SQUARE ROOTS OF HAMILTONIAN DIFFEOMORPHISMS

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Abstract. In this article we prove that on any closed symplectic manifold there exists an arbitrarily $C^\infty$-small Hamiltonian diffeomorphism not admitting a square root.

1. Introduction

Let $(M, \omega)$ be a closed symplectic manifold, i.e. $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \to \mathbb{R}$ we associate the Hamiltonian vector field $X_L$ by setting

$$\omega(X_{L_t}, \cdot) = -dL_t(\cdot)$$

where $L_t(x) := L(t, x)$. The flow $\phi^t_L : M \to M$ of the vector field $X_L$ is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi^1_L.$$ (2)

The Hamiltonian diffeomorphisms form the Lie group $\text{Ham}(M, \omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [MS98] for the basics in symplectic geometry.

In this article we prove the following Theorem.

Theorem 1. In any $C^\infty$-neighborhood of the identity in $\text{Ham}(M, \omega)$ there exists a Hamiltonian diffeomorphism $\phi$ which has no square root, i.e. for all Hamiltonian diffeomorphism $\psi$ (not necessarily close to the identity)

$$\psi^2 \neq \phi$$

holds.

An immediate corollary of Theorem 1 is the following.

Corollary 2. The exponential map

$$\exp : C^\infty(M, \mathbb{R})/\mathbb{R} \to \text{Ham}(M, \omega)$$

$L \mapsto \phi_L$ (4)

is not a local diffeomorphism.

In the proof of the Theorem we use the following beautiful observation by Milnor [Mil84, Warning 1.6]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2k$-cycles, see next section for details. The main work in this article is to construct an example in the symplectic category.

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2. Milnor’s observation

We define
\[ \text{CM}^k := M^k / (\mathbb{Z}/k) \quad (5) \]
where \( \mathbb{Z}/k \) acts by cyclic shifts on \( M^k \). We write elements of \( \text{CM}^k \) as
\[ [x_1, \ldots, x_k] \in \text{CM}^k. \quad (6) \]
The space of \( k \)-cycles of a diffeomorphism \( \phi: M \to M \) is
\[ \mathcal{C}^k(\phi) := \{ [x_1, \ldots, x_k] \in \text{CM}^k | \phi^j(x_i) \neq x_i \forall j = 1, \ldots, k-1, \phi(x_i) = x_{i+1} \}. \quad (7) \]
We point out that if \( [x_1, \ldots, x_k] \in \mathcal{C}^k(\phi) \) then \( \phi^k(x_i) = x_i \) for \( i = 1, \ldots, k \).

**Proposition 3** (Milnor [Mil84]). If \( \phi = \psi^2 \) then \( \mathcal{C}^{2k}(\phi) \) admits a free \( \mathbb{Z}/2 \)-action. In particular, \( \# \mathcal{C}^{2k}(\phi) \) is even if \( \mathcal{C}^{2k}(\phi) \) is a finite set.

For the convenience of the reader we include a proof of Milnor’s ingenious observation.

**Proof.** We define
\[ I: \mathcal{C}^{2k}(\phi) \to \mathcal{C}^{2k}(\phi) \]
\[ [x_1, \ldots, x_{2k}] \mapsto [\psi(x_1), \ldots, \psi(x_{2k})]. \quad (8) \]
Since \( \psi \circ \phi = \phi \circ \psi \) and \( \psi^2 = \phi \) the map \( I \) is well-defined and an involution. We assume by contradiction that \( [x_1, \ldots, x_{2k}] \) is a fixed point of \( I \), i.e. there exists \( 0 \leq r \leq 2k-1 \)
\[ \psi(x_i) = x_{i+r} \quad (9) \]
where we read indices \( \mathbb{Z}/2k \)-cyclically. Using \( x_{i+r} = \phi^r(x_i) \) we get
\[ \psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \quad (10) \]
and thus
\[ \psi^{2r-1}(x_i) = x_i. \quad (11) \]
In particular,
\[ x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i). \quad (12) \]
In summary we have
\[ x_i = \phi^{2r-1}(x_i) \quad \text{and} \quad x_i = \phi^{2k}(x_i). \quad (13) \]
In general, if
\[ z = \phi^a(z) \quad \text{and} \quad z = \phi^b(z) \quad (14) \]
for \( a, b \in \mathbb{Z} \) then
\[ z = \phi^{\text{lcd}(a,b)}(z) \quad (15) \]
since by the Euclidean algorithm there exists \( n_1, n_2 \in \mathbb{Z} \) with
\[ \text{lcd}(a, b) = n_1 a + n_2 b. \quad (16) \]
In our specific situation \( 2r - 1 \) is odd and \( 2k \) is even and thus
\[ 1 \leq \text{lcd}(2r - 1, 2k) < 2k \quad (17) \]
contradicting the assumption \( \phi^j(x_i) \neq x_i \forall j = 1, \ldots, 2k - 1 \). This proves the Proposition. \( \square \)
3. Proof of Theorem

