SQUARE ROOTS OF HAMILTONIAN DIFFEOMORPHISMS

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Abstract. In this article we prove that on any closed symplectic manifold there exists an arbitrarily C^∞-small Hamiltonian diffeomorphism not admitting a square root.

1. Introduction

Let (M,ω) be a closed symplectic manifold, i.e. $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \to \mathbb{R}$ we associate the Hamiltonian vector field X_L by setting

$$\omega(X_{L_t}, \cdot) = -dL_t(\cdot) \tag{1}$$

where $L_t(x) := L(t,x)$. The flow $\phi_L^t : M \to M$ of the vector field X_{L_t} is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi_L^1 \; . \tag{2}$$

The Hamiltonian diffeomorphisms form the Lie group $\mathrm{Ham}(M,\omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [MS98] for the basics in symplectic geometry.

In this article we prove the following Theorem.

Theorem 1. *In any C^∞-neighborhood of the identity in $\mathrm{Ham}(M,\omega)$ there exists a Hamiltonian diffeomorphism ϕ which has no square root, i.e. for all Hamiltonian diffeomorphism ψ (not necessarily close to the identity)*

$$\psi^2 \neq \phi \tag{3}$$

holds.

An immediate corollary of Theorem 1 is the following.

Corollary 2. *The exponential map*

$$\begin{aligned} \mathrm{Exp} : C^\infty(M,\mathbb{R})/\mathbb{R} &\to \mathrm{Ham}(M,\omega) \\ [L] &\mapsto \phi_L \end{aligned} \tag{4}$$

is not a local diffeomorphism.

In the proof of the Theorem we use the following beautiful observation by Milnor [Mil84, Warning 1.6]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2k$-cycles, see next section for details. The main work in this article is to construct an example in the symplectic category.

Acknowledgements. The authors are indebted to Kaoru Ono for invaluable discussions and drawing our attention to Milnor's article [Mil84].

This material is supported by the SFB 878 – Groups, Geometry and Actions (PA) and by the Basic Research fund 20100007669 funded by the Korean government basic (UF).

2. Milnor's observation

We define

$$CM^k := M^k/(\mathbb{Z}/k) \tag{5}$$

where $\mathbb{Z}/k$ acts by cyclic shifts on M^k. We write elements of CM^k as

$$[x_1, \dots, x_k] \in CM^k\ . \tag{6}$$

The space of k-cycles of a diffeomorphism $\phi : M \to M$ is

$$\mathscr{C}^k(\phi) := \big\{[x_1, \dots, x_k] \in CM^k \mid \phi^j(x_i) \neq x_i\, \forall j = 1, \dots, k-1,\ \phi(x_i) = x_{i+1}\big\}\ . \tag{7}$$

We point out that if $[x_1, \dots, x_k] \in \mathscr{C}^k(\phi)$ then $\phi^k(x_i) = x_i$ for $i = 1, \dots, k$.

Proposition 3 (Milnor [Mil84]). *If $\phi = \psi^2$ then $\mathscr{C}^{2k}(\phi)$ admits a free $\mathbb{Z}/2$-action. In particular, $\#\mathscr{C}^{2k}(\phi)$ is even if $\mathscr{C}^{2k}(\phi)$ is a finite set.*

For the convenience of the reader we include a proof of Milnor's ingenious observation.

Proof. We define

$$\begin{aligned} I : \mathscr{C}^{2k}(\phi) &\to \mathscr{C}^{2k}(\phi)\\ [x_1, \dots, x_{2k}] &\mapsto [\psi(x_1), \dots, \psi(x_{2k})]\ . \end{aligned} \tag{8}$$

Since $\psi \circ \phi = \phi \circ \psi$ and $\psi^2 = \phi$ the map I is well-defined and an involution. We assume by contradiction that $[x_1, \dots, x_{2k}]$ is a fixed point of I, i.e. there exists $0 \leq r \leq 2k-1$

$$\psi(x_i) = x_{i+r} \tag{9}$$

where we read indices $\mathbb{Z}/2k$-cyclically. Using $x_{i+r} = \phi^r(x_i)$ we get

$$\psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \tag{10}$$

and thus

$$\psi^{2r-1}(x_i) = x_i\ . \tag{11}$$

In particular,

$$x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i)\ . \tag{12}$$

In summary we have

$$x_i = \phi^{2r-1}(x_i) \quad \text{and} \quad x_i = \phi^{2k}(x_i)\ . \tag{13}$$

