THE HÖRMANDER INDEX OF SYMMETRIC PERIODIC ORBITS

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Abstract. A symmetric periodic orbit is a special kind of periodic orbit that can also be regarded as a Lagrangian intersection point. Therefore it has two Maslov indices whose difference is the Hörmander index. In this paper we provide a formula for the Hörmander index of a symmetric periodic orbit and its iterates in terms of Chebyshev polynomials.

1. Introduction

Symmetric periodic orbits play a crucial rule in the restricted three body problem [2, 3, 5]. In general, they can be defined for any Hamiltonian system invariant under an antisymplectic involution. Since the fixed point set of an antisymplectic involution is a Lagrangian submanifold L, symmetric periodic orbits can be interpreted either as periodic orbits or as Lagrangian intersection points. Therefore one can associate two different Maslov indices with them, namely the Conley-Zehnder index if interpreted as a periodic orbit, or the Lagrangian Maslov index if interpreted as a Lagrangian intersection point. These two Maslov indices can be obtained as the intersection number of a path of symplectic matrices with two different but homologous Maslov cycles. In particular, their difference is independent of the path and only depends on the linearization of the Poincaré return map. This difference of the two Maslov indices is the Hörmander index. The purpose of this paper is to give an explicit formula how to compute the Hörmander index of a symmetric periodic orbit and its iterates.

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2. Definitions and results

Assume that (M,ω) is a symplectic manifold and $\rho \in \mathrm{Diff}(M)$ is an antisymplectic involution of M, i.e.

$$\rho^2 = \mathrm{Id}, \quad \rho^*\omega = -\omega.$$

Suppose that $H \in C^\infty(M,\mathbb{R})$ is an autonomous Hamiltonian which is invariant under the involution ρ, i.e.

$$H \circ \rho = H.$$

In particular, the Hamiltonian vector field of H defined by the equation $dH = \omega(X_H, \cdot)$ satisfies $\rho^* X_H = -X_H$. For $\eta \in \mathbb{R}$ denote by $\phi^\eta = \phi^\eta_{X_H}$ the time-η flow of the Hamiltonian vector field. Note that because of the anti invariance of the Hamiltonian vector field under the involution ρ we obtain

$$\phi^\eta_{X_H} = \phi^{-\eta}_{-X_H} = \phi^{-\eta}_{\rho^* X_H} = \rho \phi^{-\eta}_{X_H} \rho. \tag{1}$$

Denote by $\mathbb{R}_+$ the set of positive real numbers. A *periodic orbit* is a pair $(x, \eta) \in M \times \mathbb{R}_+$ satisfying $\phi^\eta(x) = x$. It follows from (1) that if (x, η) is a periodic orbit, then $(\rho(x), \eta)$ is a periodic orbit as well. A periodic orbit (x, η) is called *symmetric*, if $x = \rho(x)$, i.e. x lies in the fixed point set $\mathscr{L} = \mathrm{Fix}(\rho)$ of the antisymplectic involution ρ. The fixed point set of an antisymplectic involution is a Lagrangian submanifold and hence a symmetric periodic orbit (x, η) also gives rise to a Lagrangian intersection point $x \in \mathscr{L} \cap \phi^\eta \mathscr{L}$.

In the following let us assume that (x, η) is a symmetric periodic orbit satisfying $dH(x) \neq 0$, i.e. x is not a critical point. Since the Hamiltonian H is autonomous the energy hypersurface $\Sigma = H^{-1}\big(H(x)\big)$ is invariant under the flow of X_H. We further choose a symplectic subspace $V \subset T_x\Sigma$ satisfying

- **(i):** $V \oplus \langle X_H(x) \rangle = T_x\Sigma$.
- **(ii):** V is $d\rho(x)$ invariant.

That such a subspace exists can be seen as follows. Since x is not a critical point of H there exists $w \in T_xM$ satisfying $dH(x)w \neq 0$. Set $v = w + d\rho(x)w$. Since ρ is an involution, we have $d\rho(x)v = v$. Because H is invariant under ρ we have $dH(x)v = 2dH(x)w \neq 0$. Choose $V = \langle v, X_H(x) \rangle^\omega$. Since V is symplectically orthogonal to $X_H(x)$ it is a codimension one subspace of $T_x\Sigma$ and therefore (i) holds. Moreover, since $\langle v, X_H(x) \rangle$ is invariant under $d\rho(x)$ its symplectic orthogonal complement is invariant as well. We abbreviate

