Floer homology for magnetic fields with at most linear growth on the universal cover

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Abstract

The Floer homology of a cotangent bundle is isomorphic to loop space homology of the underlying manifold, as proved by Abbondandolo-Schwarz, Salamon-Weber, and Viterbo. In this paper we show that in the presence of a Dirac magnetic monopole which admits a primitive with at most linear growth on the universal cover, the Floer homology in atoroidal free homotopy classes is again isomorphic to loop space homology. As a consequence we prove that for any atoroidal free homotopy class and any sufficiently small $\tau > 0$, any magnetic flow associated to the Dirac magnetic monopole has a closed orbit of period τ belonging to the given free homotopy class. In the case where the Dirac magnetic monopole admits a bounded primitive on the universal cover we also prove the Conley conjecture for Hamiltonians that are quadratic at infinity, i.e., we show that such Hamiltonians have infinitely many periodic orbits.

1 Introduction

We are interested in Hamiltonian systems of the following form. The configuration space M is a closed connected oriented manifold of dimension $n \geq 2$. The Hamiltonian H is a smooth function on the phase space T^*M which might in addition depend periodically on time. The Dirac magnetic monopole is a closed two-form $\sigma \in \Omega^2(M)$ which gives rise to a twisted symplectic form [6, 11] on the cotangent bundle

$$\omega_\sigma = d\lambda + \pi^*\sigma,$$

where λ is the Liouville one-form on T^*M and $\pi\colon T^*M \to M$ is the footpoint projection. The flow is generated by the time dependent Hamiltonian vector field $X_{H,\sigma}$ defined implicitly by the equation

$$-dH = \omega_\sigma(X_{H,\sigma}, \cdot).$$

Floer's semi-infinite dimensional Morse homology associates to a Hamiltonian system a chain complex which is generated by the periodic orbits of a given fixed period $\tau > 0$ and a given free homotopy class $\alpha \in [S^1, M]$, and defines a boundary operator by counting perturbed holomorphic cylinders which asymptotically converge to the periodic orbits. A priori it is far form obvious that this recipe gives a well defined boundary operator. Indeed, the question if Floer's boundary operator is well-defined or not depends on a difficult compactness result for the perturbed holomorphic curve equation. Tentatively we write $HF_*^\alpha(H, \sigma, \tau)$ for the Floer homology with the τ-periodic Hamiltonian H, the magnetic monopole σ, and the free homotopy class α. In order to avoid discussions about orientations of moduli spaces we take coefficients in $\mathbb{Z}_2$. In the case where the magnetic monopole vanishes and the Hamiltonian satisfies some asymptotic fibrewise quadratic growth condition considered by Abbondandolo-Schwarz the following remarkable result holds true.

Theorem 1. [Abbondandolo-Schwarz [3], Salamon-Weber [21], Viterbo [22]]. *If the τ-periodic H satisfies the Abbondandolo-Schwarz growth conditions, then Floer homology $HF_*^\alpha(H, 0, \tau)$ is well-defined in every free homotopy class α, and is isomorphic to the singular homology of the space of τ-periodic loops on M belonging to the given free homotopy class α.*

The precise definition of the Abbondandolo-Schwarz growth condition is given in Definition 5 below. In this paper we will prove the following extension of Theorem 1.

Theorem A. *Assume that the τ-periodic Hamiltonian H satisfies the Abbondandolo-Schwarz growth conditions and that σ admits a primitive of at most linear growth on the universal cover of M. Then there exists $\delta_0(H,\sigma) > 0$ such that if $|\delta|\tau < \delta_0(H,\sigma)$ then the Floer homology $HF_*^\alpha(H,\delta\sigma,\tau)$ is well defined for all σ-atoroidal classes $\alpha \in [S^1, M]$, and is again isomorphic to the singular homology of the space of τ-periodic loops on M belonging to the given free homotopy class α. If moreover σ admits a bounded primitive on the universal cover then $\delta_0(H,\sigma) = \infty$.*

We now explain the two new terms in the statement of Theorem A: a primitive of at most linear growth, and σ-atoroidal free homotopy class.

Definition. We say that σ admits a **primitive with at most linear growth** if the following condition holds:

($\boldsymbol{\sigma}$1) The 2-form σ is **weakly exact**. This means that the lift $\widetilde{\sigma}$ of σ to $\widetilde{M}$ is exact (in particular, σ is closed). Moreover $\widetilde{\sigma}$ admits a primitive with **at most linear growth**: there exists $\theta \in \Omega^1(\widetilde{M})$ such that $d\theta = \widetilde{\sigma}$ and such that for any $z \in \widetilde{M}$ there exists a constant $\Theta_z > 0$ such that for all $r \geq 0$,

$$\sup_{q \in B(z,r)} |\theta_q| \leq \Theta_z(r+1). \tag{1.1}$$

Here $B(z,r)$ denotes the geodesic ball of radius r in $\widetilde{M}$ about z, and both the geodesic metric and the norm $|\cdot|$ are defined using the lift of some Riemannian metric g on M to $\widetilde{M}$. Asking whether $\widetilde{\sigma}$ has a primitive with at most linear growth does not depend on the choice of metric g on M. Moreover as soon as (1.1) holds for some point $z \in \widetilde{M}$, it holds for all $z \in \widetilde{M}$.

Remarks.

1. The condition **($\boldsymbol{\sigma}$1)** includes the following stronger condition:

   ($\boldsymbol{\sigma}$0) The 2-form σ is **weakly exact**, and $\widetilde{\sigma}$ admits a **bounded** primitive: there exists $\theta \in \Omega^1(\widetilde{M})$ such that $d\theta = \widetilde{\sigma}$ and such that

   $$\sup_{q \in \widetilde{M}} |\theta_q| < \infty. \tag{1.2}$$

2. A classical result of Gromov [13] tells us that if M admits a metric of negative curvature then every closed 2-form σ satisfies **($\boldsymbol{\sigma}$0)**. In contrast, if σ is not exact and $\pi_1(M)$ is **amenable** then σ never satisfies **($\boldsymbol{\sigma}$0)** [19, Corollary 5.4]. The main examples of pairs (M, σ) where σ satisfies **($\boldsymbol{\sigma}$1)** are given by manifolds that admit a metric of non-positive curvature [9]. For tori $\mathbb{T}^n$ any closed non-exact 2-form σ satisfies **($\boldsymbol{\sigma}$1)** but not **($\boldsymbol{\sigma}$0)**.

Set $\mathbb{S}_\tau := \mathbb{R}/\tau\mathbb{Z}$. We often identify S^1 and $\mathbb{S}_1$. We denote by $\Lambda_\tau M := C^\infty(\mathbb{S}_\tau, M)$ the free τ-periodic loop space of M. The space $\Lambda_\tau M$ splits as a direct sum $\Lambda_\tau M = \bigoplus_{\alpha \in [S^1,M]} \Lambda_\tau^\alpha M$, where for a given free homotopy class $\alpha \in [S^1, M] \cong [\mathbb{S}_\tau, M]$, we define $\Lambda_\tau^\alpha M := \{q \in \Lambda_\tau M : [q] = \alpha\}$. Consider the 1-form $a_\sigma \in \Omega^1(\Lambda_\tau M)$ defined by

$$a_\sigma(q)(\xi) := \int_{\mathbb{S}_\tau} \sigma(\dot{q}, \xi)dt, \tag{1.3}$$

Since σ is closed, a_σ is closed, that is, the integral of a_σ over a closed path in $\Lambda_\tau M$ depends only the homology class of the path.

Definition. We say a class $\alpha \in [S^1, M]$ is a **σ-atoroidal class** if any map $f : S^1 \to \Lambda_1^\alpha M$ with $[f] = \alpha$ satisfies $\int_{S^1} f^* a_\sigma = 0$. Equivalently, α is a σ-atoroidal class if $a_\sigma|_{\Lambda_\tau^\alpha M}$ is exact (see (2.2)). Note that under the assumption that σ is weakly exact, the class 0 of nullhomotopic loops is atoroidal, since both statements are equivalent to the statement that $\sigma|_{\pi_2(M)} = 0$.

Let us briefly comment how these assumptions enter the proof of Theorem A:

1. The mere fact that σ admits a primitive on the universal cover implies that σ vanishes on $\pi_2(M)$. Hence ω_σ is symplectically aspherical and no bubbling off of holomorphic spheres can occur. This excludes the first obstruction to the compactness results needed to define the boundary operator.

2. On σ-atoroidal classes the action functional used to define the boundary operator is real valued, and hence the energy of its gradient flow lines depends only on their asymptotes. This excludes the second obstruction to the necessary compactness to define the boundary operator.

3. The third obstruction to compactness comes from the noncompactness of T^*M. To obtain an L^∞-bound on the perturbed holomorphic curves we follow the approach by Abbondandolo and Schwarz [3]. The assumption that σ admits a primitive with at most linear growth on the universal cover gives rise to a certain quadratic isoperimetric inequality which allows us to carry over the proof of Abbondandolo-Schwarz to this more general set-up. This enables us to show that the Floer homology groups $HF_*^\alpha(H, \delta\sigma, \tau)$ are well defined. Taking advantage of the quadratic isoperimetric inequality once more we construct a continuation isomorphism from $HF_*^\alpha(H, \delta\sigma, \tau)$ to the Floer homology $HF_*^\alpha(H, 0, \tau)$, and hence Theorem 1 implies our result.

The necessity of the assumption that $|\delta|\,\tau$ is small in Theorem A can be seen from the following example. Take as configuration space $M = \mathbb{T}^2$ the two-torus and as the Hamiltonian H take kinetic energy with respect to the standard flat metric on the torus. As magnetic monopole we choose the area form σ with respect to the standard metric, and work with period $\tau = 1$. Then for each $\delta\sigma$ the flow lines are either constant orbits on $\mathbb{T}^2$ or lift to circles of period $2\pi/\delta$ on the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$. Thus as long as $\delta < 2\pi$ the only periodic solutions of period one are the constant ones. Hence the critical manifold is a two-torus and the Floer homology is isomorphic to the homology of $\mathbb{T}^2$ which coincides with the homology of the contractible component of the loop space of $\mathbb{T}^2$. However if $\delta = 2\pi$ then the critical manifold is diffeomorphic to $T^*\mathbb{T}^2$ and hence not compact anymore and one cannot define Floer homology. If δ becomes larger than 2π the critical manifold is again a two-torus. But one can check that the Conley-Zehnder index of the critical manifold jumps by two once δ goes through 2π and therefore the Floer homology now differs from the loop space homology.

However, on the torus it **is** in fact possible to define the Floer homology $HF_*^\alpha(H, \delta\sigma, \tau)$ provided $|\delta|\,\tau \notin 2\pi\mathbb{Z}$, for **any** free homotopy class $\alpha \in [S^1, \mathbb{T}^2]$. More generally, let $g = \langle\cdot,\cdot\rangle$ denote a Riemannian metric on $\mathbb{T}^2$ and $f \in C^\infty(\mathbb{T}^2, \mathbb{R})$. Set $\sigma = f\,\mu_g$, and suppose there exists $k \in \mathbb{Z}$ such that

$$\frac{2\pi(k-1)}{\tau} < f(q) < \frac{2\pi k}{\tau} \quad \text{for all } q \in \mathbb{T}^2.$$

Fix $V \in C^\infty(\mathbb{S}_\tau \times \mathbb{T}^2, \mathbb{R})$ and set $H(t,q,p) := \frac{1}{2}|p|^2 + V(t,q)$. Then $HF_*^\alpha(H, \sigma, \tau)$ is well defined for every free homotopy class $\alpha \in [S^1, \mathbb{T}^2]$, and moreover

$$HF_*^\alpha(H, \sigma, \tau) = \begin{cases} H_{*+2k}(\mathbb{T}^2; \mathbb{Z}), & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases}$$

The proof of this result is specific to tori, and as such goes along somewhat different lines to that of Theorem A. For this reason the details of this proof will be discussed in a forthcoming paper.

Theorem A has the following immediate corollary. Recall that if H is given by a Riemannian metric, the Hamiltonian flow of $X_{H,\sigma}$ is called a **magnetic flow**.