In general, if

$$z = \phi^a(z) \quad \text{and} \quad z = \phi^b(z) \tag{14}$$

for $a, b \in \mathbb{Z}$ then

$$z = \phi^{\mathrm{lcd}(a,b)}(z) \tag{15}$$

since by the Euclidean algorithm there exists $n_1, n_2 \in \mathbb{Z}$ with

$$\mathrm{lcd}(a, b) = n_1 a + n_2 b\ . \tag{16}$$

In our specific situation $2r-1$ is odd and $2k$ is even and thus

$$1 \leq \mathrm{lcd}(2r-1, 2k) < 2k \tag{17}$$

contradicting the assumption $\phi^j(x_i) \neq x_i\, \forall j = 1, \dots, 2k-1$. This proves the Proposition. $\square$

3. Proof of Theorem 1

Let (M,ω) be a closed symplectic manifold. We fix a Darboux chart $B^{2N}(R) \cong B \subset M$ where $B^{2N}(R)$ is the open ball of radius R in $\mathbb{R}^{2N}$. For an integer $k \geq 1$ and a positive number $\delta > 0$ we choose a smooth function $\rho : [0, R^2] \to \mathbb{R}$ satisfying the following

$$\begin{cases} \dfrac{\pi}{2k} \geq \rho'(r) > 0 \;, \\ \rho'(r) = \dfrac{\pi}{2k} \quad \iff \quad r = \frac{1}{2}R^2 \;, \\ \rho'|_{\left[\frac{8}{9}R^2, R^2\right]} = \delta > 0 \;. \end{cases} \tag{18}$$

We set for $1 \leq \nu \leq N$

$$\zeta(\nu) := \begin{cases} 1 & \nu = N \\ \frac{9}{10} & \text{else} \end{cases} \tag{19}$$

and define

$$\begin{aligned} H : B^{2N}(R) &\to \mathbb{R} \\ z &\mapsto \rho\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) \;. \end{aligned} \tag{20}$$

We denote by $\phi_H^t : B^{2N}(R) \to B^{2N}(R)$ the induced Hamiltonian flow. We recall that the Hamiltonian flow of $z \mapsto |z|^2$ is given by $z \mapsto \exp(2it)z$ thus

$$\left(\phi_H^t(z)\right)_\nu = \exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)t\right] z_\nu \;. \tag{21}$$

We point out that ϕ_H^t preserves the quantities $|z_\nu|$, $\nu = 1, \ldots, N$.

Lemma 4. *The fixed points of ϕ_H^{2k} are precisely $z = 0$ and the circle*

$$C := \left\{(z_1, \ldots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \tfrac{1}{2}R^2 \text{ and } z_1 = \ldots = z_{N-1} = 0\right\} . \tag{22}$$

Moreover, ϕ_H acts on C by rotation of the last coordinate by an angle of $\frac{\pi}{k}$.

Proof. Assume $\phi_H^{2k}(z) = z$ which is equivalent to

$$\exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)2k\right] z_\nu = z_\nu, \quad \nu = 1, \ldots, N \;, \tag{23}$$

thus, either $z_\nu = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 4k\zeta(\nu) \in 2\pi\mathbb{Z} \;. \tag{24}$$

From $\rho'(r) \leq \frac{\pi}{2k}$ we conclude that $z_1 = \ldots = z_{N-1} = 0$. Moreover, $z_N = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) = \rho'(|z_N|^2) = \frac{\pi}{2k} \tag{25}$$

holds. In summary, either $z = 0$ or $z \in C$. This together with (21) proves the Lemma. $\square$

