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A NOTE ON MONOTONE LIKELIHOOD RATIO OF THE TOTAL SCORE VARIABLE IN UNIDIMENSIONAL ITEM RESPONSE THEORY

Abstract

This note provides a direct, elementary proof of the fundamental result on monotone likelihood ratio of the total score variable in unidimensional item response theory (IRT). This result is very important for practical measurement in IRT, because it justifies the use of the total score variable to order participants on the latent trait. The proof relies on a basic inequality for elementary symmetric functions which is proved by means of few purely algebraic, straightforward transformations. In particular, flaws in a proof of this result by Huynh [(1994). A new proof for monotone likelihood ratio for the sum of independent Bernoulli random variables. *Psychometrika, 59*, 77–79] are pointed out and corrected, and a natural generalization of the fundamental result to nonlinear (quasi-ordered) latent trait spaces is presented. This may be useful for multidimensional IRT or knowledge space theory, in which the latent 'ability' spaces are partially ordered with respect to, for instance, coordinate-wise vector-ordering or set-inclusion, respectively.

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1 Introduction

Monotone likelihood ratio for the total score variable and latent trait plays an important role in item response theory (IRT). It implies stochastic ordering properties that can be conveniently interpreted in an IRT context (e.g., Hemker, Sijtsma, Molenaar, & Junker, 1996, 1997; Hemker, Van der Ark, & Sijtsma, 2001; Sijtsma, 1998; Van der Ark, 2001, 2005): Stochastic ordering of the total score variable by the latent trait, and stochastic ordering of the latent trait by the total score variable. The fundamental result (Ghurye & Wallace, 1959; Grayson, 1988; Huynh, 1994) states that under mild conditions the total score variable has monotone likelihood ratio in the latent trait (Section 2). The (most recent and simplified) proof of this result by Huynh (1994), however, is flawed. In this note, a direct, elementary proof of the fundamental result is given (Section 3). It is different from the proof by Huynh (1994) in that it corrects flaws in Huynh's (1994) argument (Section 4) and easily allows for a generalization of the fundamental result to abstract nonlinear latent trait spaces (Section 5).[1]

Next, the required notation and terminology are introduced, and the monotone likelihood ratio and stochastic ordering properties are briefly reviewed. Throughout this note, only dichotomous items are considered. Let X_l with realization $x_l \in \{0, 1\}$ denote the *item score variable* for an item I_l ($1 \leq l \leq m$, $m \in \mathbb{N}_{\geq 2}$), and let $X_+ := \sum_{l=1}^{m} X_l$ with realization $x_+ \in \{0, 1, \ldots, m\}$ be the *total score variable*. A function $f : \{0, 1, \ldots, m\} \to \mathbb{R}$ is *nondecreasing* if and

[1] Throughout this note, using the phrase 'nonlinear latent trait space' I refer to the definition of a nonlinear latent trait space given in Section 5.

only if (iff)

$$\forall x, y \in \{0, 1, \ldots, m\}, x \leq y : f(x) \leq f(y).$$

Let the *latent trait* be denoted by θ, $\theta \in \Theta \subseteq \mathbb{R}$. I refer to this as the assumption of *unidimensionality*. A function $f : \Theta \to \mathbb{R}$ is *nondecreasing* iff

$$\forall \theta_1, \theta_2 \in \Theta, \theta_1 \leq \theta_2 : f(\theta_1) \leq f(\theta_2).$$

Let the conditional positive response probability $P(X_l = 1|\theta)$ as a function of $\theta \in \Theta$ be the *item response function* (IRF) of an item I_l ($1 \leq l \leq m$). The assumption of *local independence* states that

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m|\theta) = \prod_{l=1}^{m} P(X_l = x_l|\theta)$$

for any $x_l \in \{0, 1\}$ ($1 \leq l \leq m$) and $\theta \in \Theta$. The assumption of *monotonicity* holds iff any IRF $P(X_l = 1|.)$ ($1 \leq l \leq m$) is nondecreasing.[2] Mokken's (1971) model of *monotone homogeneity* is based on the assumptions of unidimensionality, local independence, and monotonicity.

