_0)$ obtained by counting gradient flow lines of Rabinowitz action functional $\mathcal{A}_{H_0}^\mu$ with the boundary operator in Morse homology. If the symplectic manifold is not semipositive, then it is hard to image that such a result can be proved without the help of abstract perturbation theory. In ordinary Floer homology this was proved for example in [10, Section 22]. We expect that the argument of Fukaya and Ono can be adjusted to our situation which then leads to a proof of the conjecture.

We like to point out that the Fukaya-Ono argument using multisections proves that the Floer differential for the C^2 -small H_0 agrees with the Morse differential. The Morse differential can be counted with integer coefficients. In their further constructions \mathbb{Q} -coefficients are essential whereas our approach works over $\mathbb{Z}/2$ -coefficients.

To deduce some useful information from Conjecture 4.1 one needs to have some information of the Oni operator. Up to now we only have some clue on it under the additional assumption that the Hamiltonian satisfies the condition

$$(17) \quad H_{t+\frac{1}{2}} = H_t,$$

i.e. the time one map ϕ_H of the Hamiltonian flow admits a square root.

Conjecture 4.2. *Assume that the Hamiltonian H satisfies (17), and that ν is even and bigger than $n + \kappa(\omega)$. Then there exist linear maps*

$$F: CF_*(H_0) \rightarrow \bigoplus_{i=-2}^1 CF_{*+i}(H), \quad G: \bigoplus_{i=-2}^1 CF_{*+i}(H) \rightarrow CF_*(H_0),$$

and

$$S: CF_*(H_0) \rightarrow CF_{*-1}(H_0)$$

such that

$$(18) \quad \mathfrak{D} = G \circ F + \partial S + S\partial.$$

We give an outline of the proof of Conjecture 4.2. To consider a homotopy of homotopies we need a family of almost complex structures indexed by the homotopy parameter $r \in [0, 1]$ which in addition might depend on the parameters s and t . If this family satisfies

$$(19) \quad J_{t+\frac{1}{2},s,r}^B = J_{t,s,r}^B$$

we get an involution on the Homotopy-Oni given by rotating the loop on E by 180 degrees. Since ν is even this involution keeps critical points of $\mathcal{A}_{H_0}^\mu$ of winding number zero fixed but acts freely on critical points of winding number one. Therefore this involution is free on the Homotopy-Onis. On the other in general there is little hope to achieve transversality by keeping condition (19). To overcome this difficulty we proceed a bit different. Choose a lift $\ell: \text{crit}(\mathcal{A}_{H_0}) \rightarrow \text{crit}(\mathcal{A}_{H_0}^\mu)$ which is a section for the projection $P: \text{crit}(\mathcal{A}_{H_0}^\mu) \rightarrow \text{crit}(\mathcal{A}_{H_0})$ and satisfies $\mathfrak{w}(\ell(c)) = 1$ for every critical point $c \in \text{crit}(\mathcal{A}_{H_0})$. In the following we drop the subscripts indicating the dependence of the families of almost complex structures on the s and r -parameters. If $\theta \in S^1$ and J^B is a family of almost complex structures not necessarily satisfying (19) we set

$$(\theta_* J^B)_t = J_{t+\theta}^B.$$

Recall that on \mathcal{L}_E the circle acts by rotating the loop.

Definition 4.3. A Married Homotopy-Oni is a tuple (w, r) where $r \in [0, 1]$ and w is a finite energy gradient flow line of $\nabla_{\theta_* B} \mathcal{A}_{H_r}^\mu$ for some $\theta \in S^1$ satisfying $\text{ev}_-(w) \in \theta_* \ell(\text{crit}(\mathcal{A}_{H_0}))$ and $\mathfrak{w}(\text{ev}_+(w)) = 0$.

