ON A THEOREM BY EKELAND-HOFER

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Abstract. In [EH89, Theorem 1] Ekeland-Hofer prove that for a centrally symmetric, restricted contact type hypersurface in $\mathbb{R}^{2n}$ and for any global, centrally symmetric Hamiltonian perturbation there exists a leaf-wise intersection point. In this note we show that if we replace restricted contact type by star-shaped there exists infinitely many leaf-wise intersection points or a leaf-wise intersection point on a closed characteristic.

1. Introduction

Let $S \subset \mathbb{R}^{2n}$ be a hypersurface. Then S carries a rank-1-foliation where the tangent space to a leaf $L_S(x)$ through $x \in S$ is given by $\mathscr{L}_S(x) := \{v \in T_xS \mid \omega(v,w) = 0 \ \ \forall w \in T_xS\}$. Here ω is the standard symplectic form on $\mathbb{R}^{2n}$. A point $x \in S$ such that

$$\psi_1(x) \in L_S(x) \tag{1.1}$$

is called a leaf-wise intersection point, see [Mos78]. A hypersurface is of restricted contact type if there exists a 1-form $\lambda \in \Omega^1(\mathbb{R}^{2n})$ with

$$\begin{cases} d\lambda = \omega \\ \lambda_x(v) \neq 0 \quad \forall v \in \mathscr{L}_S(x)\ . \end{cases} \tag{1.2}$$

We call a hypersurface $\mathbb{Z}/2$-invariant or centrally symmetric if it is invariant under the symplectic involution $I : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ given by $I(x) = -x$. We set $D_{\omega,\mathbb{Z}/2} := \{\phi \in \mathrm{Symp}(\mathbb{R}^{2n}) \mid \phi \circ I = I \circ \phi\}$. In [EH89] Ekeland and Hofer prove the following theorem.

Theorem 1.1 ([EH89], Theorem 1)**.** *Assume that $S \subset \mathbb{R}^{2n}$ is a connected, compact, $\mathbb{Z}/2$-invariant hypersurface of restricted contact type. Let $t \to \psi_t$ be an isotopy of the identity in $D_{\omega,\mathbb{Z}/2}$. Then there exists a leafwise intersection point $x \in S$.*

In this article we improve Theorem 1.1 under the additional assumption that S bounds a star-shaped (with respect to the origin) region in $\mathbb{R}^{2n}$, that is, it is of restricted contact type with respect to the standard primitive $\lambda_0 = \frac{1}{2}\sum x_i dy_i - y_i dx_i$.

Theorem 1.2. *Assume that $S \subset \mathbb{R}^{2n}$ is a connected, compact, $\mathbb{Z}/2$-invariant, star-shaped hypersurface. Let $t \to \psi_t$ be an isotopy of the identity in $D_{\omega,\mathbb{Z}/2}$. Then there exist infinitely many leaf-wise intersection points on S or there exists a leaf-wise intersection point y such that the leaf $L_S(y)$ is closed, that is, $L_S(y)$ is a closed characteristic.*

Remark 1.3. If $n \geq 2$ then for a generic isotopy of $\mathbb{Z}/2$-equivariant $\psi_t \in \mathbb{D}_{\omega,\mathbb{Z}/2}$ there are no leaf-wise intersection points on closed characteristics. Hence, there exist infinitely many leaf-wise intersection points. This follows from a $\mathbb{Z}/2$-invariant version of [AF08a, Theorem

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3.3]. That Theorem 3.3 holds in the $\mathbb{Z}/2$-invariant case is due to the fact that for critical points (v,η) of $\mathcal{A}$ (see below) the loop v does not pass through the fix point 0 of I since for an invariant Hamiltonian function H the Hamiltonian flow ϕ_H^t fixes 0 for all times.

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2. Equivariant Rabinowitz Floer homology

We consider the standard symplectic space $(\mathbb{R}^{2n},\omega=d\lambda_0)$ and the symplectic involution $I:\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n}$ given by $I(x)=-x$. Let $F:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$ be the I-invariant function $F(x):=\frac{1}{2}\big(|x|^2-1\big)$. In particular, the Rabinowitz action functional

$$\begin{aligned}\mathcal{A}:C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}&\longrightarrow\mathbb{R}\\ (v,\eta)&\mapsto-\int v^*\lambda_0-\eta\int F(v)dt\end{aligned}\tag{2.1}$$

is invariant under I. Moreover, I acts freely on the critical points of $\mathcal{A}$ and on the space of gradient flow lines (in the sense of Floer) which asymptotically converge to critical points, where we use an I-invariant compatible almost complex structure J to define the gradient of $\mathcal{A}$. Therefore, we can construct equivariant Rabinowitz Floer homology easily as follows:

$$\begin{aligned}\mathrm{RFC}_k^{\mathbb{Z}/2}(S^{2n-1},\mathbb{R}^{2n})&:=\mathrm{RFC}_k(S^{2n-1},\mathbb{R}^{2n})\Big/(\mathbb{Z}/2)\\ \partial^{\mathbb{Z}/2}[x]&:=[\partial x]\end{aligned}\tag{2.2}$$