The total score variable X_+ has *monotone likelihood ratio* (MLR) in θ iff, for

[2] Here '$P(X_l = 1|.)$ ($1 \leq l \leq m$)' denotes the IRF

$$P(X_l = 1|.) : \Theta \to [0, 1], \theta \mapsto P(X_l = 1|\theta)$$

of an item I_l ($1 \leq l \leq m$).

any $0 \leq x_{+,1} \leq x_{+,2} \leq m$,[3]

$$\frac{P(X_+ = x_{+,2}|\theta)}{P(X_+ = x_{+,1}|\theta)}$$

is a nondecreasing function of $\theta \in \Theta$. Similarly, the latent trait θ has *MLR* in X_+ iff, for any $\theta_1, \theta_2 \in \Theta$, $\theta_1 \leq \theta_2$,

$$\frac{P(\theta_2|X_+ = x_+)}{P(\theta_1|X_+ = x_+)}$$

is a nondecreasing function of $0 \leq x_+ \leq m$. Because MLR of X_+ in θ is equivalent to MLR of θ in X_+ (Bayes' theorem), with abuse of terminology I simply speak of the 'property of MLR.' The fundamental result states that under mild conditions the total score variable has MLR in the latent trait (Theorem 1).

The MLR property is rather technical. Its importance in IRT actually stems

[3] Throughout this note, mathematical expressions are assumed to be defined whenever they are written in the text. Note also that expressions such as $P(X_+ = x_+|\theta)$ or $P(\theta|X_+ = x_+)$ (for $0 \leq x_+ \leq m$ and $\theta \in \Theta$) are technically defined as follows. Let $f_{X_+,\Theta}(x_+, \theta)$ as a function of $0 \leq x_+ \leq m$ and $\theta \in \Theta$ be the joint 'density function' of the random vector (X_+, Θ). Associated with this, define the marginal and conditional 'density functions' as follows:

$$\begin{aligned} f_{X_+}(x_+) &:= \int_{\theta \in \Theta} f_{X_+,\Theta}(x_+, \theta)\, d\theta \quad (0 \leq x_+ \leq m), \\ f_\Theta(\theta) &:= \sum\nolimits_{x_+=0}^{m} f_{X_+,\Theta}(x_+, \theta) \quad (\theta \in \Theta), \\ f_{X_+|\Theta}(x_+|\theta) &:= \frac{f_{X_+,\Theta}(x_+, \theta)}{f_\Theta(\theta)} \quad (0 \leq x_+ \leq m; \text{ for } \theta \in \Theta \text{ with } f_\Theta(\theta) \neq 0), \\ f_{\Theta|X_+}(\theta|x_+) &:= \frac{f_{X_+,\Theta}(x_+, \theta)}{f_{X_+}(x_+)} \quad (\theta \in \Theta; \text{ for } 0 \leq x_+ \leq m \text{ with } f_{X_+}(x_+) \neq 0). \end{aligned}$$

from the fact that it implies the following two stochastic ordering properties that are much easier to interpret in IRT (cf. Section 5). The property of MLR implies that X_+ is stochastically ordered by θ. The *stochastic ordering of the manifest variable* X_+ by θ (SOM) means that, for any $0 \leq x_+ \leq m$,

$$P(X_+ \geq x_+|\theta)$$

is a nondecreasing function of $\theta \in \Theta$. Note that this property takes the latent trait as a starting point. In practice, however, the total score variable is observed and inferences about the latent trait are required. The property of MLR also implies that θ is stochastically ordered by X_+. The *stochastic ordering of the latent trait* θ by X_+ (SOL) means that, for any $\theta_0 \in \Theta$,

$$P(\theta \geq \theta_0|X_+ = x_+)$$

is a nondecreasing function of $0 \leq x_+ \leq m$. Note that the property of SOL is very important for practical measurement, because it justifies the use of the total score variable to estimate the ordering of participants on the latent trait. In particular, this is the key result that justifies the use of Mokken's (1971) monotone homogeneity model as a measurement model for people. (The monotone homogeneity model satisfies the property of MLR and hence implies the SOL property.)