Married Homotopy-Onis are harmless since each Married Homotopy-Oni has a partner obtained by rotating the loops by 180 degrees. Therefore the Oni operator \mathfrak{D}^m obtained by counting Married Homotopy-Onis modulo two vanishes

$$\mathfrak{D}^m = 0: CF_*(H_0) \rightarrow CF_*(H_0).$$

We next construct a homotopy between Homotopy-Onis and Married Homotopy-Onis. To this end we first choose a smooth homotopy $J_{\theta,\rho}^B$ where $\theta \in S^1$ and $\rho \in [0, 1]$ such that

$$J_{\theta,0}^B = J^B, \quad J_{\theta,1}^B = \theta_* J^B.$$

Note that since J^B already depends on the three parameters t , s and r we have no a five-parameter family of almost complex structures on E . We apologize for any inconvenience this might cause to the reader. Since $\nu > \kappa(\omega) + n$ for generic choice of this five parameter family of almost complex structures there are no holomorphic spheres. In the following we abbreviate for $(\theta, \rho) \in S^1 \times [0, 1]$

$$\nabla_{\theta,\rho} = \nabla_{B_{\theta,\rho}}.$$

Note that the critical manifold of \mathcal{A}_H^μ consists of a disjoint union of circles. We choose a Morse function

$$h: \text{crit}(\mathcal{A}_H^\mu) \rightarrow \mathbb{R}$$

with the property that h restricted to each circle has precisely one maximum and one minimum. We further choose a Riemannian metric on $\text{crit}(\mathcal{A}_H^\mu)$ and denote by

$$\phi_{\nabla h}^\tau: \text{crit}(\mathcal{A}_H^\mu) \rightarrow \text{crit}(\mathcal{A}_H^\mu), \quad \tau \in \mathbb{R}$$

the gradient flow of h on the critical manifold of \mathcal{A}_H^μ .

Definition 4.4. A Kibidango² is an m -tuple

$$\mathfrak{k} = (k^i)_{1 \leq i \leq m}$$

for some positive integer m satisfying the following properties.

- (i): If $m = 1$, then $k^1 = (w, r, \rho)$ is a triple, where $r \in [0, 1)$, $\rho \in [0, 1]$, and w is a finite energy gradient flow line of $\nabla_{\theta, \rho} \mathcal{A}_{H_r}^\mu$ with positive asymptotic satisfying $\mathfrak{w}(\text{ev}_+(w)) = 0$ and $\theta \in S^1$ determined by $\text{ev}_-(w) \in \theta_* \ell(\text{crit} \mathcal{A}_{H_0})$.
- (ii): If $m > 1$, then $k^i = (z^i, \rho^i)$ is a tuple for each $i \in \{1, \dots, m\}$, with the following properties
 - (a): $0 \leq \rho^1 \leq \rho^2 \leq \dots \leq \rho^m \leq 1$,
 - (b): $z^1 = w^1$ is a finite energy flow line of $\nabla_{\theta, \rho^1} \mathcal{A}_{H^+}^\mu$ where $\theta = \theta(\mathfrak{k})$ is determined by $\text{ev}_-(w^1) \in \theta_* \ell(\text{crit}(\mathcal{A}_{H_0}))$,
 - (c): $z^i = [w^i]$ is an unparametrised finite energy flow line of $\nabla_{\theta, \rho^i} \mathcal{A}_H^\mu$ for $1 < i < m$,
 - (d): $z^m = w^m$ is a finite energy flow line of $\nabla_{\theta, \rho^m} \mathcal{A}_{H^-}^\mu$ whose positive asymptotic satisfies $\mathfrak{w}(\text{ev}_-(w^m)) = 0$,
 - (e): if $0 < \rho_i \leq \rho_{i+1} < 1$ for $1 \leq i < m$, then $\text{ev}_+(w_i) = \text{ev}_-(w_{i+1})$,
 - (f): if $\rho_i = 1$ for $1 \leq i < m$, then $\text{ev}_+(w^i) \notin \text{crit}(h)$ and there exists $\tau \geq 0$ such that $\phi_{\nabla_h}^\tau(\text{ev}_+(w^i)) = \text{ev}_-(w^{i+1})$,
 - (g): if $\rho_i = 0$ for $1 < i \leq m$, then $\text{ev}_-(w^i) \notin \text{crit}(h)$ and there exists $\tau \geq 0$ such that $\phi_{\nabla_h}^\tau(\text{ev}_+(w^{i-1})) = \text{ev}_-(w^i)$.