We now perturb H. For this we fix a smooth cut-off function $\beta : [0, R^2] \to [0, 1]$ satisfying

$$\beta\big|_{\left[\frac{1}{3}R^2, \frac{2}{3}R^2\right]} = 1 \quad \text{and} \quad \beta\big|_{\left[0, \frac{1}{9}R^2\right] \cup \left[\frac{8}{9}R^2, R^2\right]} = 0 \tag{26}$$

and set

$$F(z) := \beta(|z_N|^2) \cdot \mathrm{Re}\left(\frac{z_N^k}{|z_N|^k}\right) : B^{2N}(R) \to \mathbb{R} \tag{27}$$

where Re is the real part. If we introduce new coordinates $(z_1, \ldots, z_{N-1}, r, \vartheta)$, where $z_N = r\exp(i\vartheta)$, the function F equals

$$F(z) = \beta(r^2)\cos\big(k\vartheta\big)\ . \tag{28}$$

We point out that the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F}$ maps $B^{2N}(R)$ into itself.

Lemma 5. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$*

$$\#\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F}) = 1\ . \tag{29}$$

Proof. We set

$$D := \left\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k},\ j = 0, \ldots, 2k-1\right\} \tag{30}$$

where C is defined in Lemma 4. The same lemma implies that ϕ_H acts on D as a cyclic permutation sending $\frac{j\pi}{k}$ to $\frac{(j+1)\pi}{k}$. Moreover, we have

$$\phi_{\epsilon F} z = z \tag{31}$$

for $z \in D$ since $D \subset \mathrm{Crit}F$. In particular, D corresponds precisely to a single element in $\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F})$. It remains to show that there are no other $2k$-cycles. We prove something stronger, namely that for sufficiently small $\epsilon > 0$ the only other fixed point of $\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}$ is $z = 0$.

For $0 < a < b$ we set

$$A(a, b) := \big\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2]\big\}\ . \tag{32}$$

We observe that on $A(\frac{1}{3}, \frac{2}{3})$ we have $\beta = 1$ and thus the flow of ϵF is given by

$$(z_1, \ldots, z_{N-1}, r, \vartheta) \mapsto (z_1, \ldots, z_{N-1}, \sqrt{-2\epsilon k\sin(k\vartheta)t + r^2}, \vartheta)\ . \tag{33}$$

In particular, if we set

$$\bar{\epsilon} := \frac{7R^4}{324k^2} \tag{34}$$

then for $0 < \epsilon < \bar{\epsilon}$ we conclude that

$$\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}\big(A(\tfrac{4}{9}, \tfrac{5}{9})\big) \subset A(\tfrac{1}{3}, \tfrac{2}{3})\ , \tag{35}$$

since ϕ_H^t preserves the r coordinate. Fix $w \in A(\frac{4}{9}, \frac{5}{9})$ with $\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}(w) = w$ and set for $j = 0, \ldots, 2k$

$$\begin{aligned} z_\nu^j &:= P_{z_\nu}\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \quad \nu = 1, \ldots, N-1, \\ r^j &:= P_r\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \\ \vartheta^j &:= P_\vartheta\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \end{aligned} \tag{36}$$

where P_{z_ν}, P_r, and P_ϑ are the projections on the respective coordinates. It follows from equation (33) that

$$P_{z_\nu}\Big(\big(\phi_H\circ\phi_{\epsilon F}\big)^j(w)\Big) = P_{z_\nu}\big(\phi_H^j(w)\big) \quad \nu=1,\dots,N-1\ . \tag{37}$$

By the same argument as in the proof of Lemma 4 we conclude

$$z_\nu^j = 0 \quad \forall \nu=1,\dots,N-1 \text{ and } \forall j=0,\dots,2k\ . \tag{38}$$

Next, it follows from the flow equations (21) and (33)

$$0<\vartheta_{j+1}-\vartheta_j\leq\frac{\pi}{k} \mod 2\pi\ . \tag{39}$$

By (18) equality holds if and only if $r_{j+1}=\frac12 R^2$. Using again $\big(\phi_H\circ\phi_{\epsilon F}\big)^{2k}(w)=w$ we deduce

$$\vartheta_{2k}-\vartheta_0=0 \mod 2\pi \tag{40}$$

and therefore

$$r_0=r_1=\ldots=r_{2k}=\tfrac12 R^2\ . \tag{41}$$

In summary

$$w=(0,\dots,0,\tfrac12 R^2,\vartheta_0) \tag{42}$$

with $\vartheta_0\in\frac{\pi}{k}\mathbb{Z}$, i.e. $w\in D$. Thus, we proved that the only $2k$-cycle of $\phi_H\circ\phi_{\epsilon F}$ in the region $A(\frac49,\frac59)$ is the one corresponding to the set D. Therefore it remains to prove that after possibly shrinking $\bar\epsilon$ there are no other fixed points of $\big(\phi_H\circ\phi_{\epsilon F}\big)^{2k}$ outside $A(\frac49,\frac59)$ except for $z=0$. We argue by contradiction.