Corollary B. *Let H be an autonomous Hamiltonian satisfying the Abbondandolo-Schwarz growth conditions and assume that σ admits a primitive with at most linear growth on the universal* *cover of M. Let $\alpha \in [S^1, M]$ be a σ-atoroidal class. Then for any $\tau > 0$ sufficiently small, the Hamiltonian flow of $X_{H,\sigma}$ has a closed orbit with period τ whose projection to M belongs to the class α. In particular, the same is true for any magnetic flow associated to σ.*

To appreciate the significance of Corollary B consider the following example. Let $M = \mathbb{T}^3$ and σ any closed 2-form cohomologous to $dq_1 \wedge dq_2$, where (q_1, q_2, q_3) are linear coordinates on the torus. It is easy to check that the homotopy class $\alpha = (0, 0, n)$ for any integer n is σ-atoroidal. Then given any metric on $\mathbb{T}^3$ the magnetic flow has a closed orbit of period τ in the class α for all $\tau > 0$ sufficiently small. In fact, for the standard flat metric and $\sigma = dq_1 \wedge dq_2$, the classes $(0, 0, n)$ are the only ones that contain closed orbits of any period.

Finally, in the case where $\widetilde{\sigma}$ admits a bounded primitive, note that Theorem A tells us that in particular

$$HF_n^0(H, \sigma, \tau) \cong H_n(\Lambda_\tau^0 M; \mathbb{Z}_2) \neq 0$$

for **all** τ-periodic Hamiltonians H satisfying the Abbondandolo-Schwarz growth conditions. As a consequence, Hein's proof [14] of the Conley conjecture for the cotangent bundle (which is itself based on Ginzburg's proof [12] for closed symplectically aspherical symplectic manifolds) goes through word for word, and thus we obtain the following statement.

Corollary C. (The Conley Conjecture for twisted cotangent bundles) *Assume that $\widetilde{\sigma}$ admits a bounded primitive. Let $\varphi = \phi_1^H : T^*M \to T^*M$ denote the time-1 map of a Hamiltonian $H : S^1 \times T^*M \to \mathbb{R}$ satisfying the Abbondandolo-Schwarz growth conditions. Assume that φ has only finitely many fixed points. Then φ has simple periodic orbits of arbitrarily large period.*

This paper has an appendix in which we show that a more classical approach is possible if we restrict to Hamiltonians that are in addition strictly fibrewise convex. More precisely, we obtain a (Lagrangian) action functional on the (completed) loop space $\Lambda_\tau^\alpha M$ which we show satisfies the Palais-Smale condition provided that σ admits a primitive of at most linear growth, H is fibrewise strictly convex and satisfies the Abbondandolo-Schwarz growth conditions, and $|\delta|\,\tau$ is sufficiently small. The Palais-Smale condition allows the construction of the Morse complex so one can recover again the homology of the loop space. We expect that it would be possible to prove Corollary C in the Lagrangian setting by combining the methods of the appendix with the work of Lu [15, 16] or Mazzucchelli [17].

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2 Constructing the Floer homology $HF_*^\alpha(H, \sigma, \tau)$

2.1 Preliminaries

Denote by $\mathscr{R}(M)$ the set of Riemannian metrics on M. Suppose $g = \langle \cdot, \cdot \rangle \in \mathscr{R}(M)$. The metric defines a **horizontal-vertical** splitting of TT^*M: given $z = (q, p) \in T^*M$

$$T_zT^*M = T_z^hT^*M \oplus T_z^vT^*M \cong T_qM \oplus T_q^*M;$$

here $T_z^hT^*M = \ker(\kappa_g : T_zT^*M \to T_q^*M)$, where κ_g is the connection map of the Levi-Civita connection ∇ of g, and $T_z^vT^*M = \ker(d\pi(z) : T_zT^*M \to T_qM)$. Given $\xi \in TT^*M$ we denote by ξ^h and ξ^v the horizontal and vertical components. Technically speaking $\xi^h \in TM$ and $\xi^v \in T^*M$, although we consistently use the "musical" isomorphism $v \mapsto \langle v, \cdot \rangle$ to identify TM with T^*M. The horizontal-vertical splitting also determines an almost complex structure J_g called the **metric almost complex structure** via

$$J_g = \begin{pmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{pmatrix}.$$

Recall that an almost complex structure J on T^*M is **$d\lambda$-compatible** if the bilinear form $G_J(\cdot,\cdot) := d\lambda(J\cdot,\cdot)$ defines a Riemannian metric on T^*M. The metric almost complex structure J_g is compatible for every Riemannian metric g on M, and we abbreviate $G_g := G_{J_g}$. We denote the set of all $d\lambda$-compatible almost complex structures by $\mathscr{J}(T^*M)$ and equip it with the C^∞_{loc}-topology.

Denote by $\Lambda_\tau T^*M := C^\infty(\mathbb{S}_\tau, T^*M)$. Given $x = (q,p) \in \Lambda_\tau T^*M$ and $r \geq 1$ we define

$$\|p\|_{L^r_g(\mathbb{S}_\tau)} := \left(\int_0^\tau |p|^r \, dt\right)^{1/r} \quad \text{for } 1 \leq r < \infty,$$

and

$$\|p\|_{L^\infty_g(\mathbb{S}_\tau)} := \sup_{t\in\mathbb{S}_\tau} |p(t)| \, .$$

Similarly given $\xi \in T_x\Lambda_\tau T^*M$ and $J \in \mathscr{J}(T^*M)$ we define

$$\|\xi\|_{L^r_{G_J}(\mathbb{S}_\tau)} := \left(\int_0^\tau [G_J(\xi,\xi)]^{r/2} \, dt\right)^{1/r} .$$

Given $X \in \Gamma(\mathrm{End}(TM))$ we define

$$\|X\|_{L^\infty_g} := \sup_{q\in M} \sup \left\{ |X(q)v| \, : \, v \in T_qM, \ |v| = 1 \right\} .$$

Let us now fix a closed 2-form $\sigma \in \Omega^2(M)$, and consider the symplectic form $\omega_\sigma = d\lambda + \pi^*\sigma$ from the Introduction. We denote by $\mathscr{J}_\sigma$ the open set of almost complex structures J on T^*M that are **tamed** by ω_σ - this just means that the bilinear form $\omega_\sigma(J\cdot,\cdot)$ is positive definite. We say that $J \in \mathscr{J}_\sigma$ is **uniformly tame** if J is also $d\lambda$-compatible (i.e. $J \in \mathscr{J}_\sigma \cap \mathscr{J}(T^*M)$), and there exists some positive constant $\varepsilon > 0$ such that

$$\omega_\sigma(J\xi,\xi) \geq \varepsilon G_J(\xi,\xi) \quad \text{for all } \xi \in TT^*M.$$

The pair (σ, g) defines a bundle endomorphism $Y = Y_{\sigma,g} \in \Gamma(\mathrm{End}(TM))$ called the **Lorentz force** of σ via:

$$\sigma_q(u,v) = \langle Y(q)u, v\rangle \, .$$

The following lemma will be very useful.

Lemma 2. (Uniformly tame almost complex structures)

1. *Fix $g \in \mathscr{R}(M)$. If*
$$\|Y_{\sigma,g}\|_{L^\infty_g} \leq 1 \tag{2.1}$$
then the almost complex structure J_g is uniformly tame (with $\varepsilon = 1/2$).

2. *Denote by $\mathscr{R}_\sigma(M) \subseteq \mathscr{R}(M)$ the set of Riemannian metrics on M for which (2.1) holds. Given any $g_0 \in \mathscr{R}(M)$, if $\upsilon > \|Y_{\sigma,g}\|_{L^\infty_g}$ then the rescaled metric $g := \upsilon g_0$ lies in $\mathscr{R}_\sigma(M)$.*

3. *Given $g \in \mathscr{R}(M)$ let*
$$\mathscr{U}_g := \left\{ J \in \mathscr{J}(T^*M) \, : \, \|J - J_g\|_{L^\infty_{G_g}} \leq 1/7 \right\} .$$
Then if $g \in \mathscr{R}_\sigma(M)$ and $J \in \mathscr{U}_g$ then J is uniformly tame (with $\varepsilon = 1/4$):
$$\omega_\sigma(J\xi,\xi) \geq \frac{1}{4} G_J(\xi,\xi) \quad \text{for all } \xi \in TT^*M.$$

Proof. (1). Write $Y = Y_{\sigma,g}$ and let $\xi \in TT^*M$. Then

$$\begin{aligned}
\omega_\sigma(J_g\xi,\xi) - \frac{1}{2}G_g(\xi,\xi) &= \frac{1}{2}G_g(\xi,\xi) + \pi^*\sigma(J_g\xi,\xi) \\
&= \frac{1}{2}G_g(\xi,\xi) + \left\langle Y(J_g\xi)^h, \xi^h \right\rangle \\
&= \frac{1}{2}G_g(\xi,\xi) - \left\langle Y\xi^v, \xi^h \right\rangle, \\
&\geq \frac{1}{2}\left(\left|\xi^h\right|^2 + \left|\xi^v\right|^2 \right) - \left|\xi^v\right| \left|\xi^h\right| \\
&= \left(\frac{1}{\sqrt{2}}\left|\xi^h\right| - \frac{1}{\sqrt{2}}\left|\xi^v\right| \right)^2 \geq 0.
\end{aligned}$$

(2). For any $X \in \Gamma(\mathrm{End}(TM))$ one has

$$\|X\|_{L^\infty_{\upsilon g}} = \|X\|_{L^\infty_g},$$

and since $Y_{\sigma,\upsilon g} = \frac{1}{\upsilon}Y_{\sigma,g}$ we see that if $\upsilon \geq \|Y_{\sigma,g}\|_{L^\infty_g}$ then $\|Y_{\sigma,\upsilon g}\|_{L^\infty_{\upsilon g}} \leq 1$. (3). First note that for any $J \in \mathscr{J}(T^*M)$ we have

$$\begin{aligned}
G_J(\xi,\xi) &= \omega_0(J\xi,\xi) \\
&= \omega_0(J_gJ\xi, J_g\xi) \\
&= G_g(J\xi, J_g\xi) \\
&= G_g(J_g\xi, J_g\xi) + G_g((J-J_g)\xi, J_g\xi) \\
&\geq \left(1 - \|J-J_g\|_{L^\infty_{G_g}}\right) G_g(\xi,\xi).
\end{aligned}$$

Thus if $g \in \mathscr{R}_\sigma(M)$ and $J \in \mathscr{J}(T^*M)$ satisfies $\|J-J_g\|_{L^\infty_{G_g}} \leq 1/7$ then

$$\begin{aligned}
\omega_\sigma(J\xi,\xi) - \frac{1}{4}G_J(\xi,\xi) &= \frac{3}{4}G_J(\xi,\xi) + \pi^*\sigma(J\xi,\xi) \\
&= \frac{3}{4}G_J(\xi,\xi) + \left\langle Y(J_g\xi)^h, \xi^h \right\rangle + \left\langle Y((J-J_g)\xi)^h, \xi^h \right\rangle \\
&\overset{(*)}{\geq} \frac{3}{4}G_J(\xi,\xi) - \frac{1}{2}G_g(\xi,\xi) - \|J-J_g\|_{L^\infty_{G_g}} G_g(\xi,\xi) \\
&\geq \frac{1}{4}G_g(\xi,\xi) - \frac{3}{4}\|J-J_g\|_{L^\infty_{G_g}} G_g(\xi,\xi) - \|J-J_g\|_{L^\infty_{G_g}} G_g(\xi,\xi) \\
&\geq \frac{1}{4}\left(1 - 7\|J-J_g\|_{L^\infty_{G_g}}\right) G_g(\xi,\xi) \geq 0,
\end{aligned}$$

where $(*)$ used the first part of the lemma.

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Assume that $\alpha \in [S^1, M]$ is a σ-atoroidal class. Fix a reference point $* \in M$, and fix a reference loop $q_\alpha \in \Lambda_1^\alpha M$ such that $q_\alpha(0) = *$. Given any $q \in \Lambda_\tau^\alpha M$, we define

$$\mathscr{A}_\sigma(q) := \int_{[0,1]\times\mathbb{S}_\tau} w^*\sigma,$$

where $w : [0,1]\times\mathbb{S}_\tau \to M$ is any smooth map such that $w(0,t) = q_\alpha(t/\tau)$ and $w(1,t) = q(t)$. Since α is σ-atoroidal, $\mathscr{A}_\sigma$ is well defined (i.e. independent of the choice of w), and one sees immediately that

$$d\mathscr{A}_\sigma = a_\sigma \quad \text{on } \Lambda_\tau^\alpha M. \tag{2.2}$$

Fix a point $\widetilde{*} \in \widetilde{M}$ that projects onto our fixed reference point $* \in M$. We denote by $\widetilde{q}_\alpha : [0,1] \to \widetilde{M}$ the lift of q_α to $\widetilde{M}$ with $\widetilde{q}_\alpha(0) = \widetilde{*}$. The following lemma is based on [7, Lemma 2.4], and explains the importance of the condition $\boldsymbol{(\sigma 1)}$.