2 Fundamental result

The fundamental result on monotone likelihood ratio of the total score variable in unidimensional IRT (for dichotomous items) is as follows.

Theorem 1 (Fundamental result) *Under the assumptions of unidimensionality, local independence, and monotonicity, and the requirement that each item response function assumes values strictly between zero and one, the total score variable has monotone likelihood ratio in the (unidimensional) latent trait.*

Proof. See Section 3. □

Before giving a proof of this result (Section 3), some remarks with respect to Theorem 1 are in order.

1. The requirement of having IRFs assuming values strictly between zero and one is not that restrictive in practice. 'Boundary value' IRFs (assuming the values 0 and/or 1) may be closely approximated by IRFs that do meet this requirement.
2. Mokken's (1971) nonparametric monotone homogeneity model and hence parametric special cases such as the Rasch (1960) and Birnbaum (1968) models possess the MLR property.
3. Ghurye and Wallace (1959) first established the fundamental result. Their work stemmed from a problem posed by J. Loevinger on 'stochastic ordering of the latent trait' (for details, see Ghurye & Wallace, 1959), and the fundamental result was obtained as a corollary of general results concerning a convolutive class of MLR families. Grayson (1988) provided a direct proof of a special case of Theorem 1, further adding the assumption that the first derivative of any IRF exists everywhere on the latent trait and is positive at some point of the latent trait. Grayson's (1988) proof is not only a proof of a special case of the fundamental result, but is also rather long and intricate. In particular, the added differentiability assumption excludes

step-function IRFs as they arise from latent class measurement models such as the Lazarsfeld-Henry latent distance model (e.g., Heinen, 1996; Lazarsfeld & Henry, 1968). Huynh (1994) provided a proof of the fundamental result which does not require the extra assumption made by Grayson (1988). Unfortunately, Huynh's (1994) proof is flawed (Section 4).

3 Proof

In this section, a simple proof of the fundamental result is presented.

Let $n \in \mathbb{N}_{\geq 1}$. Denote by $S_r(x)$ $(0 \leq r \leq n)$ the rth *elementary symmetric function* of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$; that is,

$$\begin{aligned}
S_0(x) &:= 1, \\
S_1(x) &:= \sum_{i=1}^{n} x_i, \\
S_2(x) &:= \sum_{1 \leq i < j \leq n} x_i x_j, \\
S_3(x) &:= \sum_{1 \leq i < j < k \leq n} x_i x_j x_k, \\
&\vdots \\
S_n(x) &:= \prod_{i=1}^{n} x_i.
\end{aligned}$$

The proof of the fundamental result relies on the following *basic* inequality for elementary symmetric functions of which I give a simple proof. The proof is direct and elementary based on few purely algebraic, straightforward manipulations. [4]

[4] This inequality was originally suggested but justified in the wrong way by Huynh (1994); see Section 4.

Lemma 1 (Basic inequality) *If* $1 \leq r \leq n$, *then*

$$\frac{S_r(u+v)}{S_{r-1}(u+v)} \geq \frac{S_r(u)}{S_{r-1}(u)}$$

for any $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n_{\geq 0}$ *for which* $S_{r-1}(u) \neq 0$ *(e.g., for positive* $u \in \mathbb{R}^n_{>0}$*, as in the proof of Theorem 1), and for any* $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n_{\geq 0}$.