Kibidangos interpolate between Homotopy-Onis and Married Homotopy-Onis. However, the moduli space of Kibidangos does not need to be compact. Namely Kibidangos might break at critical points of $\mathcal{A}_{H_0}^\mu$ or critical points of the Morse function h on the critical manifold of \mathcal{A}_H^μ . The first occurrence should give rise to the term $\partial S + S\partial$ in (18). However, to make this precise one has to relate gradient flow lines of $\mathcal{A}_{H_0}^\mu$ with Morse gradient flow lines of H_0 which requires abstract perturbation theory and a generalization of the Theorem of Fukaya and Ono to our set-up. To see at which critical points of h a Kibidango can break one has to analyze again the shadow of a Kibidango under the projection $P: \widetilde{\mathcal{L}}_E \times \mathbb{R} \rightarrow \widetilde{\mathcal{L}}_M$. By looking at the indices it turns out that generically breaking can only happen at winding number 0 or 1. Hence for each critical point of \mathcal{A}_H^μ there are four critical points of h at which breaking might occur, namely the maximum and minimum of h on the two circles corresponding to winding number 0 and winding number 1. Hence we can identify these points with vectors in $\bigoplus_{i=-2}^1 CF_{*+1}(H)$ and the broken Kibidangos give rise to the maps F and G in (18). This finishes the outline of the proof of Conjecture 4.2. \square

As a consequence of Conjecture 4.1 and Conjecture 4.2 we obtain the following Corollary.

Corollary 4.5 (assuming Conjectures 4.1 and 4.2). *Assume that ϕ is a nondegenerate Hamiltonian symplectomorphism which has a square root. Then*

$$\#\text{Fix}(\phi) \geq \frac{1}{5} \sum_{k=0}^{2n} b_k(M; \mathbb{Z}_2)$$

where $b_k(M; \mathbb{Z}_2)$ are the \mathbb{Z}_2 -Betti numbers of M and $\text{Fix}(\phi)$ is the set of contractible fixed points.

²Eating Kibidangos Momotaro was able to fight the onis.

Proof. The proof follows from next Proposition as follows. So far, we worked with the Novikov ring over

$$\Gamma = \frac{\pi_2(M)}{\ker \omega \cap \ker c_1}.$$

This as the advantage of having a well-defined grading. Instead, if one uses the Novikov ring over

$$\Gamma_0 = \frac{\pi_2(M)}{\ker \omega}$$

then Floer homology loses its \mathbb{Z} -grading. On the other hand the Novikov ring over Γ_0 is a field Λ . The corresponding chain groups are denoted by $CF(H)$. Thus,

$$\#\text{Fix}(\phi) = \dim_{\Lambda} CF(H).$$

Moreover, $\dim_{\Lambda} HF(H_0) = \sum_{k=0}^{2n} b_k(M; \mathbb{Z}_2)$. We set

$$V := CF(H_0), \quad W := CF(H), \quad X := W \oplus W \oplus W \oplus W.$$

Furthermore, we set $R := T + S$. Then we conclude from the next Proposition that

$$\sum_{k=0}^{2n} b_k(M; \mathbb{Z}_2) = \dim H(V, \partial) \leq 5 \dim W = 5 \#\text{Fix}(\phi).$$