Lemma 3. (The quadratic isoperimetric inequality)

Assume σ satisfies condition $(\boldsymbol{\sigma 1})$ *and $\alpha \in [S^1, M]$ is a σ-atoroidal class. There exists a constant $C_0 = C_0(\sigma, g) > 0$ and a constant $C_1 = C_1(\sigma, g, \alpha) > 0$ such that for all $q \in \Lambda_\tau^\alpha M$ one has*

$$|\mathscr{A}_\sigma(q)| \le C_0 \left(\int_0^\tau |\dot{q}(t)|\, dt \right)^2 + C_1.$$

Proof. Let θ denote a primitive of $\widetilde{\sigma}$ such that for all $z \in \widetilde{M}$ there exists a constant Θ_z such that

$$\sup_{q \in B(z,r)} |\theta_q| \le \Theta_z (r+1).$$

Now let $q \in \Lambda_\tau^\alpha M$, and let $w : [0,1] \times \mathbb{S}_\tau \to M$ denote a smooth map such that $w(0,t) = q_\alpha(t/\tau)$ and $w(1,t) = q(t)$, together with the additional property that if $\widetilde{w} : [0,1] \times [0,\tau] \to \widetilde{M}$ denotes the lifting of w to the universal cover such that $\widetilde{w}(0,t) = \widetilde{q}_\alpha(t/\tau)$ then

$$\int_0^1 |\partial_s \widetilde{w}(s,i)|\, ds \le d, \quad \text{for } i = 0,1,$$

where $d := \mathrm{diam}(M,g)$. Set

$$\Theta := \Theta_{\widetilde{*}}, \quad \ell_\alpha := \int_0^1 |\dot{q}_\alpha(t)|\, dt, \quad \ell(q) := \int_0^\tau |\dot{q}(t)|\, dt.$$

Then we have

$$\begin{aligned}
|\mathscr{A}_\sigma(q)| &= \left| \int_{[0,1]\times\mathbb{S}_\tau} w^*\sigma \right| \\
&= \left| \int_{[0,1]\times[0,\tau]} \widetilde{w}^*\widetilde{\sigma} \right| \\
&\le \left| \int_0^1 \theta(\partial_s \widetilde{w}(s,0)) ds \right| + \left| \int_0^\tau \theta(\partial_t \widetilde{w}(1,t)) dt \right| + \left| \int_0^1 \theta(\partial_s \widetilde{w}(s,1)) ds \right| + \left| \int_0^\tau \theta(\partial_t \widetilde{w}(0,t)) dt \right| \\
&\le \Theta(d+1)d + \Theta(d + \ell(q) + 1)\ell(q) + \Theta(d + \ell_\alpha + 1)d + \Theta(\ell_\alpha + 1)\ell_\alpha.
\end{aligned}$$

The desired statement follows with

$$C_0 := (2+d)\Theta;$$
$$C_1 := \Theta(d+1)d + \Theta(d+1) + \Theta(d + \ell_\alpha + 1)d + \Theta(\ell_\alpha + 1)\ell_\alpha.$$

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2.2 The action functional

Throughout this section assume that σ satisfies $(\boldsymbol{\sigma 1})$ and $\alpha \in [S^1, M]$ is a σ-atoroidal class.

Fix a τ-periodic Hamiltonian $H : \mathbb{S}_\tau \times T^*M \to \mathbb{R}$. Denote by $\mathscr{P}_\tau^\alpha(H,\sigma) \subseteq \Lambda_\tau^\alpha T^*M$ the set of closed τ-periodic orbits of $X_{H,\sigma}$ belonging to $\Lambda_\tau^\alpha T^*M$:

$$\mathscr{P}_\tau^\alpha(H,\sigma) = \{x \in \Lambda_\tau^\alpha T^*M \,:\, \dot{x} = X_{H,\sigma}(t,x)\}$$

($\Lambda_\tau^\alpha T^*M$ denotes those τ-periodic loops x whose projection to M lies in $\Lambda_\tau^\alpha M$). Denote by $\phi_t^{H,\sigma} : T^*M \to T^*M$ the flow of $X_{H,\sigma}$. In order to construct the Floer complex associated to H, σ and α we need to make the following standard assumption on the triple (H, σ, α):

(N) All the elements $x \in \mathscr{P}_\tau^\alpha(H,\sigma)$ are **non-degenerate**, that is, the linear map $d\phi_\tau^{H,\sigma}(x(0)) \in \mathsf{Sp}(T_{x(0)}T^*M)$ does not have 1 as an eigenvalue.

Remark 4. In fact, as far as Theorem A is concerned, the assumption that Condition **(N)** is satisfied can be relaxed. Indeed, the point is that by making a very small perturbation of H along the 1-periodic orbits we can create a new Hamiltonian $\widetilde{H}$ which still satisfies all the other requirements of Theorem A, and also such that $\widetilde{H}$ satisfies Condition **(N)**. Then Theorem A tells us that the Floer homology $HF_*^\alpha(\widetilde{H},\delta\sigma,\tau)$ is well defined, and moreover if $\widehat{H}$ is another such perturbation then by Theorem 12 below we have $HF_*^\alpha(\widetilde{H},\sigma,\tau)\cong HF_*^\alpha(\widehat{H},\sigma,\tau)$. In other words, we can still define $HF_*^\alpha(H,\sigma,\tau)$ even when Condition **(N)** is not satisfied, by simply setting

$$HF_*^\alpha(H,\sigma,\tau)\overset{\text{def}}{=}HF_*^\alpha(\widetilde{H},\sigma,\tau)$$

for any such perturbation $\widetilde{H}$.

The action functional $\mathscr{A}_{H,\sigma}:\Lambda_\tau^\alpha T^*M\to\mathbb{R}$ that we will work with is defined by

$$\mathscr{A}_{H,\sigma}(x):=\mathscr{A}_H(x)+\mathscr{A}_\sigma(\pi\circ x),$$

where $\mathscr{A}_H$ denotes the **standard Hamiltonian action functional**

$$\mathscr{A}_H(x):=\int_{\mathbb{S}_\tau}\lambda^*x-\int_0^\tau H(t,x)dt.$$

It is not hard to check that a loop $x\in\Lambda_\tau^\alpha T^*M$ is a critical point of $\mathscr{A}_{H,\sigma}$ if and only if $x\in\mathscr{P}_\tau^\alpha(H,\sigma)$.

In order to be able to obtain the necessary compactness results needed to define the Floer homology, following Abbondandolo and Schwarz [3, Section 1.5] we impose two growth conditions on H. In the statement of the following definition, $Z\in\mathrm{Vect}(T^*M)$ denotes the **Liouville vector field**, which is uniquely defined by the equation $i_Zd\lambda=\lambda$.

Definition 5. A Hamiltonian $H\in C^\infty(\mathbb{S}_\tau\times T^*M,\mathbb{R})$ satisfies the **Abbondandolo-Schwarz growth conditions** if the following two requirements hold:

(H1) There exists $h_1>0$ and $k_1\geq 0$ such that

$$dH(t,q,p)Z(q,p)-H(t,q,p)\geq h_1|p|^2-k_1\quad\text{for all }(t,q,p)\in\mathbb{S}_\tau\times T^*M.$$

(H2) There exists $h_2>0$ and $k_2\geq 0$ such that

$$|\nabla_qH(t,q,p)|\leq h_2|p|^2+k_2\quad\text{for all }(t,q,p)\in\mathbb{S}_\tau\times T^*M,$$

$$|\nabla_pH(t,q,p)|^2\leq h_2|p|^2+k_2\quad\text{for all }(t,q,p)\in\mathbb{S}_\tau\times T^*M.\tag{2.3}$$

Here we have chosen a Riemannian metric g on M, and ∇_qH and ∇_pH denote the horizontal and vertical components of the gradient ∇H of H under the splitting $TT^*M\cong TM\oplus T^*M$ induced by the Riemannian metric (see Section A.1 for the precise definition). Whilst the constants h_i,k_i depend on the choice of metric g on M, the existence of such constants does not (see [3, p273]).

We now define a constant $\delta_0(H,\sigma)$ associated to a pair (H,σ), where H satisfies the Abbondandolo-Schwarz growth conditions, and σ satisfies **($\boldsymbol{\sigma}$1)**. This is the constant that appears in the statement of Theorem A.

Definition 6. Firstly, given a Riemannian metric $g\in\mathscr{R}(M)$, define

$$\eta_1(H,g):=\sup\{h_1>0\,:\,H\text{ satisfies }\textbf{(H1)}\text{ with respect to }h_1\text{ and some }k_1\geq 0\};$$

$$\eta_2(H,g):=\inf\{h_2>0\,:\,H\text{ satisfies (2.3) with respect to }h_2\text{ and some }k_2\geq 0\}.$$

The reason that $\eta_2(H,g)$ is the infimum over the constants h_2 for which (2.3) is satisfied (rather than over the constants h_2 for which **(H2)** is satisfied) is that this part of the argument - specifically, Lemma 10 - does not require any assumptions on the growth of ∇_qH. This assumption comes into play later on, cf. Section 3.1. Note that if H satisfies both **(H1)** and **(H2)** then $0 < \eta_1(H,g), \eta_2(H,g) < \infty$.

Now set

$$\delta_0(H,\sigma,g) := \begin{cases} \frac{\eta_1(H,g)}{2C_0(\sigma,g)\eta_2(H,g)}, & \text{if } \sigma \text{ satisfies } (\boldsymbol{\sigma 1}) \text{ but not } (\boldsymbol{\sigma 0}), \\ \infty, & \text{if } \sigma \text{ satisfies } (\boldsymbol{\sigma 0}), \end{cases} \tag{2.4}$$

where the constant $C_0(\sigma,g)$ was defined in Lemma 3. Finally set

$$\delta_0(H,\sigma) := \sup_{g \in \mathscr{R}(M)} \delta_0(H,\sigma,g) \in (0,\infty].$$

Remark 7. Observe that

$$\delta_0(H,\sigma,\upsilon g) = \delta_0(H,\sigma,g) \quad \text{for all } \upsilon > 0.$$

Thus we can alternatively define

$$\delta_0(H,\sigma) := \sup_{g \in \mathscr{R}_\sigma(M)} \delta_0(H,\sigma,g) \in (0,\infty],$$

where $\mathscr{R}_\sigma(M)$ was defined in Lemma 2.2.

2.3 The Floer equation

Fix a Riemannian metric $g \in \mathscr{R}_\sigma(M)$, and a Hamiltonian $H \in C^\infty(\mathbb{S}_\tau \times T^*M, \mathbb{R})$ satisfying the Abbondandolo-Schwarz growth conditions **(H1)** and **(H2)**. Condition **(H2)** implies that there exists a constant $h_{\sigma,g} \geq 0$ such that

$$|X_{H,\sigma}(t,q,p)| \leq h_{\sigma,g}\left(1 + |p|^2\right) \quad \text{for all } (t,q,p) \in \mathbb{S}_\tau \times T^*M. \tag{2.5}$$

Observe that

$$\lambda(X_{H,\sigma}) = dH(Z) \tag{2.6}$$

(recall Z denotes the Liouville vector field); in particular $\lambda(X_{H,\sigma})$ does **not** depend on σ. Indeed,

$$\lambda(X_{H,\sigma}) + \pi^*\sigma(Z, X_{H,\sigma}) = \omega_\sigma(Z, X_{H,\sigma}) = dH(Z),$$

and

$$\pi^*\sigma(Z, X_{H,\sigma}) = 0$$

as $d\pi(Z) \equiv 0$.