We assume that there exists a sequence $\epsilon_m\to 0$ and a sequence $(z^m)_{m\in\mathbb{N}}$ of points in $B^{2N}(R)\setminus A(\frac49,\frac59)$ with

$$\big(\phi_H\circ\phi_{\epsilon_m F}\big)^{2k}(z^m)=z^m \quad \forall m\in\mathbb{N}\ . \tag{43}$$

By compactness we may assume that $z^m\to z^*\in B^{2N}(R)\setminus \mathrm{int}A(\frac49,\frac59)$ with

$$\phi_H^{2k}(z^*)=z^*\ . \tag{44}$$

It follows from Lemma 4 that $z^*=0$ and thus for M sufficiently large

$$z^m\in B^{2N}(\tfrac13 R) \quad \forall m\geq M\ . \tag{45}$$

Then by definition of β the restriction of $\phi_{\epsilon_m F}$ to the ball $B^{2N}(\frac13 R)$ equals the identity. Moreover, since ϕ_H fixes all balls centered at zero we have

$$z^m=\big(\phi_H\circ\phi_{\epsilon_m F}\big)^{2k}(z^m)=\phi_H^{2k}(z^m) \quad \forall m\geq M\ . \tag{46}$$

Applying again Lemma 4 we conclude that $z^m=0$ for all $m\geq M$. This proves the Lemma. □

Remark 6. Proposition 3 together with Lemma 5 implies that for all $0<\epsilon<\epsilon_0$ the Hamiltonian diffeomorphism $\phi_H\circ\phi_{\epsilon F}:B^{2N}(R)\to B^{2N}(R)$ has no square root.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. We choose $k\in\mathbb{Z}$, $\delta>0$ and $0<\epsilon<\epsilon_0$ (cp. Lemma 5) so that the Hamiltonian diffeomorphism

$$\phi_H\circ\phi_{\epsilon F}:B^{2N}(R)\to B^{2N}(R) \tag{47}$$

has precisely one $2k$-cycle. By construction $\phi_H\circ\phi_{\epsilon F}$ equals the map

$$(z_1,\dots,z_N)\mapsto\Big(e^{\frac{9i\delta}{5}}z_1,\dots,e^{\frac{9i\delta}{5}}z_{N-1},e^{2i\delta}z_N\Big) \tag{48}$$

near the boundary of $B^{2N}(R)$. Indeed, if $z \in \partial B^{2N}(R)$ then we conclude

$$\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \geq \frac{9}{10}\sum_{\nu=1}^{N}|z_\nu|^2 = \frac{9}{10}R^2 > \frac{8}{9}R^2 \tag{49}$$

and therefore $\rho'(\sum_{\nu=1}^{N}\zeta(\nu)|z_\nu|^2) = \delta$. Next, we extend the Hamiltonian function of $\phi_H \circ \phi_{\epsilon F}$ to $\widetilde{H} : S^1 \times M \to \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball B. If we choose $\delta > 0$ sufficiently small we can guarantee that outside B the only periodic orbits of $\widetilde{H}$ of period less or equal to $2k$ are critical points of $\widetilde{H}$, see [HZ94], in particular line 4 & 5 on page 185. In particular, $\phi_{\widetilde{H}}$ has still precisely one $2k$-cycle. Finally, by choosing k sufficiently large and δ and ϵ sufficiently small, $\phi_H \circ \phi_{\epsilon F}$ and thus $\phi_{\widetilde{H}}$ can be chosen to lie in an arbitrary C^∞-neighborhood of the identity on $B^{2N}(R)$ resp. M. Therefore, with Proposition 3 the Theorem follows. □

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