$$R = d\rho(x)|_V \colon V \to V.$$

Note that R is a linear antisymplectic involution of the symplectic vector space (V, ω). The linear antisymplectic involution gives rise to a Lagrangian splitting $V = L_+ \times L_-$, where $L_\pm$ are the eigenspaces of R to the eigenvalue ± 1. Note that $L_+ = T_x\mathscr{L} \cap T_x\Sigma$. Let $2n$ be the dimension of V, i.e. the dimension of the original manifold M is $2n+2$. Choose a basis $\{e_1, \ldots, e_n\}$ of L_+. Using the Lagrangian splitting $V = L_+ \times L_-$ we can symplectically identify V with the cotangent bundle T^*L_+. Hence the basis $\{e_1, \ldots, e_n\}$ uniquely determines a basis $\{f_1, \ldots, f_n\}$ on L_- such that $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a symplectic basis of V. Using such a basis we identify (V, ω) with $\mathbb{R}^{2n}$ endowed with its standard symplectic structure. Under this identification the Lagrangian subspaces become $L_+ = \mathbb{R}^n \times \{0\}$ and $L_- = \{0\} \times \mathbb{R}^n$ and the Lagrangian splitting becomes the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. We abbreviate the linearization of the Poincaré return map by

$$\Phi = d\phi^\eta(x)|_V \colon V \to V.$$

With respect to the Lagrangian splitting we write

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for $n \times n$-matrices A, B, C, D.

Proposition 2.1. *The matrices A, B, C, D satisfy*

$$D = A^T, \quad B = B^T, \quad C = C^T, \quad AB = BA^T, \quad AC = CA^T, \quad A^2 - BC = \mathrm{Id}\,.$$

Remark 2.2. *In the case that $n = 1$, the proposition tells us that the matrix Φ is of the form*

$$\Phi = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \quad a^2 - bc = 1$$

a fact which can be already found in the work of G. Darwin [3, p. 146]*, see also* [5]*.*

Since a symmetric periodic orbit (x, η) is a periodic orbit as well as a Lagrangian intersection point it has a Conley-Zehnder index $\mu_{CZ}(x, \eta)$ as well as a Lagrangian Maslov index $\mu_L(x, \eta)$. The difference of these two indices is the Hörmander index

$$s(x, \eta) = \mu_{CZ}(x, \eta) - \mu_L(x, \eta).$$

Note that the iterates of a symmetric periodic orbit $(x, k\eta)$ for $k \in \mathbb{N}$ are symmetric periodic orbits as well. We say that a symmetric periodic orbit is *nondegenerate* if for any $k \in \mathbb{N}$ it holds that $\det(\Phi^k - \mathrm{Id}) \neq 0$. We further recall that the *Chebyshev polynomials of the first kind* are recursively defined by

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{k+1}(x) &= 2xT_k(x) - T_{k-1}(x) \end{aligned}$$

while the *Chebyshev polynomials of the second kind* are defined by

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{k+1}(x) &= 2xU_k(x) - U_{k-1}(x). \end{aligned}$$

We are now able to formulate our main result

Theorem 2.3. *Suppose that (x, η) is a nondegenerate, symmetric periodic orbit. Then the Hörmander indices of its iterates are given by the formula*

$$s(x, k\eta) = \frac{1}{2} \operatorname{sign}\Big((\mathrm{Id} - T_k(A)) U_{k-1}(A)^{-1} C^{-1} \Big), \quad k \in \mathbb{N}.$$

In particular,

$$s(x, \eta) = \frac{1}{2} \operatorname{sign}\Big((\mathrm{Id} - A) C^{-1} \Big).$$

3. The proof

3.1. **Proof of Proposition 2.1.** Since Φ is a symplectic matrix its inverse is given by, see [7, p. 20],

$$\Phi^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

In particular, it holds that

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = \mathrm{Id}\,. \tag{2}$$

Differentiating (1) we obtain

$$\Phi = R \Phi^{-1} R.$$

Note that

$$R = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}.$$

Hence we obtain

$$A = D^T, \quad B = B^T, \quad C = C^T. \tag{3}$$

Equations (2) and (3) imply the proposition. □

3.2. **An iteration formula.** In this subsection we prove an iteration formula for the matrix Φ which allows us to reduce the proof of Theorem 2.3 to the case $k = 1$.