Fix a σ-atoroidal class $\alpha \in [S^1, M]$ and $\delta \in \mathbb{R}$. Thus the action functional $\mathscr{A}_{H,\delta\sigma} : \Lambda^\alpha_\tau T^*M \to \mathbb{R}$ is defined. Given a family $\mathbf{J} = (J_t)_{t \in \mathbb{S}_\tau} \subseteq \mathscr{J}_\sigma$, denote by $\nabla_{\mathbf{J}}\mathscr{A}_{H,\delta\sigma}$ the vector field on $\Lambda^\alpha_\tau T^*M$ defined by

$$\nabla_{\mathbf{J}}\mathscr{A}_{H,\delta\sigma}(x) = J_t(x)(\dot{x} - X_{H,\delta\sigma}(t,x)).$$

With these definitions one has

$$d\mathscr{A}_{H,\sigma}(x)(\xi) = \langle\langle \nabla_{\mathbf{J}}\mathscr{A}_{H,\sigma}(x), \xi \rangle\rangle_{L^2_{G_{\mathbf{J}}}(\mathbb{S}_\tau)}, \tag{2.7}$$

where $\langle\langle \cdot,\cdot \rangle\rangle_{L^2_{G_{\mathbf{J}}}(\mathbb{S}_\tau)}$ denotes the possibly non-symmetric inner product given by

$$\langle\langle \xi,\zeta \rangle\rangle_{L^2_{G_{\mathbf{J}}}(\mathbb{S}_\tau)} := \int_0^\tau \omega_\sigma(J_t\xi,\zeta)dt.$$

We remind the reader that since the almost complex structures $\mathbf{J} = (J_t)_{t\in\mathbb{S}_\tau}$ are only assumed to be tamed by ω_σ (rather than compatible), the order in (2.7) is important, that is, in general

$$d\mathscr{A}_{H,\sigma}(x)(\xi) \neq \langle\langle \xi, \nabla_{\mathbf{J}}\mathscr{A}_{H,\sigma}(x)\rangle\rangle_{L^2_{G_{\mathbf{J}}}(\mathbb{S}_\tau)}.$$

Given critical points $x_-, x_+ \in \mathscr{P}^\alpha_\tau(H, \delta\sigma)$ we denote by

$$\mathscr{M}^\alpha_\tau(x_-, x_+, H, \delta\sigma, \mathbf{J}) \subseteq C^\infty(\mathbb{R}\times\mathbb{S}_\tau, T^*M)$$

the set of smooth maps $u : \mathbb{R}\times\mathbb{S}_\tau \to T^*M$ that satisfy the **Floer equation**

$$\partial_s u + \nabla_{\mathbf{J}}\mathscr{A}_{H,\delta\sigma}(u) = 0 \tag{2.8}$$

and submit to the asymptotic conditions

$$\lim_{s\to\pm\infty} u(s,t) = x_\pm(t), \quad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0, \tag{2.9}$$

both limits being uniform in t.

More generally, we denote by $\mathscr{M}^\alpha_\tau(a, b, H, \delta\sigma, \mathbf{J})$ the set of maps $u \in C^\infty(\mathbb{R}\times\mathbb{S}_\tau, T^*M)$ satisfying (2.8) and

$$a \le \mathscr{A}_{H,\delta\sigma}(u(s,t)) \le b \quad \text{for all } (s,t) \in \mathbb{R}\times\mathbb{S}_\tau.$$

Recall the definition of the the set $\mathscr{U}_g$ of almost complex structures from Lemma 2.3. The following theorem is central to defining the Floer homology $HF^\alpha_*(H,\tau,\sigma)$, and will be proved in Section 3.1 below.

Theorem 8. **(L^∞ bounds on gradient flow lines)**
Suppose $g \in \mathscr{R}_\sigma(M)$, $\tau|\delta| < \delta_0(H,\sigma,g)$ and $\alpha \in [S^1, M]$ is a σ-atoroidal class. There exists a smaller neighborhood $\mathscr{V}_g \subseteq \mathscr{U}_g$ of J_g such that for any family $\mathbf{J} = (J_t)_{t\in\mathbb{S}_\tau} \subseteq \mathscr{V}_g$, and for all $-\infty < a \le b < \infty$, there exists a compact set $K = K(a,b,\mathbf{J}) \subseteq T^*M$ such that for any $u \in \mathscr{M}^\alpha_\tau(a,b,H,\delta\sigma,\mathbf{J})$ one has

$$u(\mathbb{R}\times\mathbb{S}_\tau) \subseteq K.$$

2.4 Defining the Floer homology groups

Let us now fix:

- a closed 2-form $\sigma \in \Omega^2(M)$ that satisfies Condition $(\boldsymbol{\sigma}\mathbf{1})$,
- a σ-atoroidal class $\alpha \in [S^1, M]$,
- a Riemannian metric $g \in \mathscr{R}_\sigma(M)$,
- a Hamiltonian $H \in C^\infty(\mathbb{S}_\tau \times T^*M, \mathbb{R})$ that satisfies the Abbondandolo-Schwarz growth conditions **(H1)** and **(H2)**,
- a constant $\delta \in \mathbb{R}$ such that $\tau|\delta| < \delta_0(H,\sigma,g)$, and such that $(H,\delta\sigma,\alpha)$ satisfies Condition **(N)**.

We will now explain how Theorem 8 allows us to define the Floer homology groups $HF^\alpha_*(H,\delta\sigma,\tau)$. All of this material is now standard (and essentially identical to [3, Section 1.7]), and we refer the reader to any of a number standard sources (e.g. Salamon's lecture notes [20]) for more details.

For each $x \in \mathscr{P}^\alpha_\tau(H,\delta\sigma)$, let $\mu_{\mathrm{CZ}}(x)$ denote the **Conley-Zehnder index** of x. In order to define the Conley-Zehnder index we choose a vertical preserving symplectic trivialization (see [3]); the fact that $c_1(T^*M,\omega_\sigma) = 0$ means that the value of $\mu_{\mathrm{CZ}}(x)$ is independent of this choice of trivialization. Note however that our sign conventions match those of [5] not [3]. The non-degeneracy condition **(N)** implies that $\mu_{\mathrm{CZ}}(x)$ is always an integer.

Given $k \in \mathbb{Z}$ let

$$\mathscr{P}_\tau^\alpha(H, \delta\sigma)_k := \{x \in \mathscr{P}_\tau^\alpha(H, \delta\sigma) \,:\, \mu_{\mathrm{CZ}}(x) = k\}.$$

The moduli spaces $\mathscr{M}_\tau^\alpha(x_-, x_+, H, \delta\sigma, \mathbf{J})$ all carry a free $\mathbb{R}$-action given by $(s_0 \cdot u)(s,t) := u(s - s_0, t)$, and we denote by $\mathscr{M}_\tau^\alpha(x_-, x_+, H, \delta\sigma, \mathbf{J})/\mathbb{R}$ the quotient space under this action. For a generic choice of $\mathbf{J} = (J_t)_{t \in \mathbb{S}_\tau} \subseteq \mathscr{V}_g$, it follows from Theorem 8 and standard Floer-theoretic arguments that the quotient moduli spaces $\mathscr{M}_\tau^\alpha(x_-, x_+, H, \delta\sigma, \mathbf{J})/\mathbb{R}$ all carry the structure of a $(\mu_{\mathrm{CZ}}(x_-) - \mu_{\mathrm{CZ}}(x_+) - 1)$-dimensional manifold. Moreover if $\mu_{\mathrm{CZ}}(x_-) = \mu_{\mathrm{CZ}}(x_+) + 1$ then $\mathscr{M}_\tau^\alpha(x_-, x_+, H, \delta\sigma, \mathbf{J})/\mathbb{R}$ is actually compact (and hence a finite set).

We define the **Floer chain group** $CF_k^\alpha(H, \delta\sigma, \tau)$ to be the free $\mathbb{Z}_2$-module generated by the elements of $\mathscr{P}_\tau^\alpha(H, \delta\sigma)_k$. Note that $CF_\tau^\alpha(H, \delta\sigma, \tau)$ may not be finitely generated. The boundary operator $\partial(\mathbf{J}) : CF_k^\alpha(H, \delta\sigma, \tau) \to CF_{k-1}^\alpha(H, \delta\sigma, \tau)$ is defined by

$$\partial(\mathbf{J})(x) := \sum_{y \in \mathscr{P}_\tau^\alpha(H,\delta\sigma)_{k-1}} n(x,y)y, \quad x \in \mathscr{P}_\tau^\alpha(H, \delta\sigma)_k,$$

where

$$n(x,y) := \#_2 \left(\mathscr{M}_\tau^\alpha(x, y, H, \delta\sigma, \mathbf{J})/\mathbb{R}\right)$$

denotes the parity of the finite set $\mathscr{M}_\tau^\alpha(x, y, H, \delta\sigma, \mathbf{J})/\mathbb{R}$. This is well defined since the sum contains only finitely many non-zero terms, thanks to the forthcoming Remark 11.

The usual argument [20], tells us that $\partial(\mathbf{J}) \circ \partial(\mathbf{J}) = 0$, and hence we may define the **Floer homology** $HF_*^\alpha(H, \delta\sigma, \tau)$ to be the homology of the chain complex $\{CF_*^\alpha(H, \delta\sigma, \tau), \partial(\mathbf{J})\}$. It is acceptable to omit the $\mathbf{J}$ from the notation for the homology $HF_*^\alpha(H, \delta\sigma, \tau)$, as any two (generically chosen) families $\mathbf{J}$ and $\mathbf{J}'$ produce chain homotopic chain complexes (see [3, Theorem 1.20]).

3 Proofs

3.1 The proof of Theorem 8

As mentioned in the Introduction, our proof of Theorem 8 will closely follow Abbondandolo and Schwarz' method in [3]. Their method has two distinct stages. The first stage appears as Lemma 1.12 in [3], and asserts that under the hypotheses of the theorem, there exists a constant $R = R(a,b) > 0$ such that for any $u = (q,p) \in \mathscr{M}_\tau^\alpha(a, b, H, \delta\sigma, \mathbf{J})$ and any interval $I \subseteq \mathbb{R}$ it holds that

$$\|p\|_{W_g^{1,2}(I \times \mathbb{S}_\tau)} \le R\left(|I|^{1/2} + 1\right). \tag{3.1}$$

This stage uses heavily the fact that H satisfies conditions **(H1)** and **(H2)**. The second stage appears as Theorem 1.14 in [3]. Roughly speaking, the second stage works as follows: firstly, by Nash's Theorem, we may isometrically embed the Riemannian manifold (M, g) into $(\mathbb{R}^N, g_{\mathrm{eucl}})$. This embedding in turn induces an isometric embedding of (TT^*M, G_g) into $(\mathbb{R}^{2N}, g_{\mathrm{eucl}})$. Under this embedding if $\mathtt{i}$ denotes the canonical almost complex structure on $\mathbb{R}^{2N}$ given by

$$\mathtt{i} = \begin{pmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{pmatrix}$$

then $\mathtt{i}|_{T^*M} = J_g$. The proof then uses Calderon-Zygmund estimates for the Cauchy-Riemann operator, together with certain interpolation inequalities, to upgrade equation (3.1) to the full statement of Theorem 8. These estimates only work for $\mathbf{J}$ contained in a sufficiently small neighborhood $\mathscr{W}_g$ of J_g: the set $\mathscr{V}_g$ in the statement of Theorem 8 is then defined by $\mathscr{V}_g := \mathscr{U}_g \cap \mathscr{W}_g$. The proof of this stage goes through word for word in our situation, and thus in order to prove Theorem 8 it suffices to prove the first stage, namely equation (3.1).

The proof of (3.1) (Lemma 1.12 in [3]) consists of six claims. A careful inspection of their proof shows that everything apart from Claim 1 and Claim 2 goes through verbatim in our case. Claims 1 and 2 however require a little more work. The following lemma proves Claim 1.