□

Proposition 4.6. *Let V , W , and X finite dimensional vector spaces over some fixed field. We consider maps*

$$\begin{aligned} \Phi &: V \rightarrow W \\ \Psi &: W \rightarrow V \\ F &: V \rightarrow X \\ G &: X \rightarrow V \\ \partial &: V \rightarrow V \\ R &: V \rightarrow V \end{aligned}$$

satisfying $\partial^2 = 0$ and

$$\Psi\Phi + GF = \text{id}_V + R\partial + \partial R.$$

Then the following inequality holds

$$\dim H(V, \partial) \leq \dim W + \dim X.$$

Proof. First we explain that we may assume that $X = \{0\}$ is the trivial vector space. For that we set

$$\begin{aligned} \tilde{\Phi} &: V \rightarrow W \oplus X \\ v &\mapsto (\Phi v, Fv) \end{aligned}$$

$$\begin{aligned} \tilde{\Psi} &: W \oplus X \rightarrow V \\ (w, x) &\mapsto \Psi w + Gx \end{aligned}$$

and compute

$$\tilde{\Psi}\tilde{\Phi} = \Psi\Phi + GF = \text{id}_V + R\partial + \partial R.$$

Thus, the assertion of the Proposition in the case $X = \{0\}$ implies the general case since $\dim W \oplus X = \dim W + \dim X$. It remains to prove

$$\dim H(V, \partial) \leq \dim W .$$

whenever

$$\Psi\Phi = \text{id}_V + R\partial + \partial R .$$

First we show that

$$\ker(\Phi|_{\ker \partial}) \subset \text{im } \partial .$$

Indeed, if $v \in V$ satisfies $\partial v = 0$ and $\Phi v = 0$ then

$$0 = \Psi\Phi v = v + R\partial v + \partial Rv = v + \partial Rv .$$

Then we can estimate

$$\begin{aligned} \dim H(V, \partial) &= \dim \ker \partial - \dim \text{im } \partial \\ &= \underbrace{\dim \ker(\Phi|_{\ker \partial})}_{\leq \dim \text{im } \partial} + \dim \text{im}(\Phi|_{\ker \partial}) - \dim \text{im } \partial \\ &\leq \dim \text{im}(\Phi|_{\ker \partial}) \\ &\leq \dim W \end{aligned}$$

and this proves the Proposition. \square

APPENDIX A. VORTICES

A.1. The vortex equation. As vortices we refer to solutions $(u, \eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}) \times C^\infty(\mathbb{R}, \mathbb{R})$ of the problem

$$(20) \quad \left. \begin{aligned} \partial_s u + i\partial_t u - 2\pi\eta u &= 0 \\ \partial_s \eta - \pi \int_0^1 |u|^2(t, \cdot) dt + \pi &= 0 \end{aligned} \right\}$$

whose energy is finite

$$E(u, \eta) := \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds dt + \int_{\mathbb{R}} |\partial_s \eta|^2 ds < \infty .$$

Vortices arise as gradient flow lines of Rabinowitz action functional

$$\mathcal{A}^\mu : C^\infty(S^1, \mathbb{C}) \times \mathbb{R} \rightarrow \mathbb{R}$$

given by

$$\mathcal{A}^\mu(u, \eta) = - \int u^* \lambda - \eta \int \mu(u) dt$$

where $\lambda = xdy$ is the primitive of the standard symplectic structure on \mathbb{C} . The gradient is taken with respect to the product metric on $C^\infty(S^1, \mathbb{C}) \times \mathbb{R}$ which on the first factor is given by the L^2 -metric and on the second by the standard inner product of \mathbb{R} .

The differential of Rabinowitz action function $d\mathcal{A}^\mu$ is invariant under the $S^1 \times \mathbb{Z}$ -action on $C^\infty(S^1, \mathbb{C}) \times \mathbb{R}$ given by

$$(r, k)_*(v, \eta) = ((r, k)_*v, \eta + k), \quad (r, k) \in S^1 \times \mathbb{Z}$$

where

$$(r, k)_*v(t) = e^{-2\pi ir} e^{-2\pi ikt} v(t), \quad t \in S^1 .$$

Due to the invariance under this group action Rabinowitz action functional \mathcal{A}^μ is not Morse but only Morse-Bott. The critical manifold consists of one single $S^1 \times \mathbb{Z}$ orbit, namely