Let \((M, \omega)\) be a closed symplectic manifold. We fix a Darboux chart \(B^{2N}(R) \cong B \subset M\) where \(B^{2N}(R)\) is the open ball of radius \(R\) in \(\mathbb{R}^{2N}\). For an integer \(k \geq 1\) and a positive number \(\delta > 0\) we choose a smooth function \(\rho: [0, R^2] \to \mathbb{R}\) satisfying the following

\[
\begin{align*}
\frac{\pi}{2k} &\geq \rho'(r) > 0 , \\
\rho'(r) &= \frac{\pi}{2k} \iff r = \frac{1}{2} R^2 , \\
\rho'|_{\left[\frac{4}{9} R^2, R^2\right]} &= \delta > 0 .
\end{align*}
\]

We set for \(1 \leq \nu \leq N\)

\[
\zeta(\nu) := \begin{cases} 
\frac{1}{9} & \nu = N \\
\text{else}
\end{cases}
\]

and define

\[
H : B^{2N}(R) \to \mathbb{R}
\]

\[
z \mapsto \rho \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \right).
\]

We denote by \(\phi_H^t : B^{2N}(R) \to B^{2N}(R)\) the induced Hamiltonian flow. We recall that the Hamiltonian flow of \(z \mapsto |z|^2\) is given by \(z \mapsto \exp(2it)z\) thus

\[
(\phi_H^t(z))_\nu = \exp \left[ \rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \right) 2i\zeta(\nu)t \right] z_\nu .
\]

We point out that \(\phi_H^t\) preserves the quantities \(|z_\nu|\), \(\nu = 1, \ldots, N\).

**Lemma 4.** The fixed points of \(\phi_H^{2k}\) are precisely \(z = 0\) and the circle

\[
C := \left\{ (z_1, \ldots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \frac{1}{2} R^2 \text{ and } z_1 = \ldots = z_{N-1} = 0 \right\} .
\]

Moreover, \(\phi_H\) acts on \(C\) by rotation of the last coordinate by an angle of \(\frac{\pi}{k}\).

**Proof.** Assume \(\phi_H^{2k}(z) = z\) which is equivalent to

\[
\exp \left[ \rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \right) 2i\zeta(\nu)2k \right] z_\nu = z_\nu, \quad \nu = 1, \ldots, N ,
\]

thus, either \(z_\nu = 0\) or

\[
\rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \right) 4k \zeta(\nu) \in 2\pi\mathbb{Z} .
\]

From \(\rho'(r) \leq \frac{\pi}{2k}\) we conclude that \(z_1 = \ldots = z_{N-1} = 0\). Moreover, \(z_N = 0\) or

\[
\rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \right) = \rho'(|z_N|^2) = \frac{\pi}{2k}
\]

holds. In summary, either \(z = 0\) or \(z \in C\). This together with (21) proves the Lemma. □
We now perturb $H$. For this we fix a smooth cut-off function $\beta : [0,R^2] \to [0,1]$ satisfying
\begin{equation}
\beta|_{\frac{1}{2}R^2,\frac{3}{2}R^2} = 1 \quad \text{and} \quad \beta|_{\left[\frac{1}{2}R^2,\frac{3}{2}R^2\right]} = 0
\end{equation}
and set
\begin{equation}
F(z) := \beta(|z_N|^2) \cdot \text{Re} \left( \frac{z_N^k}{|z_N|^k} \right) : B^{2N}(R) \to \mathbb{R}
\end{equation}
where $\text{Re}$ is the real part. If we introduce new coordinates $(z_1, \ldots, z_{N-1}, r, \vartheta)$, where $z_N = r \exp(i\vartheta)$, the function $F$ equals
\begin{equation}
F(z) = \beta(r^2) \cos(k\vartheta).
\end{equation}
We point out that the Hamiltonian diffeomorphism $\phi_H \circ \phi_F$ maps $B^{2N}(R)$ into itself.