Lemma 9. *Fix $g \in \mathscr{R}_\sigma(M)$. If $\mathbf{J} = (J_t)_{t \in \mathbb{S}_\tau} \subseteq \mathscr{U}_g$ and $u : \mathbb{R} \times \mathbb{S}_\tau \to T^*M$ satisfies (2.8) and (2.9) with respect to $\mathbf{J}$, then*

$$\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} \le 4 \sup_{t\in\mathbb{S}_\tau} \|J_t\|^2_{L^\infty_{G_g}} (\mathscr{A}_{H,\sigma}(x_-) - \mathscr{A}_{H,\sigma}(x_+)).$$

Proof. The proof is a simple computation using Lemma 2.3 and (2.7).

$$\begin{aligned}
\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} &\le \sup_{t\in\mathbb{S}_\tau} \|J_t\|^2_{L^\infty_{G_g}} \|\partial_s u\|^2_{L^2_{G_{J_t}}(\mathbb{R}\times\mathbb{S}_\tau)} \\
&\le 4 \sup_{t\in\mathbb{S}_\tau} \|J_t\|^2_{L^\infty_{G_g}} \int_{-\infty}^{\infty}\int_0^\tau \omega_\sigma(J_t\partial_s u, \partial_s u)dtds \\
&= 4 \sup_{t\in\mathbb{S}_\tau} \|J_t\|^2_{L^\infty_{G_g}} \int_{-\infty}^{\infty} (-d\mathscr{A}_{H,\sigma}(u(s)))(\partial_s u)ds \\
&= 4 \sup_{t\in\mathbb{S}_\tau} \|J_t\|^2_{L^\infty_{G_g}} \left(\mathscr{A}_{H,\sigma}(x_-) - \mathscr{A}_{H,\sigma}(x_+)\right).
\end{aligned}$$

■

The proof of Claim 2 is somewhat trickier, and we state this below as a separate lemma. It is this lemma that explains why in our case the constant $\delta_0(H,\sigma,g)$ enters the picture.

Lemma 10. *Fix $g \in \mathscr{R}_\sigma(M)$. Assume $\tau|\delta| < \delta_0(H,\sigma,g)$ and $\alpha \in [S^1, M]$ is a σ-atoroidal class. Fix $\mathbf{J} = (J_t)_{t\in\mathbb{S}_\tau} \subseteq \mathscr{U}_g$. Then for all $a \in \mathbb{R}$ there exists a constant $S = S(a) > 0$ such that for any $u = (q,p) \in \mathscr{M}^\alpha_\tau(-\infty, a, H, \delta\sigma, \mathbf{J})$ one has*

$$\|p(s,\cdot)\|_{L^2_g(\mathbb{S}_\tau)} \le S\left(1 + \|\partial_s u(s,\cdot)\|_{L^2_{G_g}(\mathbb{S}_\tau)}\right).$$

Proof. We begin with the more difficult case where σ satisfies **($\boldsymbol{\sigma}$1)** but not **($\boldsymbol{\sigma}$0)**, so that by (2.4), we have

$$\delta_0(H,\sigma,g) = \frac{\eta_1(H,g)}{2C_0\eta_2(H,g)}.$$

Set

$$T := \sup_{t\in\mathbb{S}_\tau} \|J_t\|_{L^\infty_{G_g}}.$$

Fix $u = (q,p) \in \mathscr{M}^\alpha_\tau(-\infty, a, H, \sigma, \mathbf{J})$ as in the statement of the lemma. Observe that by (2.6) we have:

$$\begin{aligned}
\lambda(\partial_t u) &= \lambda(X_{H,\sigma}(t,u)) + \lambda(J_t(u)\partial_s u) \\
&= dH(t,u)Z(u) + d\lambda(Z(u), J_t(\partial_s u)) \\
&\ge dH(t,u)Z(u) - T|p|\, G_g(\partial_s u, \partial_s u)^{1/2}.
\end{aligned}$$

Thus if H satisfies **(H1)** with respect to $h_1 > 0$ and $k_1 \ge 0$, then

$$\lambda(\partial_t u) - H(t,u) \ge h_1|p|^2 - k_1 - T|p|\, G_g(\partial_s u, \partial_s u)^{1/2},$$

and hence

$$\mathscr{A}_H(u(s,\cdot)) \ge h_1 \|p(s,\cdot)\|^2_{L^2_g(\mathbb{S}_\tau)} - k_1\tau - T\|p(s,\cdot)\|_{L^2_g(\mathbb{S}_\tau)} \|\partial_s u(s,\cdot)\|_{L^2_{G_g}(\mathbb{S}_\tau)}.$$

Taking horizontal components of the equation

$$\partial_t u = J_t(u)\partial_s u + X_{H,\sigma}(t,u)$$

gives

$$\partial_t q = (J_t(u)\partial_s u)^h + \nabla_p H(t,q,p),$$

and hence if H satisfies the second of the two conditions needed for **(H2)** with $h_2 > 0$ and $k_2 \geq 0$ then

$$\begin{aligned}|\partial_t q|^2 &\leq 2\left|(J_t(u)\partial_s u)^h\right|^2 + 2\left|\nabla_p H(t,q,p)\right|^2 \\ &\leq 2\left\|J_t\right\|^2_{L^\infty_{G_g}} G_g(\partial_s u, \partial_s u) + 2h_2 |p|^2 + 2k_2.\end{aligned}$$

Thus

$$\begin{aligned}\left(\int_0^\tau |\partial_t q(s,\cdot)|\, dt\right)^2 &\leq \tau \int_0^\tau |\partial_t q(s,\cdot)|^2\, dt \\ &\leq 2\tau T^2 \left\|\partial_s u(s,\cdot)\right\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} + 2\tau h_2 \left\|p(s,\cdot)\right\|^2_{L^2_g(\mathbb{S}_\tau)} + 2\tau k_2.\end{aligned}$$

Thus by Lemma 3,

$$\begin{aligned}|\mathscr{A}_{\delta\sigma}(q(s,\cdot))| &\leq |\delta|\left(C_0\left(\int_0^\tau |\partial_t q(s,\cdot)|\, dt\right)^2 + C_1\right) \\ &\leq 2|\delta|\, C_0\tau T^2 \left\|\partial_s u(s,\cdot)\right\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} + 2|\delta|\, C_0\tau h_2 \left\|p(s,\cdot)\right\|^2_{L^2_g(\mathbb{S}_\tau)} + |\delta|\,(2C_0\tau k_2 + C_1),\end{aligned}$$

and hence

$$\begin{aligned}a &\geq \mathscr{A}_{H,\delta\sigma}(u(s,\cdot)) \\ &\geq \mathscr{A}_H(u(s,\cdot)) - |\mathscr{A}_{\delta\sigma}(q(s,\cdot))| \\ &\geq (h_1 - 2|\delta|\, C_0\tau h_2)\left\|p(s,\cdot)\right\|^2_{L^2_g(\mathbb{S}_\tau)} - T\left\|p(s,\cdot)\right\|_{L^2_g(\mathbb{S}_\tau)}\left\|\partial_s u(s,\cdot)\right\|_{L^2_{G_g}(\mathbb{S}_\tau)} \\ &\quad - 2|\delta|\, C_0\tau T^2\left\|\partial_s u(s,\cdot)\right\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} - k_1\tau - |\delta|\,(2C_0\tau k_2 + C_1).\end{aligned}$$

Using the fact that for any $c, d, \mu > 0$ it holds that

$$cd \leq \mu c^2 + \frac{1}{4\mu} d^2,$$

we have that for any $\mu > 0$ it holds that

$$\begin{aligned}a &\geq (h_1 - 2|\delta|\,\tau C_0 h_2 - \mu)\left\|p(s,\cdot)\right\|^2_{L^2_g(\mathbb{S}_\tau)} - \left(2|\delta|\, C_0\tau T^2 + \frac{1}{4\mu}T^2\right)\left\|\partial_s u(s,\cdot)\right\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} \\ &\quad - k_1\tau - |\delta|\,(2C_0\tau k_2 + C_1).\end{aligned}$$

Our choice of δ implies that

$$h_1 - 2|\delta|\,\tau C_0 h_2 > 0,$$

and hence for suitably small μ we obtain an equality of the desired form.

Finally consider the case where σ satisfies the stronger condition $(\boldsymbol{\sigma 0})$. In this case $\widetilde{\sigma}$ admits a bounded primitive θ, and Lemma 3 can be upgraded to a **linear isoperimetric inequality** - see [7, Lemma 4.4]. It is then easy to improve the proof above to work for any $\delta \in \mathbb{R}$, and we omit the details. ■

We have now verified Claim 2 of Lemma 1.12 in [3]. As discussed above, the remaining parts of the proof of Lemma 1.12 go through without change in our situation, and thus this concludes the proof of equation (3.1), and hence also of Theorem 8.

Remark 11. This argument also proves that if $\tau\,|\delta| < \delta_0(H,\sigma,g)$ and the triple $(H,\delta\sigma,\alpha)$ satisfies Condition **(N)**, then for any $a \in \mathbb{R}$, there are at most finitely many critical points $x \in \mathscr{P}_\tau^\alpha(H,\delta\sigma)$ with $\mathscr{A}_{H,\delta\sigma}(x) \le a$. Indeed, the proof shows that if

$$\mathbb{P} := \{x \in \mathscr{P}_\tau^\alpha(H,\delta\sigma)\,:\,\mathscr{A}_{H,\delta\sigma}(x) \le a\}\,,$$

then there exists a uniform bound on $\|p\|^2_{L^2_g(\mathbb{S}_\tau)}$ for all $x = (q,p) \in \mathbb{P}$.

Since

$$|\dot{x}| = |X_{H,\sigma}(t,x)| \le h_{\sigma,g}\left(1 + |p|^2\right)$$

by (2.5), we see that $\mathbb{P}$ is bounded in $W^{1,1}_g$, and hence in L^∞_g. In particular, the set

$$\{x(0)\,:\,x \in \mathbb{P}\}$$

is precompact in T^*M, and since it is discrete by Condition **(N)**, it is finite.

3.2 Invariance

The following result completes the proof of Theorem A from the Introduction, whose proof is similar to [3, Lemma 1.21] and [7, Theorem 2.7]. Indeed, to obtain Theorem A from Theorem 12, simply take $\sigma_0 = \sigma$ and $\sigma_1 = 0$, and apply Theorem 1.

Theorem 12. (Invariance of Floer homology under homotopies) *Fix a Riemannian metric g on M, $\alpha \in [S^1, M]$ and $\tau > 0$. Suppose we are given:*

1. *2-forms σ_0 and σ_1 that both satisfy $(\boldsymbol{\sigma 1})$ and are such that α is both σ_0-atoroidal and σ_1-atoroidal, and such that $g \in \mathscr{R}_{\sigma_0}(M) \cap \mathscr{R}_{\sigma_1}(M)$. Set*

$$\sigma_s := (1-s)\sigma_0 + s\sigma_1.$$

2. *Hamiltonians H_0 and H_1 satisfying the Abbondandolo-Schwarz growth conditions. Set*

$$H_s := (1-s)H_0 + sH_1,$$

Choose a smooth function $\delta : [0,1] \to \mathbb{R}$ such that

$$\tau\,|\delta(s)| < \delta_0(H_s,\sigma_s,g)$$

for each $s \in [0,1]$, and suppose that both $(H_0,\delta(0)\sigma_0,\alpha)$ and $(H_1,\delta(1)\sigma_1,\alpha)$ satisfy Condition **(N)**. *Then there exists a continuation map*

$$\Psi : CF_*^\alpha(H_0,\delta(0)\sigma_0,\tau) \to CF_*^\alpha(H_1,\delta(1)\sigma_1,\tau)$$

inducing an isomorphism

$$\psi : HF_*^\alpha(H_0,\delta(0)\sigma_0,\tau) \to HF_*^\alpha(H_1,\delta(1)\sigma_1,\tau).$$

Before getting started on the proof, we will introduce some notation. Our assumption

$$\tau\,|\delta(s)| < \delta_0(H_s,\sigma_s,g) \quad \text{for all } s \in [0,1] \tag{3.2}$$

implies that we can choose bounded functions $\eta_1(s), \eta_2(s)$ and constants $k_1, k_2 \ge 0$ and $\chi > 0$ such that for all $s \in [0,1]$:

1. H_s satisfies **(H1)** with respect to $\eta_1(s)$ and k_1;
2. H_s satisfies (2.3) with respect to $\eta_2(s)$ and k_2;

3. If $C_0(\sigma_s, g)$ and $C_1(\sigma_s, g, \alpha)$ denote the constants associated to σ_s from Lemma 3 then

$$\eta_1(s) - 2\left|\delta(s)\right| \tau C_0(\sigma_s, g)\eta_2(s) > \chi \quad \text{for all } s \in [0,1]. \tag{3.3}$$

Set

$$\eta_1 := \max_{s\in[0,1]} \eta_1(s), \quad \eta_2 := \max_{s\in[0,1]} \eta_2(s);$$
$$C_0 := \max_{s\in[0,1]} C_0(\sigma_s, g), \quad C_1 := \max_{s\in[0,1]} C_1(\sigma_s, g, \alpha);$$
$$d := \max_{s\in[0,1]} \left|\delta(s)\right|.$$