$$\text{crit}(\mathcal{A}^\mu) = (S^1 \times \mathbb{Z})_*(1, 0).$$

Note that since the metric is invariant under the action as well, the vortex equations itself are $S^1 \times \mathbb{Z}$ -invariant. The action value for the critical point corresponding to $(r, k) \in S^1 \times \mathbb{Z}$ computes to be

$$\mathcal{A}^\mu((r, k)_*(1, 0)) = \pi k$$

namely the area of the unit disk times the winding number around it. The assumption that the energy is finite guarantees that vortices exponentially converge to critical points of \mathcal{A}^μ at both asymptotic ends. In particular, if (u, η) is a vortex there exist $(r^\pm, k^\pm) \in S^1 \times \mathbb{Z}$ such that

$$\lim_{s \rightarrow \pm\infty} (u, \eta)(s) = (r^\pm, k^\pm)_*(1, 0).$$

Since the action is nonincreasing along gradient flow lines, we observe that

$$(21) \quad k^- \geq k^+.$$

Alternatively, this fact can also be deduced via positivity of intersections by interpreting $-k^\pm$ as asymptotic winding numbers of the vortex. We further note, that the inequality is strict, unless the vortex is constant.

A.2. Transversality. In this section we show that the standard complex structure on \mathbb{C} given by multiplication with i is regular, i.e. the linearization of the vortex equation at each vortex is surjective. Since Rabinowitz action functional is only Morse-Bott we have to consider the linearization in suitable weighted Sobolev spaces in order that it becomes a Fredholm operator. Because critical points of \mathcal{A}^μ consist of a single $S^1 \times \mathbb{Z}$ -orbit, the spectrum of the Hessian is independent of the critical point. We choose $\delta > 0$ smaller than the spectral gap at zero, i.e. smaller than the minimum of the absolute value of all nonzero eigenvalues of the Hessian. We further choose a smooth function $\beta \in C^\infty(\mathbb{R}, [-1, 1])$, for which there exist $T > 0$ with the property that

$$\beta(s) = \begin{cases} -1 & s < -T \\ 1 & s > T \end{cases}$$

We define

$$\gamma_\delta \in C^\infty(\mathbb{R}, \mathbb{R}), \quad \gamma_\delta(s) = e^{\beta(s)\delta}, \quad s \in \mathbb{R}.$$

We abbreviate

$$W_\delta^{1,2} = \{f \in W_{loc}^{1,2} : f\gamma_\delta \in W^{1,2}\}$$

the space of all $W^{1,2}$ -functions which at both asymptotics exponentially decay with weight at least δ . Note that this definition is independent of the choice of the function β . Using these spaces the linearization along a vortex gives rise to a Fredholm operator

$$D = D_{(u,\eta)} : W_{-\delta}^{1,2}(\mathbb{R} \times S^1, \mathbb{C}) \times W_{-\delta}^{1,2}(\mathbb{R}, \mathbb{R}) \rightarrow L_{-\delta}^2(\mathbb{R} \times S^1, \mathbb{C}) \times L_{-\delta}^2(\mathbb{R}, \mathbb{R})$$

given for $(\hat{u}, \hat{\eta}) \in W_{-\delta}^{1,2}(\mathbb{R} \times S^1, \mathbb{C}) \times W_{-\delta}^{1,2}(\mathbb{R}, \mathbb{R})$ by

$$D(\hat{u}, \hat{\eta}) = \begin{cases} \partial_s \hat{u} + i\partial_t \hat{u} - 2\pi\eta\hat{u} - 2\pi u\hat{\eta} \\ \partial_s \hat{\eta} - 2\pi \int u\hat{u}. \end{cases}$$