Proof. Without restriction, let $2 \leq r \leq n$. I have to show that

$$\frac{\sum_{1\leq i_1<\ldots<i_r\leq n} \prod_{j=1}^{r}(u_{i_j}+v_{i_j})}{\sum_{1\leq i'_1<\ldots<i'_{r-1}\leq n} \prod_{j'=1}^{r-1}(u_{i'_{j'}}+v_{i'_{j'}})} \geq \frac{\sum_{1\leq i''_1<\ldots<i''_r\leq n} \prod_{j''=1}^{r} u_{i''_{j''}}}{\sum_{1\leq i'''_1<\ldots<i'''_{r-1}\leq n} \prod_{j'''=1}^{r-1} u_{i'''_{j'''}}}.$$

This inequality is equivalent to

$$\sum_{1\leq i'''_1<\ldots<i'''_{r-1}\leq n} \sum_{1\leq i_1<\ldots<i_r\leq n} \left(\prod_{j'''=1}^{r-1} u_{i'''_{j'''}}\right)\left(\prod_{j=1}^{r}(u_{i_j}+v_{i_j})\right)$$
$$\geq \sum_{1\leq i'_1<\ldots<i'_{r-1}\leq n} \sum_{1\leq i''_1<\ldots<i''_r\leq n} \left(\prod_{j'=1}^{r-1}(u_{i'_{j'}}+v_{i'_{j'}})\right)\left(\prod_{j''=1}^{r} u_{i''_{j''}}\right).$$

For any choices of $1 \leq i'_1 < \ldots < i'_{r-1} \leq n$ and $1 \leq i''_1 < \ldots < i''_r \leq n$, there is an appropriate $i''_{\widehat{j''}}$ $(1 \leq \widehat{j''} \leq r)$ such that $i''_{\widehat{j''}} \neq i'_{j'}$ for any $1 \leq j' \leq r-1$. For such a list of indices, for the right-hand side of the previous inequality I finally have

$$\sum_{1\leq i'_1<\ldots<i'_{r-1}\leq n} \sum_{1\leq i''_1<\ldots<i''_r\leq n} \left(\prod_{j'=1}^{r-1}(u_{i'_{j'}}+v_{i'_{j'}})\right)\left(\prod_{j''=1}^{r} u_{i''_{j''}}\right)$$
$$\leq \sum_{1\leq i'_1<\ldots<i'_{r-1}\leq n} \sum_{1\leq i''_1<\ldots<i''_r\leq n} (u_{i''_{\widehat{j''}}}+v_{i''_{\widehat{j''}}})\left(\prod_{j'=1}^{r-1}(u_{i'_{j'}}+v_{i'_{j'}})\right)\left(\prod_{1\leq j''\leq r, j''\neq\widehat{j''}} u_{i''_{j''}}\right)$$
$$\leq \sum_{1\leq i'''_1<\ldots<i'''_{r-1}\leq n} \sum_{1\leq i_1<\ldots<i_r\leq n} \left(\prod_{j'''=1}^{r-1} u_{i'''_{j'''}}\right)\left(\prod_{j=1}^{r}(u_{i_j}+v_{i_j})\right).$$

□

The basic inequality is at the core of the following proof of Theorem 1.

Proof of Theorem 1. Let $\theta \in \Theta$ and $0 \leq x_+ \leq m$. As described in Huynh (1994), the term $P(X_+ = x_+|\theta)$ can be expressed as

$$P(X_+ = x_+|\theta) = f(\theta) \cdot \sum_{x \in \mathcal{R}_{x_+}} \prod_{l=1}^{m} \left[\xi_l(\theta)^{x_l}\right] = f(\theta) \cdot S_{x_+}(\xi(\theta)), \tag{1}$$

where

$$\begin{aligned}
f(\theta) &:= \prod_{l=1}^{m} \left[1 - P(X_l = 1|\theta)\right], \\
\mathcal{R}_{x_+} &:= \{x = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m : \sum_{l=1}^{m} x_l = x_+\}, \\
\xi_l(\theta) &:= \frac{P(X_l = 1|\theta)}{1 - P(X_l = 1|\theta)} \quad (1 \leq l \leq m),
\end{aligned}$$