Lemma 3.1. *For $k \in \mathbb{N}$ the k-th iterate of Φ satisfies*

$$\Phi^k = \begin{pmatrix} T_k(A) & U_{k-1}(A)B \\ CU_{k-1}(A) & T_k(A^T) \end{pmatrix}.$$

Proof. We show that the entries of the iterates Φ^k satisfy the mutual recursion formula for Chebyshev polynomials as in Lemma A.2, namely

$$T_{k+1}(A) = xT_k(A) - (1 - x^2)U_{k-1}(A)$$
$$U_k(A) = AU_{k-1}(A) + T_k(A).$$

We proceed by induction. The claim holds for $k = 1$. For the induction step, we compute

$$\begin{aligned} \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}^{k+1} &= \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \begin{pmatrix} T_k(A) & U_{k-1}(A)B \\ CU_{k-1}(A) & T_k(A^T) \end{pmatrix} \\ &= \begin{pmatrix} AT_k(A) + BCU_{k-1}(A) & AU_{k-1}(A)B + BT_k(A^T) \\ CT_k(A) + A^T CU_{k-1}(A) & CU_{k-1}(A)B + A^T T_k(A^T) \end{pmatrix} \\ &= \begin{pmatrix} AT_k(A) - (\mathrm{Id} - A^2)U_{k-1}(A) & AU_{k-1}(A)B + T_k(A)B \\ CT_k(A) + CAU_{k-1}(A) & -(\mathrm{Id} - (A^T)^2)U_{k-1}(A^T) + A^T T_k(A^T) \end{pmatrix}. \end{aligned}$$

In the last step, we have used the identities, $A^2 - \mathrm{Id} = BC$, $BA^T = AB$ and $A^T C = CA$. □

3.3. **Nondegeneracy.** In this subsection we prove that the assumption that the symmetric periodic orbit is nondegenerate guarantees that the formula in Theorem 2.3 is well defined, namely

Lemma 3.2. *Assume that (x, η) is a nondegenerate symmetric periodic orbit. Then C is invertible.*

Proof: It suffices to show that C is injective. Let us assume that v lies in the kernel of C, i.e. $Cv = 0$. Since $A^2 - BC = \mathrm{Id}$ we conclude that $A^2 v = v$. We first check that

$$w := \begin{pmatrix} Av + v \\ 0 \end{pmatrix} \in \ker(\Phi - \mathrm{Id}).$$

To see that we compute

$$\begin{aligned} (\Phi - \mathrm{Id})w &= \begin{pmatrix} A - \mathrm{Id} & B \\ C & D - \mathrm{Id} \end{pmatrix} \begin{pmatrix} Av + v \\ 0 \end{pmatrix} = \begin{pmatrix} A^2 v + Av - Av - v \\ CAv + Cv \end{pmatrix} \\ &= \begin{pmatrix} v + Av - Av - v \\ A^T Cv + Cv \end{pmatrix} = 0. \end{aligned}$$

Hence $w \in \ker(\Phi - \mathrm{Id})$ and since the symmetric periodic orbit is nondegenerate we conclude that

$$Av = -v.$$

We claim that this implies that

$$z = \begin{pmatrix} v \\ 0 \end{pmatrix} \in \ker(\Phi^2 - \mathrm{Id}).$$

Indeed,

$$\begin{aligned} (\Phi^2 - \mathrm{Id})z &= 2\begin{pmatrix} A^2 - \mathrm{Id} & AB \\ CA & (A^T)^2 - \mathrm{Id} \end{pmatrix}\begin{pmatrix} v \\ 0 \end{pmatrix} = 2\begin{pmatrix} A^2 v - v \\ CAv \end{pmatrix} \\ &= 2\begin{pmatrix} v - v \\ -Cv \end{pmatrix} = 0. \end{aligned}$$

Since the symmetric orbit is nondegenerate this implies that z vanishes and therefore v is zero as well. This proves that C is injective and the lemma follows. □