Proposition A.1. *Along any vortex the Fredholm operator D is surjective.*

Proof: Showing surjectivity of D is equivalent to showing injectivity for the adjoint operator D^* . The adjoint operator

$$D^* : W_\delta^{1,2}(\mathbb{R} \times S^1, \mathbb{C}) \times W_\delta^{1,2}(\mathbb{R}, \mathbb{R}) \rightarrow L_\delta^2(\mathbb{R} \times S^1, \mathbb{C}) \times L_\delta^2(\mathbb{R}, \mathbb{R})$$

is given by

$$D^*(\hat{u}, \hat{\eta}) = \begin{cases} -\partial_s \hat{u} + i\partial_t \hat{u} - 2\pi\eta \hat{u} - 2\pi u \hat{\eta} \\ -\partial_s \hat{\eta} - 2\pi \int u \hat{u}. \end{cases}$$

Assume that

$$(\hat{u}, \hat{\eta}) \in \ker D^*.$$

We first show that $\hat{\eta}$ vanishes. For this purpose we compute using (20)

$$\begin{aligned} \partial_s^2 \hat{\eta} &= -2\pi \int (\partial_s u) \hat{u} - 2\pi \int u (\partial_s \hat{u}) \\ &= 2\pi \int \langle i\partial_t u, \hat{u} \rangle - 4\pi^2 \int \eta \langle u, \hat{u} \rangle \\ &\quad - 2\pi \int \langle u, i\partial_t \hat{u} \rangle + 4\pi^2 \int \eta \langle u, \hat{u} \rangle + 4\pi^2 \int \hat{\eta} \langle u, u \rangle \\ &= 4\pi^2 \hat{\eta} \int_0^1 |u|^2 dt \end{aligned}$$

Taking the product of this expression with $\hat{\eta}$ and integrating over \mathbb{R} we obtain via integration by parts

$$4\pi^2 \int_{-\infty}^{\infty} |\hat{\eta}(s)|^2 \left(\int_0^1 \hat{u}(s, t) |dt| \right) ds = - \int_{-\infty}^{\infty} |\partial_s \hat{\eta}|^2 ds.$$

The lefthandside is nonnegative and the righthandside is nonpositive therefore both sides have to vanish and we conclude that $\partial_s \hat{\eta}$ vanishes identically. Since $\hat{\eta} \in W_\delta^{1,2}(\mathbb{R}, \mathbb{R})$ we conclude

$$\hat{\eta} = 0.$$

Using again that $(\hat{u}, \hat{\eta})$ is in the kernel of D^* we conclude that \hat{u} is a solution of the PDE

$$-\partial_s \hat{u} + i\partial_t \hat{u} - 2\pi\eta \hat{u} = 0.$$

We expand \hat{u} into a time dependent Fourier series

$$\hat{u}(s, t) = \sum_{k \in \mathbb{Z}} a_k(s) e^{2\pi i k t}.$$

The Fourier coefficients are solutions of the ODE

$$\partial_s a_k + 2\pi(k + \eta) a_k = 0.$$

Asymptotically η converges to minus the asymptotic winding numbers of the vortex (u, η)

$$\lim_{s \rightarrow \pm\infty} \eta(s) = k^\pm.$$

which satisfy $k^- \geq k^+$ by (21). We claim that

$$(22) \quad a_k = 0, \quad k \in \mathbb{Z}.$$

Otherwise, since a_k decays exponentially at the positive end we would obtain

$$k > -k^+.$$

Since a_k also decays exponentially at the negative end the inequality

$$k < -k^-$$

has to hold true as well. Together we conclude

$$k^+ < k^-$$

contradicting (21). This proves (22) and therefore

$$\hat{u} = 0.$$

We have shown that $(\hat{u}, \hat{\eta}) = (0, 0)$ and the proof of the Proposition is complete. \square