Now fix $\varepsilon > 0$, which we will specify precisely later. Choose a natural number

$$N \geq \frac{2\max_{s\in[0,1]}\left|\delta'(s)\right|}{\varepsilon}, \tag{3.4}$$

and choose a subdivision $0 = r_0 < r_1 < \cdots < r_N = 1$ such that $|r_i - r_{i+1}| < 2/N$ for each $i = 0, \dots, N-1$ and such that for each $i = 0, \dots, N-1$ the following two inequalities hold [1]:

$$\begin{cases} \left|H_{r_{i+1}}(t,q,p) - H_{r_i}(t,q,p)\right| \leq \varepsilon\left(1 + |p|^2\right); \\ C_0(\sigma_{r_{i+1}} - \sigma_{r_i}, g) < \varepsilon. \end{cases} \tag{3.5}$$

Let $\beta : \mathbb{R} \to [0,1]$ denote a smooth cut-off function such that $\beta(s) \equiv 0$ for $s \leq 0$ and $\beta(s) \equiv 1$ for $s \geq 1$, with $0 \leq \beta'(s) \leq 2$ for all $s \in \mathbb{R}$. Now define:

$$F^i_s := H_{r_i} + \beta(s)(H_{r_{i+1}} - H_{r_i});$$
$$\nu^i_s := \sigma_{r_i} + \beta(s)(\sigma_{r_{i+1}} - \sigma_{r_i});$$
$$f_i(s) := \delta(r_i + \beta(s)(r_{i+1} - r_i));$$
$$\omega^i_s := d\lambda + f_i(s)\pi^*\nu^i_s.$$

Note that by (3.4),

$$\max_{s\in[0,1]} \left|f_i'(s)\right| < 2\varepsilon \quad \text{for all } i \in \{0,1,\dots,N-1\}.$$

Let

$$\mathscr{A}^i : \Lambda^\alpha_\tau T^*M \to \mathbb{R}$$

be defined by

$$\mathscr{A}^i(x) := \mathscr{A}_{H_{r_i}, \delta(r_i)\sigma_{r_i}}(x) = \mathscr{A}_{H_{r_i}}(x) + \mathscr{A}_{\delta(r_i)\sigma_{r_i}}(\pi \circ x),$$

and let

$$\mathscr{A}^i_s : \Lambda^\alpha_\tau T^*M \to \mathbb{R}$$

be defined by

$$\mathscr{A}^i_s(x) := \mathscr{A}_{F^i_s, f_i(s)\nu^i_s}(x) = \mathscr{A}_{F^i_s}(x) + \mathscr{A}_{f_i(s)\nu^i_s}(\pi \circ x).$$

Fix $\mathbf{J} = (J_t)_{t\in\mathbb{S}_\tau} \subseteq \mathscr{V}_g$ (where $\mathscr{V}_g$ is as in the statement of Theorem 8).

Given $i \in \{0,1,\dots,N-1\}$ and $-\infty < a \leq b < \infty$, denote by

$$\mathscr{N}^\alpha_\tau(a, b, F^i_s, f_i(s)\nu^i_s, \mathbf{J})$$

the set of maps $u \in C^\infty(\mathbb{R} \times \mathbb{S}_\tau, T^*M)$ that satisfy the s-dependent Floer equation

$$\partial_s u + \nabla_{\mathbf{J}}\mathscr{A}^i_s(u) = 0$$

and which satisfy

$$a \leq \mathscr{A}^i_s(u(s,t)) \leq b \quad \text{for all } (s,t) \in \mathbb{R} \times \mathbb{S}_\tau.$$

The following statement constitutes most of the work needed to prove Theorem 12.

[1] That it is possible to choose such a subdivision so that the first inequality holds is explained in [3, p289], and uses the fact that both H_0 and H_1 satisfy **(H2)**, and that M is compact.

Lemma 13. *If $\varepsilon > 0$ is sufficiently small then given any $i \in \{0, 1, \dots, N-1\}$ and any $-\infty < a \leq b < \infty$ there exists a compact set $K_i = K_i(a, b, \mathbf{J}) \subseteq T^*M$ such that for all $u \in \mathscr{N}_\tau{}^\alpha(a, b, F_s^i, f_i(s)\nu_s^i, \mathbf{J})$ one has $u(\mathbb{R} \times \mathbb{S}_\tau) \subseteq K_i$.*

Proof. Fix $i \in \{0, 1, \dots, N-1\}$, and fix $u = (q, p) \in \mathscr{N}_\tau{}^\alpha(a, b, F_s^i, f_i(s)\nu_s^i, \mathbf{J})$. Firstly, note that by Lemma 3 we have that for all $s \in \mathbb{R}$,

$$\left|\mathscr{A}_{\nu_s^i}(q(s, \cdot))\right| \leq C_0 \left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 + C_1; \tag{3.6}$$

$$\left|\mathscr{A}_{(\sigma_{r_{i+1}} - \sigma_{r_i})}(q(s, \cdot))\right| \leq \varepsilon \left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 + C_2, \tag{3.7}$$

for some constant $C_2 > 0$, where the second equation used (3.5).

The key term we wish to estimate is:

$$\Delta(u) := \int_{-\infty}^\infty \left|\left(\frac{\partial}{\partial s}\mathscr{A}_s^i\right)(u(s, \cdot))\right| ds.$$

We compute

$$\begin{aligned}\left|\left(\frac{\partial}{\partial s}\mathscr{A}_s^i\right)(u(s, \cdot))\right| &= \left|-\int_0^1 \left(\frac{\partial}{\partial s}F_s^i\right)(u(s, t))dt + \frac{\partial}{\partial s}\mathscr{A}_{f_i(s)\nu_s^i}(q(s, \cdot))\right| \\ &\leq \beta'(s)\int_0^1 \left|(H_{r_{i+1}} - H_{r_i})(t, u)\right| dt + \left|\frac{\partial}{\partial s}\mathscr{A}_{f_i(s)\nu_s^i}(q(s, \cdot))\right|.\end{aligned}$$

We can estimate the first term from (3.5) by

$$\beta'(s)\int_0^1 \left|(H_{r_{i+1}} - H_{r_i})(t, u)\right| dt \leq 2\varepsilon\left(1 + \|p(s, \cdot)\|^2_{L^2_g(\mathbb{S}_\tau)}\right)$$

As for the second term, we compute using (3.6) and (3.7) that

$$\begin{aligned}\left|\frac{\partial}{\partial s}\mathscr{A}_{f_i(s)\nu_s^i}(u(s, \cdot))\right| &= \left|f_i'(s)\mathscr{A}_{\nu_s^i}(q(s, \cdot)) + \beta'(s)f_i(s)\mathscr{A}_{(\sigma_{r_{i+1}} - \sigma_{r_i})}(q(s, \cdot))\right| \\ &\leq 2\varepsilon\left(C_0\left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 + C_1\right) + 2d\left(\varepsilon\left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 + C_2\right) \\ &\leq 2\varepsilon(C_0 + d)\left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 + 2\varepsilon C_1 + 2dC_2.\end{aligned}$$

Arguing as in the proof of Lemma 10, we have

$$\left(\int_0^\tau |\partial_t q(s, \cdot)|\, dt\right)^2 \leq 2\tau T^2 \|\partial_s u(s, \cdot)\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} + 2\tau\eta_2 \|p(s, \cdot)\|^2_{L^2_g(\mathbb{S}_\tau)} + 2\tau k_2,$$

where as before,

$$T := \sup_{t \in \mathbb{S}_\tau} \|J_t\|_{L^\infty_{G_g}}.$$

Thus

$$\begin{aligned}\left|\frac{\partial}{\partial s}\mathscr{A}_{f_i(s)\nu_s^i}(q(s, \cdot))\right| &\leq 4\varepsilon(C_0 + d)\tau T^2 \|\partial_s u(s, \cdot)\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} + 4\varepsilon(C_0 + d)\tau\eta_2 \|p(s, \cdot)\|^2_{L^2_g(\mathbb{S}_\tau)}. \\ &\quad + 4\varepsilon(C_0 + d)\tau k_2 + 2\varepsilon C_1 + 2dC_2.\end{aligned}$$

Putting this together and integrating we conclude

$$\Delta(u) \le \underbrace{(2\varepsilon + 4\varepsilon(C_0+d)\tau\eta_2)}_{:=c_1} \|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} + \underbrace{4\varepsilon(C_0+d)\tau T^2}_{:=c_2} \|\partial_s u\|^2_{L^2_{G_g}([0,1]\times\mathbb{S}_\tau)}$$
$$+ \underbrace{2\varepsilon + 4\varepsilon(C_0+d)\tau k_2 + 2\varepsilon C_1 + 2dC_2}_{:=c_3}.$$

Arguing as in Lemma 9 we have

$$\begin{aligned}\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} &\le 4T^2(b-a+\Delta(u)) \\ &\le 4T^2 c_1 \|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} + 4T^2 c_2 \|\partial_s u\|^2_{L^2_{G_g}([0,1]\times\mathbb{S}_\tau)} + 4T^2(b-a+c_3),\end{aligned}$$

and thus provided $\varepsilon > 0$ is small enough such that

$$4T^2 c_2 \le \frac{1}{2},$$

we conclude that

$$\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} \le 8T^2 c_1 \|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} + 8T^2(b-a+c_3). \tag{3.8}$$

Similarly one has

$$\begin{aligned}\sup_{s\in\mathbb{R}} \mathscr{A}^i_s(u(s,\cdot)) &\le b + \Delta(u) \\ &\le b + c_1 \|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} + c_2 \|\partial_s u\|^2_{L^2_{G_g}([0,1]\times\mathbb{S}_\tau)} + c_3.\end{aligned} \tag{3.9}$$

Arguing as in the proof of Lemma 10 we discover that

$$\begin{aligned}c_1 \|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} + c_2 \|\partial_s u\|^2_{L^2_{G_g}([0,1]\times\mathbb{S}_\tau)} + c_3 + b &\ge \mathscr{A}^i_s(u(s,\cdot)) \\ &= \int_0^\tau (\lambda(X_{F^i_s, f_i(s)\nu^i_s}(t,u)) - F_s(t,u))dt + \mathscr{A}_{f_i(s)\nu^i_s}(q(s,\cdot)) \\ \overset{(*)}{\ge} (\chi-\mu)\|p(s,\cdot)\|^2_{L^2_g(\mathbb{S}_\tau)} - \left(2|\delta| C_0\tau T^2 + \frac{1}{4\mu}T^2\right) &\|\partial_s u(s,\cdot)\|^2_{L^2_{G_g}(\mathbb{S}_\tau)} - k_1\tau - d(2C_0\tau k_2 + C_1),\end{aligned}$$

where $\mu > 0$ is any positive number and $(*)$ used (3.3). Take $\mu = \chi/2$. Integrating this expression over $[0,1]$ and rearranging gives

$$\begin{aligned}\left(\frac{\chi}{2} - c_1\right)\|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} &\le \left(c_2 + \left(2|\delta| C_0\tau T^2 + \frac{1}{2\chi}T^2\right)\right)\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} \\ &\quad + b + c_3 + k_1\tau + d(2C_0\tau k_2 + C_1).\end{aligned}$$

Substituting in the expression (3.8) for $\|\partial_s u\|^2_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)}$ we obtain

$$\underbrace{\left(\frac{\chi}{2} - c_1 - 8T^2 c_1\left(c_2 + \left(2|\delta| C_0\tau T^2 + \frac{1}{2\chi}T^2\right)\right)\right)}_{:=c_4}\|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)}$$
$$\le \underbrace{8T^2(b-a+c_3)\left(c_2 + \left(2|\delta| C_0\tau T^2 + \frac{1}{2\chi}T^2\right)\right) + b + c_3 + k_1\tau + d(2C_0\tau k_2 + C_1)}_{:=c_5}.$$

We can choose $\varepsilon > 0$ sufficiently small such[2] that $c_4 > \chi/4$. Assuming this is so, we have proved that for any $u = (q,p) \in \mathscr{N}_\tau^\alpha(a, b, F_s^i, f_i(s)\nu_s^i, \mathbf{J})$ one has

$$\|p\|^2_{L^2_g([0,1]\times\mathbb{S}_\tau)} \leq \frac{4c_5}{\chi}.$$

Feeding this into (3.8) and (3.9) we find constants $c_6, c_7 > 0$ such that for all such maps u,

$$\|\partial_s u\|_{L^2_{G_g}(\mathbb{R}\times\mathbb{S}_\tau)} \leq c_6, \quad \sup_{s\in\mathbb{R}} \mathscr{A}_s^i(u(s,\cdot)) \leq c_7.$$

This proves the analogue of Lemma 9, and allows us to prove the analogue of Lemma 10, for elements of $\mathscr{N}_\tau^\alpha(a, b, F_s^i, f_i(s)\nu_s^i, \mathbf{J})$. We can proceed exactly as in the proof of Theorem 8 to obtain the desired compact set K_i. This completes the proof of Lemma 13. ■

Armed with Lemma 13, the proof of Theorem 12 is very standard.