and $S_{x_+}(\xi(\theta))$ is the x_+th elementary symmetric function of the positive vector $\xi(\theta) := (\xi_1(\theta), \xi_2(\theta), \ldots, \xi_m(\theta)) \in \mathbb{R}^m_{>0}$. (At this point, the assumption of local independence and the requirement that IRFs assume values strictly between zero and one are applied.) Let $0 \leq x_{+,1} < x_{+,2} \leq m$, and let $\theta_1, \theta_2 \in \Theta$ with $\theta_1 < \theta_2$. By the assumption of monotonicity, $P(X_l = 1|\theta_1) \leq P(X_l = 1|\theta_2)$ for any $1 \leq l \leq m$. Hence, $\xi_l(\theta_1) \leq \xi_l(\theta_2)$ for any $1 \leq l \leq m$. In particular, $\xi_l(\theta_2) = \xi_l(\theta_1) + v_l$, for $v_l \in \mathbb{R}_{\geq 0}$ $(1 \leq l \leq m)$. That is, $\xi(\theta_2) = \xi(\theta_1) + v$, for $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m_{\geq 0}$.

Based on Lemma 1, the property of MLR can now be proved:

$$\begin{aligned}
&\frac{P(X_+ = x_{+,2}|\theta_2)}{P(X_+ = x_{+,1}|\theta_2)} \\
&\overset{\text{Eq. (1)}}{=} \frac{S_{x_{+,2}}(\xi(\theta_2))}{S_{x_{+,1}}(\xi(\theta_2))} \\
&= \frac{S_{x_{+,2}}(\xi(\theta_1) + v)}{S_{x_{+,2}-1}(\xi(\theta_1) + v)} \cdot \frac{S_{x_{+,2}-1}(\xi(\theta_1) + v)}{S_{x_{+,2}-2}(\xi(\theta_1) + v)} \cdot \ldots \cdot \frac{S_{x_{+,1}+1}(\xi(\theta_1) + v)}{S_{x_{+,1}}(\xi(\theta_1) + v)} \\
&\overset{\text{Lemma 1}}{\geq} \frac{S_{x_{+,2}}(\xi(\theta_1))}{S_{x_{+,2}-1}(\xi(\theta_1))} \cdot \frac{S_{x_{+,2}-1}(\xi(\theta_1))}{S_{x_{+,2}-2}(\xi(\theta_1))} \cdot \ldots \cdot \frac{S_{x_{+,1}+1}(\xi(\theta_1))}{S_{x_{+,1}}(\xi(\theta_1))}
\end{aligned}$$

$$= \frac{S_{x_{+,2}}(\xi(\theta_1))}{S_{x_{+,1}}(\xi(\theta_1))}$$
$$\overset{\text{Eq. (1)}}{=} \frac{P(X_+ = x_{+,2}|\theta_1)}{P(X_+ = x_{+,1}|\theta_1)}.$$

□

Before contrasting this proof with Huynh's (1994) argument (Section 4), it is instructive to see what prerequisites in Theorem 1 are actually necessary to prove the theorem. Obviously, the proof rests on the assumptions of local independence and monotonicity, and the requirement that IRFs assume values strictly between zero and one. However, the algebraic (e.g., addition) and *extra* order-theoretic (e.g., completeness) properties underlying the assumption of unidimensionality are not required for the proof:[5] The property of MLR is solely formulated 'locally,' that is, for 'pairs' of latent trait points $\theta_1, \theta_2 \in \Theta$ which are in $\leq$-relation $\theta_1 < \theta_2$. In fact, a variant of Theorem 1 also applies in a more general setting where the assumption of unidimensionality is weakened to allow for abstract nonlinear latent trait spaces (Section 5).

4 Correction to Huynh's (1994) argument

Huynh's (1994) proof of the fundamental result (Theorem 1) is wrongly based on the following inequality for elementary symmetric functions by Marcus and Lopes (1957).

[5] The assumption of unidimensionality is here understood as the mathematical structure $\langle \Theta \subseteq \mathbb{R}, +, -, 0, \cdot, ^{-1}, 1, \leq \rangle$.