For details on the construction of Rabinowitz Floer homology and its relation with leaf-wise intersection points we refer to [CF09, AF08b]

Theorem 2.1 ([CF09])**.** *Since S^{2n-1} is Hamiltonianly displaceable*

$$\mathrm{RFH}_*(S^{2n-1},\mathbb{R}^{2n})\cong 0\tag{2.3}$$

Theorem 2.2. *For all $k\in\mathbb{Z}$ we have*

$$\mathrm{RFH}_k^{\mathbb{Z}/2}(S^{2n-1},\mathbb{R}^{2n})\cong\mathbb{Z}/2\ .\tag{2.4}$$

Proof. For a critical point (v,η) of $\mathcal{A}$ the η-periodic loop $v(t/\eta)$ is a Reeb orbit on S^{2n-1} with respect to the standard contact form or in case $(v,0)$ the loop $v(t)$ is constant and represents a point on S^{2n-1}. Since the Reeb flow φ^t on S^{2n-1} is periodic the action functional $\mathcal{A}$ is Morse-Bott with critical manifolds

$$C_k\cong S^{2n-1}\quad k\in\mathbb{Z}\tag{2.5}$$

where a point $x\in S^{2n-1}$ is identified with $(t\mapsto\varphi^{2\pi kt}(x),2\pi k)\in C_k$. The Conley-Zehnder index μ_{CZ} equals $2nk$ on C_k. We fix on S^{2n-1} the Morse function

$$f(x_1,\dots,x_{2n}):=\sum_{i=1}^{2n}ix_i^2\ .\tag{2.6}$$

f descends to a $\mathbb{Z}/2$-perfect Morse function $\bar{f}$ on $\mathbb{RP}^{2n-1} = S^{2n-1}/(\mathbb{Z}/2)$. In particular, $\bar{f}$ has precisely one critical point in each degree $0, \dots, 2n-1$ and therefore, f has critical points y_l, z_l of degree $l = 0, \dots, 2n-1$ satisfying $-y_l = z_l$. The Morse differential δ computes to

$$\delta y_l = y_{l-1} + z_{l-1} = \delta z_l, \quad \forall\, l = 1, \dots, 2n-1 \tag{2.7}$$

and therefore in the quotient using the notation $\xi_l := [y_l] = [z_l]$

$$\delta \xi_l = 2\xi_{l-1} = 0 \quad \forall\, l = 1, \dots, 2n-1\ . \tag{2.8}$$

We define Morse functions

$$f^k : C_k \longrightarrow \mathbb{R}, \quad k \in \mathbb{Z} \tag{2.9}$$

by $f^k := f$ via the identification $C_k \cong S^{2n-1}$. We denote the critical points of f^k by y_l^k, z_l^k, $l = 0, \dots, 2n-1$.

The boundary operator ∂ in Rabinowitz Floer homology is defined by counting gradient flows lines with cascades, see [Fra04, CF09]. Since the Conley-Zehnder index equals $2nk$ on the critical manifolds C_k the complex $\mathrm{RFC}_*(S^{2n-1}, \mathbb{R}^{2n})$ has exactly two generators in each degree. By index and energy reasons

$$\partial y_l^k = \delta y_l^k \text{ and } \partial z_l^k = \delta z_l^k \quad \forall l = 1, \dots, 2n-1,\ \forall k \in \mathbb{Z}\ . \tag{2.10}$$

Again, by index reasons and by symmetry there exists $a^k, b^k \in \mathbb{Z}/2$ with

$$\partial y_0^{k+1} = a^k y_{2n-1}^k + b^k z_{2n-1}^k = \partial z_0^{k+1} \quad \forall k \in \mathbb{Z}\ . \tag{2.11}$$

From $\partial \circ \partial = 0$ and (2.10) we conclude $a^k = b^k$. According to Theorem 2.1 by Cieliebak-Frauenfelder we have $\mathrm{RFH}_*(S^{2n-1}, \mathbb{R}^{2n}) \cong 0$. This implies that $a^k = b^k = 1$ since otherwise y_0^{k+1} is a cycle but not a boundary, compare figure 1.

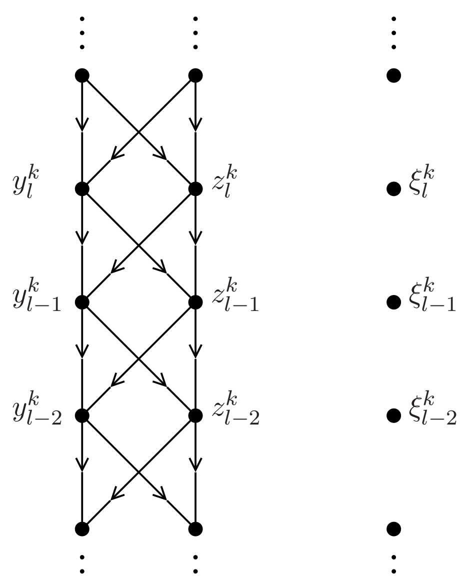

FIGURE 1. The non-equivariant and the equivariant chain complexes.