3.4. **Proof of Theorem 2.3.** Let (V, ω) be a symplectic vector space, and denote the Lagrangian Grassmannian of V by $\mathcal{L} = \mathcal{L}(V)$. This space is the manifold consisting of all Lagrangian subspaces of V. The Maslov index associates a half integer [8] with any two paths $\Lambda, \Lambda' \colon [0,1] \to \mathcal{L}$. We denote this index by

$$\mu(\Lambda, \Lambda') \in \frac{1}{2}\mathbb{Z}.$$

If $\Psi \colon [0,1] \to \mathrm{Sp}(V)$ is a path of linear symplectic transformations of V satisfying $\Psi(0) = \mathrm{Id}$ and $\det(\Psi(1) - \mathrm{Id}) \neq 0$, then the Conley-Zehnder index associated to this path is defined as

$$\mu_{CZ}(\Psi) = \mu(\mathrm{Gr}(\Psi), \Delta)$$

where $\mathrm{Gr}(\Psi)$ is the path in the Lagrangian Grassmannian of $\big(V \times V, (-\omega) \times \omega\big)$ obtained from the graph of Ψ and $\Delta \subset V \times V$ is the diagonal. If $L \in \mathcal{L}(V)$ is a Lagrangian subspace of V, then the Lagrangian Maslov index of Ψ with respect to L can be defined as (see [8, Theorem 3.2])

$$\mu_L(\Psi) = \mu(\mathrm{Gr}(\Psi), L \times L).$$

Abbreviate $\Phi = \Psi(1)$. According to [8, Theorem 3.5] the Hörmander index can be defined by

$$s\big(L \times L, \Delta; \Delta, \mathrm{Gr}(\Phi)\big) = \mu_{CZ}(\Psi) - \mu_L(\Psi).$$

It is also shown in [8, Theorem 3.5] that this index only depends on the endpoints of the path Ψ. For symmetric orbits, the Hörmander index can be computed with the following lemma.

Lemma 3.3. *Assume Φ satisfies the conditions of Proposition 2.1 with C invertible. Then*

$$s\big(L \times L, \Delta; \Delta, \mathrm{Gr}(\Phi)\big) = \frac{1}{2}\,\mathrm{sign}\Big((\mathrm{Id} - A)C^{-1}\Big)$$

where $L = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n} = V$.

Proof. We first invoke [8, Theorem 3.5] again or [6, Formula 3.3.7] to get that

$$s\big(L \times L, \Delta; \Delta, \mathrm{Gr}(\Phi)\big) = -s\big(\Delta, \mathrm{Gr}(\Phi); L \times L, \Delta\big). \tag{4}$$

By [4, Formula 2.10] the Hörmander index can be computed as

$$s\big(\Delta, \mathrm{Gr}(\Phi); L \times L, \Delta\big) = \frac{1}{2}\Big(\mathrm{sign}\, Q\big(\Delta, \mathrm{Gr}(\Phi); L \times L\big) - \mathrm{sign}\, Q\big(\Delta, \mathrm{Gr}(\Phi); \Delta\big)\Big). \tag{5}$$

If W is a Lagrangian subspace of $(V \times V, \Omega)$ with $\Omega = -\omega \times \omega$, then the quadratic form $Q\big(\Delta, \mathrm{Gr}(\Phi); W\big)$ is defined as follows. Since the Lagrangians Δ and $\mathrm{Gr}(\Phi)$ are transverse by assumption there exists a linear map $\Gamma\colon \Delta \to \mathrm{Gr}(\Phi)$ such that

$$W = \{z + \Gamma z : z \in \Delta\}.$$

We define[1]

$$Q(\Delta, \mathrm{Gr}(\Phi); W)\colon \Delta \times \Delta \to \mathbb{R}$$

by the formula

$$(z, z') \mapsto \Omega(z, \Gamma z').$$

Note that since W is Lagrangian, the form Q is actually symmetric. We immediately see that $Q(\Delta, \mathrm{Gr}(\Phi); \Delta)$ vanishes and therefore

$$\operatorname{sign} Q(\Delta, \mathrm{Gr}(\Phi); \Delta) = 0. \tag{6}$$

It therefore remains to compute $Q(\Delta, \mathrm{Gr}(\Phi); L \times L)$. For this purpose we pick $z = (u, u) \in \Delta$ with $u = (u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. We define $v(u) \in \mathbb{R}^{2n}$ implicitly by the condition that

$$\big(u + v(u), u + \Phi(v(u))\big) \in L \times L.$$

To obtain an explicit formula for the vector $v(u)$ we decompose $v(u) = (v_1(u), v_2(u)) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. The condition that $u + v(u) \in L$ immediately implies that

$$v_2 = -u_2.$$

To meet the requirement $u + \Phi(v(u)) \in L$ we obtain the equation

$$0 = u_2 + Cv_1 + Dv_2 = u_2 + Cv_1 - A^T u_2$$

and therefore

$$v_1 = C^{-1}(A^T - \mathrm{Id})u_2 = (A - \mathrm{Id})C^{-1}u_2.$$

Summarizing we obtained

$$v(u) = \big((A - \mathrm{Id})C^{-1}u_2, -u_2\big). \tag{7}$$

The map $\Gamma\colon \Delta \to \mathrm{Gr}(\Phi)$ is given by

$$\Gamma(u, u) = \big(v(u), \Phi(v(u))\big).$$

We compute

$$\Phi(v(u)) = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \begin{pmatrix} (A - \mathrm{Id})C^{-1}u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} \big(A(A - \mathrm{Id})C^{-1} - B\big)u_2 \\ -u_2 \end{pmatrix}.$$