A.3. The vortex number. If the asymptotic winding numbers k^\pm of a vortex (u, η) satisfy

$$k^- = k^+ + 1,$$

then the Fredholm index of the Fredholm operator $D = D_{(u, \eta)}$ is

$$\text{ind}(D) = 3.$$

We denote by $\tilde{\mathfrak{V}}(k^-, k^+)$ the moduli space of all vortices with asymptotic winding numbers k^- and k^+ . By Proposition A.1 it is a three dimensional manifold. The group \mathbb{R} acts on vortices by timeshift and the group S^1 acts on vortices by rotation of the domain of the loop as well as by rotation on the target \mathbb{C} . Since all this actions commute we get an $\mathbb{R} \times S^1 \times S^1$ action on $\tilde{\mathfrak{V}}(k^-, k^+)$. Since the two asymptotics are different, the action is free. Therefore the quotient

$$\mathfrak{V}(k^-, k^+) = \tilde{\mathfrak{V}}(k^-, k^+) / (\mathbb{R} \times S^1 \times S^1)$$

is a zero dimensional manifold. Since it is compact, see [3, 8] it is a finite set and we define the vortex number as

$$\mathfrak{v} = \#\mathfrak{V}(k^-, k^+) \bmod 2 \in \mathbb{Z}_2.$$

Note that since the vortex equation is invariant under the \mathbb{Z} -action the vortex number does not depend on the asymptotic winding numbers. The vortex number has the following interpretation. If (r^-, k^-) and (r^+, k^+) are two elements in $S^1 \times \mathbb{Z}$ satisfying $k^- = k^+ + 1$ then \mathfrak{v} is the modulo 2 number of vortices (u, η) subject to the asymptotic conditions

$$\lim_{s \rightarrow \pm\infty} (u, \eta)(s) = (r^\pm, k^\pm)_*(1, 0) \in \text{crit}(\mathcal{A}^\mu).$$

Theorem A.2. *The vortex number equals one.*

Proof: Since the unit circle in the complex plane is Hamiltonian displaceable, Rabinowitz Floer homology vanishes by [3]. If the vortex number were zero, Rabinowitz Floer homology would be equal to the homology of the critical manifold of \mathcal{A}^μ which is a countable disjoint union of circles. Therefore the vortex number has to be equal to one. For an alternative argument based on finite dimensional approximation we refer to [8]. \square

APPENDIX B. SQUARE ROOTS IN SIMPLE GROUPS

In this section we prove that in the Hamiltonian symplectomorphism group of a closed connected symplectic manifold each element can be written as a finite product of elements which admit a square root. This result is a straightforward consequence of a deep result due to Banyaga.

Theorem B.1. *Assume (M, ω) is a closed connected symplectic manifold. Then for each $\phi \in \text{Ham}(M, \omega)$ there exists $n \in \mathbb{N}$ and $\psi_i \in \text{Ham}(M, \omega)$ for $1 \leq i \leq n$ such that*

$$\phi = \psi_1^2 \cdots \psi_n^2.$$

Before embarking on the proof of the Theorem we first consider an arbitrary group G . We denote by G^2 the subgroup of G generated by all squares in G , i.e.

$$G^2 = \{g_1^2 \cdots g_n^2 : g_1, \dots, g_n \in G, n \in \mathbb{N}\}.$$

Lemma B.2. *G^2 is a normal subgroup in G .*

Proof: Let $g \in G$ and $g_1^2 \cdots g_n^2 \in G^2$. Then

$$gg_1^2 \cdots g_n^2 g^{-1} = (gg_1 g^{-1})^2 \cdots (gg_n g^{-1})^2 \in G^2.$$

This finishes the proof of the Lemma. □

Corollary B.3. *If G is simple and not two-torsion, then $G^2 = G$.*

Proof: Since G is not two-torsion G^2 is nontrivial. Therefore by Lemma B.2 G^2 is a nontrivial normal subgroup of G . Since G is simple $G^2 = G$. □

Proof of Theorem B.1: We can assume without loss of generality that M has positive dimension and therefore $\text{Ham}(M, \omega)$ is not two-torsion. By Banyaga's theorem, see [2], the Hamiltonian symplectomorphism group is simple. Hence the Theorem follows from Corollary B.3. □

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