Theorem 2 (Marcus-Lopes inequality) *If* $1 \leq r \leq n$, *then*

$$\frac{S_r(u+v)}{S_{r-1}(u+v)} \geq \frac{S_r(u)}{S_{r-1}(u)} + \frac{S_r(v)}{S_{r-1}(v)}$$

for any $u = (u_1, u_2, \ldots, u_n)$ *and* $v = (v_1, v_2, \ldots, v_n)$ *in* $\mathbb{R}^n_{\geq 0}$, *where for each of these vectors* ***at least r*** *of its elements are assumed to be positive.*

Proof. See Marcus and Lopes (1957, pp. 306–307, Theorem 1). □

In the context of the fundamental result, the Marcus-Lopes inequality is not applicable. Huynh's (1994) conclusion of the basic inequality (Lemma 1) from the Marcus-Lopes inequality is wrong and not justified. In Theorem 1, IRFs are assumed to be nondecreasing functions. Hence in the proof one can have $\xi(\theta_2) = \xi(\theta_1) + v$ (for $\theta_1, \theta_2 \in \Theta$, $\theta_1 < \theta_2$), where $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m_{\geq 0}$ may, in principle, contain any number $1 \leq L \leq m$ of zeros. Theorem 2, however, requires *at least* r of the elements v_l ($1 \leq l \leq m$) to be positive.

An appreciation for this issue can be gained by the following example. Consider $m := 4$ dichotomous items. Let $x_{+,1} := 2$ and $x_{+,2} := 3$. Let $\theta_1, \theta_2 \in \Theta$, $\theta_1 < \theta_2$, with $P(X_1 = 1|\theta_1) < P(X_1 = 1|\theta_2)$ and $P(X_l = 1|\theta_1) = P(X_l = 1|\theta_2)$ for any $2 \leq l \leq 4$. Then, $\xi(\theta_2) = \xi(\theta_1) + v$, with $v = (v_1, 0, 0, 0)$ and $v_1 > 0$. Under these conditions,

$$\frac{P(X_+ = 3|\theta_2)}{P(X_+ = 2|\theta_2)} = \frac{S_3(\xi(\theta_2))}{S_2(\xi(\theta_2))} = \frac{S_3(\xi(\theta_1) + v)}{S_2(\xi(\theta_1) + v)}.$$

But the Marcus-Lopes inequality *cannot* be used to conclude that

$$\frac{S_3(\xi(\theta_1) + v)}{S_2(\xi(\theta_1) + v)} \geq \frac{S_3(\xi(\theta_1))}{S_2(\xi(\theta_1))} + \frac{S_3(v)}{S_2(v)},$$

in order to obtain the required statement

$$\frac{P(X_+ = 3|\theta_2)}{P(X_+ = 2|\theta_2)} \geq \frac{S_3(\xi(\theta_1))}{S_2(\xi(\theta_1))} + \frac{S_3(v)}{S_2(v)} \geq \frac{S_3(\xi(\theta_1))}{S_2(\xi(\theta_1))} = \frac{P(X_+ = 3|\theta_1)}{P(X_+ = 2|\theta_1)}.$$

The reason for this fact is that only one element of v, namely v_1, is positive, contrary to *at least three* positive elements as required for the Marcus-Lopes inequality to hold. Note also that the summand $S_3(v)/S_2(v)$ on the right-hand side of the Marcus-Lopes inequality is not well-defined since the denominator $S_2(v)$ is equal to zero.

Furthermore, Huynh's (1994) reference to Marshall and Olkin (1979, p. 80) utilises the Marcus-Lopes inequality for more restrictive vectors *all* of the elements of which are assumed to be positive.

In sum, the basic inequality is the crucial statement solely required for the proof of the fundamental result. Though this inequality is elementary, it *cannot* be derived from the Marcus-Lopes inequality. The basic inequality has to be proved separately. In the note, this is accomplished by the simple proof of Lemma 1.

5 Generalized fundamental result

Finally, the fundamental result is extended to the case of nonlinear latent trait spaces.