**Lemma 5.** There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$
\begin{equation}
\#\mathcal{C}^2(\phi_H \circ \phi_F) = 1.
\end{equation}

**Proof.** We set
\begin{equation}
D := \left\{ (z_1, \ldots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k}, j = 0, \ldots, 2k - 1 \right\}
\end{equation}
where $C$ is defined in Lemma 4. The same lemma implies that $\phi_H$ acts on $D$ as a cyclic permutation sending $\frac{4\pi}{k}$ to $\frac{(j+1)\pi}{k}$. Moreover, we have
\begin{equation}
\phi_F z = z
\end{equation}
for $z \in D$ since $D \subset \text{Crit}F$. In particular, $D$ corresponds precisely to a single element in $\mathcal{C}^2(\phi_H \circ \phi_F)$. It remains to show that there are no other $2k$-cycles. We prove something stronger, namely that for sufficiently small $\epsilon > 0$ the only other fixed point of $(\phi_H \circ \phi_F)^{2k}$ is $z = 0$.

For $0 < a < b$ we set
\begin{equation}
A(a,b) := \left\{ (z_1, \ldots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2] \right\}.
\end{equation}
We observe that on $A\left(\frac{1}{3}, \frac{2}{3}\right)$ we have $\beta = 1$ and thus the flow of $\epsilon F$ is given by
\begin{equation}
(z_1, \ldots, z_{N-1}, r, \vartheta) \mapsto (z_1, \ldots, z_{N-1}, \sqrt{-2\epsilon k \sin(k\vartheta)t + r^2}, \vartheta).
\end{equation}
In particular, if we set
\begin{equation}
\bar{\epsilon} := \frac{7R^4}{324k^2}
\end{equation}
then for $0 < \epsilon < \bar{\epsilon}$ we conclude that
\begin{equation}
(\phi_H \circ \phi_F)^{2k}(A\left(\frac{1}{3}, \frac{2}{3}\right)) \subset A\left(\frac{1}{3}, \frac{2}{3}\right),
\end{equation}
since $\phi_F^j$ preserves the $r$ coordinate. Fix $w \in A\left(\frac{4}{9}, \frac{5}{9}\right)$ with $(\phi_H \circ \phi_F)^{2k}(w) = w$ and set for $j = 0, \ldots, 2k$
\begin{equation}
\begin{aligned}
z_{\nu}^j &:= P_{z\nu} \left( (\phi_H \circ \phi_F)^j(w) \right), \quad \nu = 1, \ldots, N - 1, \\
r^j &:= P_{r} \left( (\phi_H \circ \phi_F)^j(w) \right), \\
\vartheta^j &:= P_{\vartheta} \left( (\phi_H \circ \phi_F)^j(w) \right),
\end{aligned}
\end{equation}
where \( P_z \), \( P_r \), and \( P_\theta \) are the projections on the respective coordinates. It follows from equation (33) that

\[
P_z \left( \left( \phi_H \circ \phi_{e_F} \right)^j(w) \right) = P_z \left( \phi_H^j(w) \right) \quad \nu = 1, \ldots, N - 1.
\]

By the same argument as in the proof of Lemma 4 we conclude

\[
z^j = 0 \quad \forall \nu = 1, \ldots, N - 1 \quad \text{and} \quad \forall j = 0, \ldots, 2k.
\]

Next, it follows from the flow equations (21) and (33)

\[
0 < \vartheta_{j+1} - \vartheta_j \leq \frac{\pi}{k} \mod 2\pi.
\]

By (13) equality holds if and only if \( r_{j+1} = \frac{1}{2}R^2 \). Using again \( \left( \phi_H \circ \phi_{e_F} \right)^{2k}(w) = w \) we deduce

\[
\vartheta_{2k} - \vartheta_0 = 0 \mod 2\pi
\]

and therefore

\[
r_0 = r_1 = \ldots = r_{2k} = \frac{1}{2}R^2.
\]

In summary

\[
w = (0, \ldots, 0, \frac{1}{2}R^2, \vartheta_0)
\]

with \( \vartheta_0 \in \mathbb{T} \mathbb{Z} \), i.e. \( w \in D \). Thus, we proved that the only 2k-cycle of \( \phi_H \circ \phi_{e_F} \) in the region \( A(\frac{4}{5}, \frac{5}{2}) \) is the one corresponding to the set \( D \). Therefore it remains to prove that after possibly shrinking \( \epsilon \) there are no other fixed points of \( \left( \phi_H \circ \phi_{e_F} \right)^{2k} \) except for \( z = 0 \).