Proof. *(of Theorem 12)*
Fix $N \in \mathbb{N}$ such that there exists a subdivision $0 = r_0 < r_1 < \cdots < r_N = 1$ with the property that (3.5) holds for some $\varepsilon > 0$ small enough such that Lemma 13 holds for each $i = 0, 1, \ldots, N-1$. After possibly making additional arbitrarily small perturbations of H_s for s near r_i, for each $i = 1, 2, \ldots, N-1$ (which for simplicity we omit from our notation), we may assume that $(H_{r_i}, \delta(r_i)\sigma_{r_i}, \alpha)$ satisfies Condition **(N)** for each $i = 0, 1, \ldots, N$.

Under these assumptions we define for each $i = 0, 1, \ldots, N-1$ a continuation map

$$\Psi_i(\mathbf{J}) : CF_*^\alpha(H_{r_i}, \delta(r_i)\sigma_{r_i}, \tau) \to CF_*^\alpha(H_{r_{i+1}}, \delta(r_{i+1})\sigma_{r_{i+1}}, \tau)$$

by

$$\Psi_i(\mathbf{J})(x) := \sum_{y\in\mathscr{P}_\tau^\alpha(H_{r_{i+1}}, \delta(r_{i+1})\sigma_{r_{i+1}})_k} n_i(x,y)y, \quad x \in \mathscr{P}_\tau^\alpha(H_{r_i}, \delta(r_i)\sigma_{r_i})_k,$$

where

$$n_i(x,y) := \#_2\mathscr{N}_\tau^\alpha(x, y, F_s^i, f_i(s)\nu_s^i, \mathbf{J}),$$

and $\mathscr{N}_\tau^\alpha(x, y, F_s^i, f_i(s)\nu_s^i, \mathbf{J})$ denotes the (finite) set of maps $u : \mathbb{R}\times\mathbb{S}_\tau \to T^*M$ satisfying

$$\partial_s u + \nabla_{\mathbf{J}}\mathscr{A}_s^i(u) = 0,$$

and which submit to the asymptotic conditions

$$\lim_{s\to\infty} u(s,t) = x(t), \quad \lim_{s\to-\infty} u(s,t) = y(t), \quad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0.$$

Standard Floer-theoretical arguments (see for instance [20]) tell us that the $\Psi_i(\mathbf{J})$ are chain maps that induce isomorphisms

$$\psi_i : HF_*^\alpha(H_{r_i}, \delta(r_i)\sigma_{r_i}, \tau) \to HF_*^\alpha(H_{r_{i+1}}, \delta(r_{i+1})\sigma_{r_{i+1}}, \tau)$$

for $i = 0, 1, \ldots, N-1$ on homology. The chain map Ψ from the statement of the theorem is then defined as the composition

$$\Psi := \Psi_{N-1}(\mathbf{J}) \circ \cdots \circ \Psi_1(\mathbf{J}) \circ \Psi_0(\mathbf{J}).$$

■

[2] Here of course it is important to note that this choice can be made **independently** of both N and i.

A The Lagrangian Framework

In this Appendix we outline an alternative approach to obtaining some of the results of this paper **without** using the machinery of Floer homology. Roughly speaking, this method can be used to recover all of the results proved in this paper for a more restricted class of Hamiltonian systems: the so-called **convex quadratic growth Hamiltonians**, which are those Hamiltonians $H : \mathbb{S}_\tau \times T^*M \to \mathbb{R}$ which satisfy the Abbondandolo-Schwarz growth conditions and **in addition** are **strictly fibrewise convex**.

Given such a Hamiltonian H, the idea is to study the **Lagrangian action functional** $\mathscr{S}_{L,\delta\sigma}$ on the atoroidal components of the (completed) τ-periodic loop space of M, where L is the **Fenchel dual Lagrangian** of H. The key point is to show that (for $\tau\,|\delta|$ sufficiently small), the functional $\mathscr{S}_{L,\delta\sigma}$ satisfies the **Palais-Smale condition**, and this allows one to construct the **Morse complex** of $\mathscr{S}_{L,\delta\sigma}$.

A.1 The Lagrangian action functional

Fix a Riemannian metric g on M. Suppose $L \in C^\infty(TM, \mathbb{R})$. Then $dL(q,v) \in T^*_{(q,v)}TM$, and thus its gradient $\nabla L(q,v)$ (with respect to the G_g-metric on TM) lies in $T_{(q,v)}TM$. Thus we can speak of the horizontal and vertical components

$$\nabla_q L(q,v) := \nabla L(q,v)^h \in T_qM;$$

$$\nabla_v L(q,v) := \nabla L(q,v)^v \in T_qM.$$

Thinking of $\nabla_q L$ as a map $TM \to TM$ (so its derivative is a map $d(\nabla_q L) : TTM \to TTM$), we define

$$\nabla_{qq} L(q,v)(w) := d(\nabla_q L)(q,v)(\xi_w)^v,$$

where $\xi_w \in T_{(q,v)}TM$ is the unique vector such that $\xi_w^h = w$ and $\xi_w^v = 0$. Similarly we define

$$\nabla_{qv} L(q,v)(w) := d(\nabla_q L)(q,v)(\zeta_w)^v,$$

where this time $\zeta_w \in T_{(q,v)}TM$ is the unique vector such that $\zeta_w^h = 0$ and $\zeta_w^v = w$. We define maps $\nabla_{qv}L$ and $\nabla_{vv}L$ in exactly the same way, starting with $\nabla_v L$ instead of $\nabla_q L$. Note that the operator $\nabla_{vv}L(q,v) : T_qM \to T_qM$ coincides with the second derivative of the map $v \mapsto L(q,v)$ in the vector space T_qM. If L is time-dependent then these notations still make sense, with $\nabla_{qq}L(t,q,v) := \nabla_{qq}L_t(q,v)$ etc., where $L_t(q,v) := L(t,q,v)$.

We will be interested in time-dependent Lagrangians $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ that satisfy the following **convex quadratic growth conditions**:

(L1) There exists $\ell_1 > 0$ such that for all $(t,q,v) \in \mathbb{S}_\tau \times TM$ it holds that

$$\nabla_{vv}L(t,q,v) \geq \ell_1 \mathbb{1}.$$

(L2) There exists $\ell_2 > 0$ such that for all $(t,q,v) \in \mathbb{S}_\tau \times TM$ it holds that

$$|\nabla_{vv}L(t,q,v)| \leq \ell_2,\ |\nabla_{vq}L(t,q,v)| \leq \ell_2(1+|v|),\ |\nabla_{qq}L(t,q,v)| \leq \ell_2(1+|v|^2).$$

Whilst the constants ℓ_1 and ℓ_2 depend on the choice of metric g on M, the existence of such constants does not (see [18, Proposition 3.3.1]). Note that the assumption **(L1)** implies that $\nabla_v L(t,q,\cdot) : T_qM \to T_q^*M$ is a diffeomorphism for each $(t,q) \in \mathbb{S}_\tau \times M$, and hence we may define the **Fenchel dual Hamiltonian** $H \in C^\infty(\mathbb{S}_\tau \times T^*M, \mathbb{R})$ by

$$H(t,q,p) := p(v) - L(t,q,v), \quad \text{where } \nabla_v L(t,q,v) = p. \tag{A.1}$$

It is not hard to check that asking L to satisfy **(L1)** and **(L2)** implies that H satisfies the Abbondandolo-Schwarz growth conditions **(H1)** and **(H2)**. Going the other way round, if $H \in$

$C^\infty(\mathbb{S}_\tau \times T^*M, \mathbb{R})$ satisfies **(H1)** and **(H2)** and in addition is **strictly fibrewise convex**, then there is a unique Lagrangian $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ called the **Fenchel dual Lagrangian** of H for which $\nabla_v L(t, q, \cdot)$ is a diffeomorphism for each $(t, q) \in \mathbb{S}_\tau \times M$, and which is related to H by (A.1). Moreover, this Lagrangian L satisfies **(L1)** and **(L2)**.

Denote by $\mathscr{L}_\tau M := W^{1,2}(\mathbb{S}_\tau, M)$ the **Sobolev completion** of the free loop space $\Lambda_\tau M = C^\infty(\mathbb{S}_\tau, M)$, and as before given $\alpha \in [S^1, M]$ denote by $\mathscr{L}_\tau^\alpha M$ the component of $\mathscr{L}_\tau M$ belonging to α. Unlike $\Lambda_\tau M$, the space $\mathscr{L}_\tau M$ carries the structure of a Hilbert manifold, and therefore is much better suited for doing Morse homology on. As before we denote by $\|\cdot\|_{W_g^{1,2}(\mathbb{S}_\tau)}$ the $W_g^{1,2}$-metric on $\mathscr{L}_\tau^\alpha M$.

In this Appendix we study the **Lagrangian action functional** $\mathscr{S}_{L,\sigma} : \mathscr{L}_\tau^\alpha M \to \mathbb{R}$ associated to a Lagrangian L satisfying **(L1)** and **(L2)**, together with a 2-form σ satisfying $(\boldsymbol{\sigma}\mathbf{1})$ on a σ-atoroidal class $\alpha \in [S^1, M]$. As with the Hamiltonian action functional $\mathscr{A}_{H,\sigma}$, the Lagrangian action functional $\mathscr{S}_{L,\sigma}$ is defined as the sum

$$\mathscr{S}_{L,\sigma}(q) := \mathscr{S}_L(q) + \mathscr{A}_\sigma(q),$$

where $\mathscr{S}_L$ is the **standard** Lagrangian action functional

$$\mathscr{S}_L(q) := \int_0^\tau L(t, q(t), \dot{q}(t))dt$$

(note $\mathscr{S}_L$ is defined on all of $\mathscr{L}_\tau M$), and $\mathscr{A}_\sigma$ is defined as before (only now on the completed loop space $\mathscr{L}_\tau^\alpha M$).

A standard computation (which does not use assumptions **(L1)** and **(L2)** and only requires that α is a σ-atoroidal class) tells us that if $q \in \mathscr{L}_\tau^\alpha M$ and $(q_s)_{s\in(-\varepsilon,\varepsilon)} \subseteq \mathscr{L}_\tau^\alpha M$ is a variation of q with $\frac{\partial}{\partial s}\big|_{s=0} q_s(t) =: \xi(t)$ then, writing $Y = Y_{\sigma,g}$ for the Lorentz force defined in Section 2.1, we have:

$$\frac{\partial}{\partial s}\Big|_{s=0} \mathscr{S}_{L,\sigma}(q_s) = \int_0^\tau \langle \nabla_q L(t, q, \dot{q}), \xi\rangle + \langle \nabla_v L(t, q, \dot{q}), \nabla_t \xi\rangle + \langle Y(q)\dot{q}, \xi\rangle \, dt, \tag{A.2}$$

which we can rewrite as

$$\frac{\partial}{\partial s}\Big|_{s=0} \mathscr{S}_{L,\sigma}(q_s) = \int_0^\tau \langle \nabla_q L(t, q, \dot{q}) - \nabla_t(\nabla_v L(t, q, \dot{q})) + Y(q)\dot{q}, \xi\rangle \, dt.$$

Thus $\frac{\partial}{\partial s}\big|_{s=0} \mathscr{S}_{L,\sigma}(q_s) = 0$ for all such variations q_s if and only if q satisfies the **Euler-Lagrange equations**

$$\nabla_q L(t, q, \dot{q}) - \nabla_t(\nabla_v L(t, q, \dot{q})) + Y(q)\dot{q} = 0. \tag{A.3}$$

Since $\nabla_{vv} L(t, q, v)$ is invertible by **(L1)**, we can rewrite this as

$$\nabla_t \dot{q} = [\nabla_{vv} L(t, q, \dot{q})]^{-1} \left(\nabla_q L(t, q, \dot{q}) - \nabla_{qv} L(t, q, \dot{q})\dot{q} + Y(q)\dot{q}\right).$$

In the special case $\sigma = 0$, the following theorem is due Abbondandolo and Schwarz [4] (see also [18, Proposition 3.4.1] for a detailed proof). However a careful inspection of their proof reveals that everything still goes through in our setting.