With this we can compute the $\mathbb{Z}/2$-equivariant complex $\big(\mathrm{RFC}_k^{\mathbb{Z}/2}(S^{2n-1}, \mathbb{R}^{2n}), \partial^{\mathbb{Z}/2}\big)$ as follows. We have generators $\xi_l^k := [y_l^k] = [z_l^k]$, $l = 0, \dots, 2n-1$, $k \in \mathbb{Z}$, of degree $\deg \xi_l^k =$

$l+2nk$. In particular, there is exactly one critical point in each degree. We compute

$$\begin{aligned}\partial^{\mathbb{Z}/2}\xi_l^k=[\partial y_l^k]&=\begin{cases}[y_{l-1}^k+z_{l-1}^k] & \text{for } l=1,\dots,2n-1\\ [y_{2n-1}^{k-1}+z_{2n-1}^{k-1}] & \text{for } l=0\end{cases}\\ &=\begin{cases}2\xi_{l-1}^k & \text{for } l=1,\dots,2n-1\\ 2\xi_{2n-1}^{k-1} & \text{for } l=0\end{cases}\\ &=0\ .\end{aligned}\tag{2.12}$$

That is, the equivariant complex is acyclic. This proves the Theorem. □

In [AF10] we associated spectral values $\sigma(\xi)$ to homology classes ξ in Rabinowitz Floer homology. We define

$$\mathfrak{S}:=\{\sigma(\xi_l^k)\mid \xi_l^k\in \mathrm{RFH}_*^{\mathbb{Z}/2}(S^{2n-1},\mathbb{R}^{2n})\}\ .\tag{2.13}$$

From the proof of Theorem 2.1 is follows immediately

$$\mathfrak{S}=2\pi\mathbb{Z}\tag{2.14}$$

since $\mathcal{A}(\xi_l^k)=-2\pi k$.

3. Proof of Theorem 1.2

We first assume that the isotopy ψ_t is generated by a compactly supported Hamiltonian function $H:\mathbb{R}^{2n}\times[0,1]\longrightarrow\mathbb{R}$. Since $\psi_t\circ I=I\circ\psi_t$ we can assume

$$H(t,I(x))=H(t,x)\ .\tag{3.1}$$

Moreover, since S is star-shaped it is a graph over the standard sphere S^{2n-1}. Therefore, we can find a family of functions $F_r:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$, $r\in[0,1]$, such that $F_1^{-1}(0)=S$, $F_0=F=\frac{1}{2}\big(x^2-1\big)$, and all hypersurfaces $F_r^{-1}(0)$ are I-invariant and graphs over S^{2n-1}. Thus, all Rabinowitz action functionals

$$\begin{aligned}\mathcal{A}_r:C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}&\longrightarrow\mathbb{R}\\ (v,\eta)&\mapsto-\int v^*\lambda_0-\eta\int F_r(v)dt-\int rH(t,v)dt\end{aligned}\tag{3.2}$$

are I-invariant. Moreover, I acts freely on the critical points and gradient flow lines for each $r\in[0,1]$. Equality 2.14 implies that $\mathcal{A}_0$ has critical points of arbitrarily large critical value. [AF10, Corollary 5.13] implies that then also $\mathcal{A}_1$ has to have critical points with arbitrarily large critical value. In particular, $\mathcal{A}_1$ has infinitely many critical points. It follows from [AF08b, Proposition 2.4] that critical points of $\mathcal{A}_1$ give rise to leaf-wise intersections. Moreover, the map from critical points to leaf-wise intersection points is injective unless there exists a leaf-wise intersection on a closed characteristic. This proves the theorem in case that ψ_t is generated by a compactly supported Hamiltonian function.

A general isotopy $\psi_t\in\mathrm{Symp}(\mathbb{R}^{2n})$ is generated by a Hamiltonian function $\widetilde{H}:\mathbb{R}^{2n}\times[0,1]\longrightarrow\mathbb{R}$ which however is not necessarily compactly supported. The set

$$K:=\{\psi_t(x)\mid x\in S,\ t\in[0,1]\}\subset\mathbb{R}^{2n}\tag{3.3}$$

is compact since S is compact. In particular, all critical points (v,η) of $\mathcal{A}$ satisfy

$$v(t)\in K\quad\forall t\in S^1\ .\tag{3.4}$$

Thus, we can cut-off $\widetilde{H}$ outside K to make it into a compactly supported Hamiltonian without changing the critical points of $\mathcal{A}$. Thus, by first part of the proof we are done.

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