A *nonlinear latent trait space* is a pair $(\Theta, \preceq)$, with Θ a nonempty set of *latent trait points*, and $\preceq$ a quasi-order (reflexive and transitive binary relation) on Θ. I refer to this as the assumption of *nonlinear dimensionality.*[6] The unidimensional concepts introduced in Section 1 can be formulated for abstract nonlinear latent trait spaces. For instance, a function $f : \Theta \to \mathbb{R}$ is *isotonic* iff

$$\forall \theta_1, \theta_2 \in \Theta, \theta_1 \preceq \theta_2 : f(\theta_1) \leq f(\theta_2).$$[7]

Or, the assumption of *isotonicity* means that all IRFs are isotonic.

The *generalized* fundamental result (for dichotomous items) is as follows.

Corollary 1 (Generalized fundamental result) *Under the assumptions of nonlinear dimensionality, local independence, and isotonicity, and the requirement that each item response function assumes values strictly between zero and one, the total score variable has monotone likelihood ratio in the (nonlinear) latent trait.*

Proof. The proof of Theorem 1 carries over easily (cf. Section 3). □

Discussion. What can be gained from such a generalization of the fundamental result?

1. The generalized fundamental result is at the interface of IRT and knowledge

[6] Important examples of nonlinear latent trait spaces are provided by Euclidean subsets $\Theta \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}_{\geq 2}$) which are partially ordered with respect to, for instance, coordinate-wise vector-ordering (multidimensional IRT), or by set-families which are partially ordered with respect to set-inclusion (knowledge space theory). The class of knowledge space theory examples is discussed in more detail below.

[7] For $\preceq$-*incomparable* latent trait points $\theta_1, \theta_2 \in \Theta$, that is, $\theta_1 \not\preceq \theta_2$ and $\theta_2 \not\preceq \theta_1$, no restrictions are imposed on the relationship of the function values $f(\theta_1)$ and $f(\theta_2)$ to each other.

space theory (KST), two modern but still split directions of psychological test theories.[8] It provides a first connection between the two theories on a *nonparametric* probabilistic basis. The KST variant of the generalized fundamental result (for dichotomous items) takes the following form.

Corollary 2 (KST variant of generalized fundamental result) *Let Q denote the item set $\{I_l : 1 \leq l \leq m\}$. Let $\mathcal{K}$ be a knowledge structure on Q, that is, a family of subsets of Q containing at least the empty set $\emptyset$ and Q. Then, for the nonlinear latent trait space $(\mathcal{K}, \subseteq)$, under the assumptions of local independence and isotonicity, and the requirement that $0 < P(X_l = 1|K) < 1$ for any $K \in \mathcal{K}$ and $1 \leq l \leq m$, the total score variable has monotone likelihood ratio in the (nonlinear) latent trait $K \in \mathcal{K}$.*

Proof. Here, $\Theta := \mathcal{K}$ and $\preceq := \subseteq$. □

2. As mentioned in Section 1, the MLR property is rather technical and its (hence the fundamental result's) importance in IRT actually stems from the fact that it implies the SOM and SOL properties (unidimensional case). Whether the generalized MLR property still implies the generalized SOM and/or generalized SOL properties, that is, whether the generalized fundamental result may still apply, (nonlinear case) is thoroughly investigated in Ünlü (2006). Some of the relevant findings are (for details, see Ünlü, 2006): (a) The generalized fundamental result implies stochastic ordering of the total score variable by the nonlinear latent trait, but may fail to imply

[8] KST was introduced by Doignon and Falmagne (1985); see also Doignon and Falmagne (1999) and Falmagne, Koppen, Villano, Doignon, and Johannesen (1990).

stochastic ordering of the nonlinear latent trait by the total score variable. (b) The reason for this fact (order-theoretic completeness property) and (technical) conditions under which the implication holds are specified. (c) Simulations demonstrate that violations of the generalized SOL property occur only for extreme (unrealistic) parameter vectors.

In any case, an extension of unidimensional ordering properties to the case of incomparabilities among latent trait points may prove valuable in nonlinear settings such as multidimensional IRT or KST.

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