Proposition 14. *Let $\sigma \in \Omega^2(M)$ denote a closed 2-form and $\alpha \in [S^1, M]$ a σ-atoroidal class, and let $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ satisfy* **(L1)** *and* **(L2)***. Then $\mathscr{S}_{L,\sigma} : \mathscr{L}_\tau^\alpha M \to \mathbb{R}$ is of class C^1, and its differential $d\mathscr{S}_{L,\sigma}$ is Gâteau differentiable and locally Lipschitz continuous. Moreover its critical points are precisely the (smooth) solutions of the Euler-Lagrange equation* (A.3)*, and the second Gâteau differential $d^2\mathscr{S}_{L,\sigma}(q)$ at a critical point q is a Fredholm operator of finite Morse index.*

Recall that a C^1-functional $\mathscr{S} : \mathscr{M} \to \mathbb{R}$ on a Riemannian Hilbert manifold $\mathscr{M}$ satisfies the **Palais-Smale condition** if every sequence $(q_m)_{m\in\mathbb{N}} \subseteq \mathscr{M}$ for which $\mathscr{S}(q_m)$ is bounded and $\|d\mathscr{S}(q_m)\| \to 0$ admits a convergent subsequence (here $\|\cdot\|$ denotes the dual norm on $T^*_{q_m}\mathscr{M}$). The main result we wish to prove in this Appendix is the following statement.

Theorem 15. *Let $\sigma \in \Omega^2(M)$ satisfy* **(σ1)***, let $\alpha \in [S^1, M]$ denote a σ-atoroidal class, and let $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ satisfy* **(L1)** *and* **(L2)***. Then there exists $\delta_0(L, \sigma, g) > 0$ such that if $\tau |\delta| < \delta_0(L, \sigma, g)$ then $\mathscr{S}_{L,\delta\sigma} : \mathscr{L}_\tau^\alpha M \to \mathbb{R}$ satisfies the Palais-Smale condition.*

This theorem was proved for the case $\sigma = 0$ originally by Benci [8]; our proof however will closely follow that of Abbondandolo and Figalli [1, Appendix A]. The proof of Theorem 15 makes use of Lemma 3.

Proof. (of Theorem 15)

It follows from **(L1)** that there exists a constant $D > 0$ such that $L(t, q, v) \geq \ell_0 |v|^2 - D$ for all $(t, q, v) \in \mathbb{S}_\tau \times TM$. Thus for any $q \in \mathscr{L}_\tau^\alpha M$ by Lemma 3 one has

$$\mathscr{S}_{L,\delta\sigma}(q) \geq \mathscr{S}_L(q) - |\mathscr{A}_{\delta\sigma}(q)| \geq (\ell_0 - |\delta| C_0 \tau) \|\dot{q}\|^2_{L^2_g(\mathbb{S}_\tau)} - (|\delta| C_1 + D). \tag{A.4}$$

Define

$$\delta(L, \sigma, g) := \frac{\ell_0}{C_0}, \tag{A.5}$$

and fix $\delta \in \mathbb{R}$ such that $\tau |\delta| < \delta(L, \sigma, g)$. Suppose $(q_m)_{m \in \mathbb{N}} \subseteq \mathscr{L}_\tau^\alpha M$ is a sequence such that $\mathscr{S}_{L,\delta\sigma}(q_m)$ is bounded and $\|d\mathscr{S}_{L,\delta\sigma}(q_m)\| \to 0$ in the dual norm of $T^*_{q_m} \mathscr{L}_\tau^\alpha M$. Then (A.4) implies that the sequence $(\dot{q}_m)$ is bounded in L^2_g. Since

$$\mathrm{dist}(q_m(t), q_m(s)) \leq \int_s^t |\dot{q}_m| \, dr \leq |s - t|^{1/2} \|\dot{q}_m\|_{L^2_g(\mathbb{S}_\tau)},$$

the sequence (q_m) is equicontinuous, and the Arzelà-Ascoli theorem implies that up to passing to a subsequence, we may assume q_m converges uniformly to some $q \in C^0(\mathbb{S}_\tau, M)$.

We now employ the **localization** argument of Abbondandolo and Figalli, which allows us to reduce the problem to one on $\mathbb{R}^n$ (roughly speaking, this involves making an intelligent choice of a chart on $\mathscr{L}_\tau^\alpha M$ about q - see [18, Remark 3.4.1]). As a result, from now on let us assume L is defined on $\mathbb{S}_\tau \times U \times \mathbb{R}^n$ for some open set U of $\mathbb{R}^n$, with $\sigma \in \Omega^2(U)$, and that $(q_m) \subseteq \mathscr{L}_\tau U$ is a sequence such that $\mathscr{S}_{L,\delta\sigma}(q_m)$ is bounded and $\|d\mathscr{S}_{L,\delta\sigma}(q_m)\| \to 0$ in the dual norm on $T^*_{q_m} \mathscr{L}_\tau U$, with $(\dot{q}_m)$ bounded in L^2 and q_m converging uniformly to some $q \in C^0(\mathbb{S}_\tau, U)$.

This automatically implies that $q \in \mathscr{L}_\tau^\alpha U$, and up to passing to a subsequence, q_m converges weakly to q in $\mathscr{L}_\tau \mathbb{R}^n$. To complete the proof we need to show that this convergence is strong in $W^{1,2}$. Since (q_m) is bounded in $W^{1,2}$, we have $d\mathscr{S}_{L,\delta\sigma}(q_m)(q_m - q) \to 0$, and hence by (A.2) (expressed now in the simpler setting of $\mathbb{R}^n$)

$$\int_0^\tau (\partial_q L_t(q_m, \dot{q}_m) \cdot (q_m - q) + \partial_v L_t(q_m, \dot{q}_m) \cdot (\dot{q}_m - \dot{q}) + \delta Y(q_m) \dot{q}_m \cdot (q_m - q)) \, dt \to 0.$$

The term $\partial_q L_t(q_m, \dot{q}_m)$ is bounded in L^2 by **(L2)**. Similarly $Y(q_m)\dot{q}_m$ is bounded in L^2, and consequently we have

$$\int_0^\tau \partial_v L_t(q_m, \dot{q}_m) \cdot (\dot{q}_m - \dot{q}) dt \to 0.$$

From this it is straightforward to show that $\|\dot{q}_m - \dot{q}\|_{L^2_g(\mathbb{S}_\tau)} \to 0$ using **(L1)** and **(L2)**; the proof is identical to [18, Proposition 3.5.2], and hence we omit the details. ■

In general the functional $\mathscr{S}_{L,\sigma}$ is **not** of class C^2. In fact, arguing as in [4, Proposition 3.2], one sees that $\mathscr{S}_{L,\sigma}$ is of class C^2 if and only if the function $v \mapsto L(t, q, v)$ is a polynomial of degree at most 2 for each $(t, q) \in \mathbb{S}_\tau \times M$. One would think that this means that in general there is no hope of doing infinite dimensional Morse theory with $\mathscr{S}_{L,\sigma}$. Indeed, such a Morse theory needs at least C^2-regularity - for example, the Morse Lemma requires C^2-regularity - see [10]. Nevertheless, under a suitable non-degeneracy assumption (see Condition **(N)** below), it **is** still possible to construct a **Morse complex** for $\mathscr{S}_{L,\delta\sigma}$ (see Theorem 17 below). The only missing ingredient we still need for this is the existence of a **pseudo-gradient** for $\mathscr{S}_{L,\sigma}$, which we will discuss shortly in Proposition 16.

The final condition we impose is a non-degeneracy condition:

(N') Every solution q of the Euler-Lagrange equations (A.3) is **non-degenerate**, which means that there are no nonzero periodic Jacobi fields along q.

Asking for q to be a non-degenerate solution is equivalent to requiring q to be a non-degenerate critical point of $\mathscr{S}_{L,\sigma}$, in the sense that the symmetric bilinear form $d^2\mathscr{S}_{L,\sigma}(q)$ on $T_q\mathscr{L}_\tau^\alpha M$ is non-degenerate. Moreover if H is the corresponding Fenchel dual Hamiltonian then (L, σ, α) satisfies Condition **(N')** if and only if (H, σ, α) satisfies Condition **(N)**.

The following result can be proved in exactly the same way as [4, Theorem 4.1].

Proposition 16. *Let $\sigma \in \Omega^2(M)$ satisfy* **($\boldsymbol{\sigma}$1)***, and let $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ satisfy* **(L1)** *and* **(L2)***. Fix a σ-atoroidal class $\alpha \in [S^1, M]$, and assume that (L, σ, α) satisfies Condition* **(N')***. Then there exists a* **pseudo-gradient** *for $\mathscr{S}_{L,\sigma}$. That is, there exists a smooth bounded vector field $\mathscr{G}$ on $\mathscr{L}_\tau^\alpha M$ whose zeros are precisely the smooth solutions of the Euler-Lagrange equations* (A.3)*, together with a continuous function $\varepsilon \in C(\mathbb{R}, \mathbb{R}^+)$ such that*

$$d\mathscr{S}_{L,\sigma}(q)\mathscr{G}(q) \geq \varepsilon(\mathscr{S}_{L,\sigma}(q)) \, \|d\mathscr{S}_{L,\sigma}(q)\| \quad \text{for all } q \in \mathscr{L}_\tau^\alpha M,$$

and such that for any solution $q \in \Lambda_\alpha^\tau M$ of (A.3) *one has*

$$d^2\mathscr{S}_{L,\sigma}(q)(\xi, \zeta) = \langle \nabla\mathscr{G}(q)\xi, \zeta \rangle_{W_g^{1,2}(\mathbb{S}_\tau)} \quad \text{for all } \xi, \zeta \in W^{1,2}(q^*TM)$$

(here $\nabla\mathscr{G}(q) : T_q\mathscr{L}_\tau^\alpha M \to T_q\mathscr{L}_\tau^\alpha M$ is defined by $\nabla\mathscr{G}(q)\xi := [\mathscr{G}, X](q)$, where X is any vector field on $\mathscr{L}_\tau^\alpha M$ such that $X(q) = \xi$).

As mentioned above, Proposition 14, Theorem 15, and Proposition 16 imply that one can define the Morse complex of $\mathscr{S}_{L,\delta\sigma}$ for $\tau\,|\delta| < \delta_0(L, \sigma, g)$. We refer the reader to [2] for more information on the construction of the Morse complex, and for the proof of the following **Morse homology theorem**.

Theorem 17. *Let $\sigma \in \Omega^2(M)$ satisfy* **($\boldsymbol{\sigma}$1)***, and let $L \in C^\infty(\mathbb{S}_\tau \times TM, \mathbb{R})$ satisfy* **(L1)** *and* **(L2)***. Fix a σ-atoroidal class $\alpha \in [S^1, M]$, and fix $\delta \in \mathbb{R}$ such that $\tau\,|\delta| < \delta_0(L, \sigma, g)$, and assume that $(L, \delta\sigma, \alpha)$ satisfies Condition* **(N')***. Denote by $CM_*^\alpha(L, \delta\sigma, \tau)$ the free $\mathbb{Z}_2$-module generated by the solutions q of the Euler-Lagrange equations* (A.3)*, graded by their Morse index (as a critical point of $\mathscr{S}_{L,\delta\sigma}$). Then it is possible to define a map $\partial^{\mathrm{Morse}} : CM_*^\alpha(L, \delta\sigma, \tau) \to CM_{*-1}^\alpha(L, \delta\sigma, \tau)$ such that $\partial^{\mathrm{Morse}} \circ \partial^{\mathrm{Morse}} = 0$, and such that the associated* **Morse homology**

$$HM_*^\alpha(L, \delta\sigma, \tau) := H_*(CM_*^\alpha(L, \delta\sigma, \tau); \partial^{\mathrm{Morse}})$$

is isomorphic to the singular homology $H_*(\mathscr{L}_\tau^\alpha M; \mathbb{Z}_2)$.

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