To simplify the first factor we derive

$$A(A - \mathrm{Id})C^{-1} - B = (BC + \mathrm{Id})C^{-1} - AC^{-1} - B = (\mathrm{Id} - A)C^{-1}.$$

Hence we get

$$\Phi(v(u)) = \begin{pmatrix} (\mathrm{Id} - A)C^{-1}u_2 \\ -u_2 \end{pmatrix}. \tag{8}$$

[1] Our conventions differ from those of Duistermaat [4]: the Maslov cycle is oriented differently.

Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$. From (7) and (8) we obtain the following expression for the symmetric form $Q = Q\big(\Delta, \mathrm{Gr}(\Phi); L\times L\big) \colon \Delta\times\Delta \to \mathbb{R}$ if we insert the vectors $z=(u,u)$, $z'=(u',u')\in\Delta$,

$$\begin{aligned} Q(z,z') &= \Omega(z,\Gamma z') \\ &= \Big\langle u_2, (A-\mathrm{Id})C^{-1}u_2'\Big\rangle + \Big\langle u_1, u_2'\Big\rangle - \Big\langle u_2, (\mathrm{Id}-A)C^{-1}u_2'\Big\rangle - \Big\langle u_1, u_2'\Big\rangle \\ &= 2\Big\langle u_2, (A-\mathrm{Id})C^{-1}u_2'\Big\rangle. \end{aligned}$$

In particular, we obtain

$$\operatorname{sign} Q\big(\Delta, \mathrm{Gr}(\Phi); L\times L\big) = \operatorname{sign}\Big((A-\mathrm{Id})C^{-1}\Big). \tag{9}$$

Combining equations (4), (5), (6), and (9) the lemma follows. □

The proof of Theorem 2.3 is now immediate. For $k=1$ the theorem follows from Lemma 3.2 and Lemma 3.3. The general case follows from the case $k=1$ by using the iteration formula from Lemma 3.1. This finishes the proof of the theorem. □

Appendix A. Identities for Chebyshev polynomials

The Chebyshev polynomials have many remarkable properties. Some identities are particularly relevant for us.

Lemma A.1. *Let $a=\cos\alpha$. Then*

$$T_n(a) = \cos n\alpha$$
$$U_n(a) = \frac{\sin(n+1)\alpha}{\sin\alpha}$$

Proof. To see this, use induction: for $n=0,1$, the identities hold true. Then we compute

$$\begin{aligned} T_{n+1}(a) &= 2\cos\alpha T_n(\cos\alpha) - T_{n-1}(\cos\alpha) = 2\cos\alpha\cos n\alpha - \cos(n-1)\alpha \\ &= \cos 2\alpha\cos(n-1)\alpha - \sin 2\alpha \sin(n-1)\alpha = \cos(n+1)\alpha, \\ U_{n+1}(a) &= 2\cos\alpha\frac{\sin(n+1)\alpha}{\sin\alpha} - \frac{\sin n\alpha}{\sin\alpha} = \frac{1}{\sin\alpha}\left(2\cos^2\alpha\sin n\alpha - \sin n\alpha + 2\cos\alpha\sin\alpha\cos n\alpha\right) \\ &= \frac{1}{\sin\alpha}\left(\sin n\alpha\cos 2\alpha + \cos n\alpha \sin 2\alpha\right) = \frac{\sin(n+2)\alpha}{\sin\alpha}. \end{aligned}$$

□

From here, it is also straightforward to derive the mutual recurrence relations that appear in the proof of Lemma 3.1. These are given by

Lemma A.2 (Mutual recursion formulas)**.** *The Chebyshev polynomials satisfy*

$$T_{n+1}(x) = xT_n(x) - (1-x^2)U_{n-1}(x)$$
$$U_n(x) = xU_{n-1} + T_n(x).$$

Indeed, one can simply apply the standard sum formulas for cos and sin.

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