We argue by contradiction.

We assume that there exists a sequence \( \epsilon_m \to 0 \) and a sequence \((z^m)_{m \in \mathbb{N}}\) of points in \( B^{2N}(R) \setminus A(\frac{4}{5}, \frac{5}{2}) \) with

\[
(\phi_H \circ \phi_{e_m})(z^m) = z^m \quad \forall m \in \mathbb{N}.
\]

By compactness we may assume that \( z^m \to z^* \in B^{2N}(R) \setminus \text{int}A(\frac{4}{5}, \frac{5}{2}) \) with

\[
\phi_H^{2k}(z^*) = z^*.
\]

It follows from Lemma 4 that \( z^* = 0 \) and thus for \( M \) sufficiently large

\[
z^m \in B^{2N}(\frac{1}{2}R) \quad \forall m \geq M.
\]

Then by definition of \( \beta \) the restriction of \( \phi_{e_m} \) to the ball \( B^{2N}(\frac{1}{2}R) \) equals the identity. Moreover, since \( \phi_H \) fixes all balls centered at zero we have

\[
z^m = \left( \phi_H \circ \phi_{e_m} \right)^{2k}(z^m) = \phi_H^{2k}(z^m) \quad \forall m \geq M.
\]

Applying again Lemma 4 we conclude that \( z^m = 0 \) for all \( m \geq M \). This proves the Lemma. \( \square \)

**Remark 6.** Proposition 3 together with Lemma 5 implies that for all \( 0 < \epsilon < \epsilon_0 \) the Hamiltonian diffeomorphism \( \phi_H \circ \phi_{e_F} : B^{2N}(R) \to B^{2N}(R) \) has no square root.

We are now in the position to prove Theorem 1.

**Proof of Theorem 1.** We choose \( k \in \mathbb{Z} \), \( \delta > 0 \) and \( 0 < \epsilon < \epsilon_0 \) (cp. Lemma 5) so that the Hamiltonian diffeomorphism

\[
\phi_H \circ \phi_{e_F} : B^{2N}(R) \to B^{2N}(R)
\]

has precisely one 2k-cycle. By construction \( \phi_H \circ \phi_{e_F} \) equals the map

\[
(z_1, \ldots, z_N) \mapsto \left( e^{\frac{2\pi i}{k}} z_1, \ldots, e^{\frac{2\pi i}{k}} z_{N-1}, e^{2\pi i \delta} z_N \right)
\]
near the boundary of $B^{2N}(R)$. Indeed, if $z \in \partial B^{2N}(R)$ then we conclude

$$\sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^{2} \geq \frac{9}{10} \sum_{\nu=1}^{N} |z_{\nu}|^{2} = \frac{9}{10} R^{2} > \frac{8}{9} R^{2}$$

(49)

and therefore $\rho'(\sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^{2}) = \delta$. Next, we extend the Hamiltonian function of $\phi_{H} \circ \phi_{\epsilon F}$ to $\tilde{H} : S^{1} \times M \to \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball $B$. If we choose $\delta > 0$ sufficiently small we can guarantee that outside $B$ the only periodic orbits of $\tilde{H}$ of period less or equal to $2k$ are critical points of $\tilde{H}$, see [HZ94], in particular line 4 \\
& 5 on page 185. In particular, $\phi_{\tilde{H}}$ has still precisely one $2k$-cycle. Finally, by choosing $k$ sufficiently large and $\delta$ and $\epsilon$ sufficiently small, $\phi_{H} \circ \phi_{\epsilon F}$ and thus $\phi_{\tilde{H}}$ can be chosen to lie in an arbitrary $C^{\infty}$-neighborhood of the identity on $B^{2N}(R)$ resp. $M$. Therefore, with Proposition 3 the Theorem follows.

References


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